De Bruijn Digraphs and Affine Transformations *

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Abstract Let $\mathbb{Z}_d^n$ be the additive group of $1\times n$ row vectors over $\mathbb{Z}_d$. For an $n\times n$ matrix T over $\mathbb{Z}_d$ and $\omega\in\mathbb{Z}_d^n$, the affine transformation $F_{T,\omega}$ of $\mathbb{Z}_d^n$ sends x to $xT+\omega$. Let $\langle\alpha\rangle$ be the cyclic group generated by a vector $\alpha\in\mathbb{Z}_d^n$. The affine transformation coset pseudo-digraph $TCP(\mathbb{Z}_d^n,\alpha,F_{T,\omega})$ has the set of cosets of $\langle\alpha\rangle$ in $\mathbb{Z}_d^n$ as vertices and there are c arcs from $x+\langle\alpha\rangle$ to $y+\langle\alpha\rangle$ if and only if the number of $z\in x+\langle\alpha\rangle$ such that $F_{T,\omega}(z)\in y+\langle\alpha\rangle$ is c. We prove that the following statements are equivalent: (a) $TCP(\mathbb{Z}_d^n,\alpha,F_{T,\omega})$ is isomorphic to the d-nary $(n-1)$-dimensional De Bruijn digraph; (b) $TCP(\mathbb{Z}_d^n,\alpha,F_{T,\omega})$ is primitive; (c) α is a cyclic vector for T. This strengthens a result conjectured by Fiduccia and Jacobson [Universal multistage networks via linear permutations, in: Proceedings of the 1991 ACM/IEEE Conference on Supercomputing, ACM Press, 1991, New York, pp. 380–389]. Under the further assumption that T is invertible we show that each component of $TCP(\mathbb{Z}_d^n,\alpha,F_{T,\omega})$ is a conjunction of a cycle and a De Bruijn digraph, namely a generalized wrapped butterfly. Finally, we discuss the affine TCP digraph representations for a class of digraphs introduced by Coudert, Ferreira and Perennes [Isomorphisms of the De Bruijn digraph and free-space optical networks, Networks 40 (2002) 155–164].

Keywords– affine transformation, De Bruijn digraph, wrapped butterfly, transformation coset pseudo-digraph.

1 Introduction

This paper is about an interesting phenomenon, namely sometimes some digraphs arising from seemingly very general algebraic constructions or those restricted by a simple algebraic requirement turn out to be of a very regular pattern and have close connection with a family of seemingly very special digraphs, the so-called De Bruijn digraphs [2]. Perhaps we should call this special type of digraphs universal digraphs as they already appear in a wide range of research [1–22]. In this sense, our work here will confirm the assertion that the mysterious De Bruijn digraphs are really universal. Let us postpone a more accurate

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description of the phenomenon referred above to the end of this section and review first some preliminary definitions to be used in this paper.

As usual, $\mathbb{Z}_d$ denotes the ring of integers modulo d and $\mathbb{Z}_d^n$ represents the set of $1 \times n$ matrices over $\mathbb{Z}_d$. $\mathbb{Z}_d^n$ can be viewed as the n-dimensional free module over $\mathbb{Z}_d$ and has a *standard basis* $\{e_0, e_1, \cdots, e_{n-1}\}$, where e_i is the vector with a single 1 in the $(i+1)$th position and 0's elsewhere. Unless otherwise specified, e_i is always of n-dimensional. A subset $\{\alpha_1, \cdots, \alpha_t\}$ of $\mathbb{Z}_d^n$ is called *linearly independent* over $\mathbb{Z}_d^n$ if and only if whenever $\sum_{i=1}^t k_i \alpha_i = 0$ with $k_i \in \mathbb{Z}_d$, then $k_1 = k_2 = \cdots = k_t = 0$.

Let $Mat_n(\mathbb{Z}_d)$ be the set of all $n \times n$ matrices over $\mathbb{Z}_d$. The set of invertible matrices in $Mat_n(\mathbb{Z}_d)$ is denoted $GL_n(\mathbb{Z}_d)$. Arbitrarily picking $T \in Mat_n(\mathbb{Z}_d)$ and $\omega \in \mathbb{Z}_d^n$, the *affine transformation* $F_{T,\omega}$ on $\mathbb{Z}_d^n$ is defined by $F_{T,\omega}(x) = xT + \omega, \forall x \in \mathbb{Z}_d^n$. For $i = 0, 1, \cdots, n-1$, let $T(i, \cdot) = e_i T$ and $T(\cdot, i) = T e_i^\top$, respectively.

For $S \subseteq \mathbb{Z}_d^n$, the submodule generated (or spanned) by S is $\langle S \rangle \doteq \{\sum_{i=1}^m c_i s_i : \ c_i \in \mathbb{Z}_d, \ s_i \in S, \ m \geq 0\}$. If a submodule M is spanned by a set S, we call S a *generating set* of M. For any nonzero vector $\alpha \in \mathbb{Z}_d^n$ and $T \in Mat_n(\mathbb{Z}_d)$, the *$T$-cyclic submodule* generated by α is the submodule $\mathbb{Z}_d(\alpha; T) \doteq \langle \{\alpha T^k, \ k \geq 0\} \rangle$. A vector α is a *cyclic vector* for T provided $\mathbb{Z}_d(\alpha; T) = \mathbb{Z}_d^n$. For any finite set S, $\#S$ denotes its cardinality.

Let Γ be a digraph. The vertex set and the arc set of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. For a subset V_0 of $V(\Gamma)$, we write $N_\Gamma(V_0)$ for the out-neighbor set of V_0, which is $\{w \in V(\Gamma) : \ \exists \ u \in V_0, \ e \in E(\Gamma), \ e \text{ starts from } u \text{ and ends at } w\}$. We let $N_\Gamma^0(V_0) = V_0$ and define inductively that $N_\Gamma^k(V_0) = N_\Gamma(N_\Gamma^{k-1}(V_0))$ for any positive integer k. A digraph Γ is *strongly connected* if for any two vertices x and y of Γ, there always exists in Γ a path from x to y. We say that a digraph is *connected* if its underlying undirected graph is connected. The *components* of a digraph refer to its connected components. A digraph Γ is said to be *balanced* if the in-degree and out-degree of each of its vertices are equal. We write $G \cong H$ to denote that G and H are isomorphic digraphs. If $G \cong H$, we think of them as different representations of the same object and thus often do not distinguish between them.

Given an $\alpha \in \mathbb{Z}_d^n$ and a transformation F on $\mathbb{Z}_d^n$, the *transformation coset pseudo-digraph* (TCP digraph, for short) of $\mathbb{Z}_d^n$ with respect to them, denoted $TCP(\mathbb{Z}_d^n, \alpha, F)$, is the digraph whose vertex set is $\mathbb{Z}_d^n / \langle \alpha \rangle$ and the number of arcs from vertex $x + \langle \alpha \rangle$ to vertex $y + \langle \alpha \rangle$ is $\#\{z \in x + \langle \alpha \rangle : F(z) \in y + \langle \alpha \rangle\}$. Let S be a union of several cosets of $\langle \alpha \rangle$ in $\mathbb{Z}_d^n$. If there are no arcs between $S/\langle \alpha \rangle$ and $(\mathbb{Z}_d^n - S)/\langle \alpha \rangle$ in $TCP(\mathbb{Z}_d^n, \alpha, F)$, then we use the notation $TCP(S, \alpha, F)$ to represent the subdigraph induced by the vertex set $S/\langle \alpha \rangle$, which is a TCP digraph on S.

The *conjunction* $\Gamma_1 \otimes \Gamma_2$ of two digraphs Γ_1 and Γ_2 has $V(\Gamma_1) \times V(\Gamma_2)$ as the vertex set and $E(\Gamma_1 \otimes \Gamma_2)$ has $(x_1, x_2)(y_1, y_2)$ as an element of multiplicity $m_1 m_2$, where m_i is the multiplicity of (x_i, y_i) in $E(\Gamma_i)$, $i = 1, 2$. We write C_t for the cycle of length t. A *generalized t-cycle* [13, 20] is a digraph which permits a homomorphism from it to C_t. Clearly, for any digraph Γ, $C_t \otimes \Gamma$ is a generalized t-cycle. If G is a generalized t-cycle with a given homomorphism f in our mind, we refer to a subset of $V(G)$ of the form $f^{-1}(v)$ for $v \in V(C_t)$ as a *stage* of G.

For any two positive integers d and n, the d-nary n-dimensional *De Bruijn digraph* $B(d,n)$ [2] has $\mathbb{Z}_d^n$ as its vertex set and there is an arc from $x = (x_0, x_1, \cdots, x_{n-1})$ to $y = (y_0, y_1, \cdots, y_{n-1})$ if and only if $y_i = x_{i+1}$ for $i = 0, \cdots, n-2$. We also define $B(d,0)$ to be the digraph with one vertex and d loops. The r-stage d-nary n-dimensional *generalized wrapped butterfly* [17, 20], denoted by $GWBY(r,d,n)$, is the generalized r-cycle $C_r \otimes B(d,n)$. When $r = n$, $GWBY(r,d,n)$ becomes the well-known d-nary n-dimensional *wrapped butterfly* and is commonly denoted by $WBY(d,n)$ [20]. The conjunction of a path with a De Bruijn digraph is called a generalized butterfly [11, 14, 21].

In the course of investigating highly arc transitive digraphs, Praeger [17, Theorem 2.9] proved a surprising result that if a digraph admits certain symmetries then it must be a generalized wrapped butterfly. The school of Fiol investigated some symmetry property of the generalized wrapped butterflies [3, 8]. Hotzel [14] established many results about generalized butterflies and then made a very interesting conjecture that these digraphs have the largest order automorphism group among all MIN digraphs with a set of equal parameters. With the help of the powerful tool of Layered Cross Product [9], Golbandi and Litman [11] worked out a series of deep characterizations for the generalized butterflies in terms of various symmetry properties. For some other characterizations for generalized butterflies or more general De Bruijn-like structures, see [21] and its references.

In the study of OTIS layouts of De Bruijn networks, Coudert, Ferreira and Perennes [6] introduced a class of digraphs and discussed when such digraphs will be isomorphic to De Bruijn digraphs. They claimed without proof that each component of that type of digraphs is the conjunction of a De Bruijn digraph with a cycle [6, p. 159]. However, recently, Coudert told us via email that the proof can be found in [5].

The following results of Fiduccia et al. are motivated by their research on nonblocking switching networks [4, 10].

Theorem 1.1. *[10] For any $T \in GL_n(\mathbb{Z}_d)$ and any $\alpha \in \mathbb{Z}_d^n$, $TCP(\mathbb{Z}_2^n, \alpha, F_{T,0}) \cong B(2, n-1)$ if and only if any of the following statements holds:*

(a) α is a cyclic vector for T;

(b) The digraph $TCP(\mathbb{Z}_2^n, \alpha, F_{T,0})$ is primitive;

(c) The digraph $TCP(\mathbb{Z}_2^n, \alpha, F_{T,0})$ is strongly connected;

(d) The digraph $TCP(\mathbb{Z}_2^n, \alpha, F_{T,0})$ is connected.

Conjecture 1.2. *[10] Let d be a prime number, $\omega \in \mathbb{Z}_d^n$ and $T \in GL_n(\mathbb{Z}_d)$. For any $\alpha \in \mathbb{Z}_d^n$, $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega}) \cong B(d, n-1)$ if and only if α is a cyclic vector for T.*

Following the same line of research as described above, our paper aims to further study the De Bruijn-like structure hidden in affine TCP digraphs.

In Section 2, we prove that for any positive integers d, n, vectors $\alpha, \omega \in \mathbb{Z}_d^n$ and matrix $T \in Mat_n(\mathbb{Z}_d)$, the following statements are equivalent: (a) $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is isomorphic to the d-nary $(n-1)$-dimensional De Bruijn digraph; (b) $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is primitive; (c) α is a cyclic vector for T. This is much stronger than the statement of Conjecture 1.2. In Section 3, with the further assumption that $T \in GL_n(\mathbb{Z}_d)$ we show that each component of $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is a conjunction of a cycle and a De Bruijn digraph, i.e., a generalized wrapped butterfly. In Section 4 we prove that the digraphs introduced by Coudert et al. [6] turn out to be a class of TCP digraphs. We present some necessary and sufficient conditions for such digraphs to be isomorphic with affine TCP digraphs and furthermore, with the De Bruijn digraphs.

2 De Bruijn digraphs and affine transformations

In what follows, we fix the notation that $\alpha, \omega \in \mathbb{Z}_d^n$, $d_0 = \#\langle\alpha\rangle$, $T \in Mat_n(\mathbb{Z}_d)$. For $i \geq 0$, instead of αT^i, we simply write α^i. Note that $\alpha \neq \alpha^1$ if α is not fixed by T and it always holds $\alpha = \alpha^0$. If $T \in GL_n(\mathbb{Z}_d)$, we also write α^{-i} for αT^{-i}. Let $m_{T,\alpha} = min\{l > 0 : \alpha^l \in \langle\alpha^{l-1}, \cdots, \alpha^1, \alpha\rangle\}$. It is easy to see that $\#\mathbb{Z}_d(\alpha; T) \leq d_0^{m_{T,\alpha}}$.

Inspired by Theorem 1.1, we determine in this section some general conditions under which $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ turns out to be a De Bruijn digraph.

Let

$$P_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}.$$

Lemma 2.1. $B(d, n-1) \cong TCP(\mathbb{Z}_d^n, e_{n-1}, F_{P_n,0})$.

Proof. The verification is routine. □

We say that a digraph Γ is *nice* in case that there is a constant c such that for each vertex v of Γ and each $w \in N_\Gamma(v)$ there are exactly c arcs going from v to w. We call c the *multiplicity* of the nice digraph Γ. To get further isomorphism results about TCP digraphs, let us show that all affine TCP digraphs are nice. For $\Gamma = TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$, there are three cases to examine:

(i) $\alpha = 0$ (which is equivalent to $d_0 = 1$);

(ii) $\alpha \neq 0$, αT and α are linearly independent over $\mathbb{Z}_{d_0}$;

(iii) $\alpha \neq 0$, αT and α are linearly dependent over $\mathbb{Z}_{d_0}$.

Lemma 2.2. *$\Gamma = TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is nice with multiplicity 1 in cases (i) and (ii) and is nice with multiplicity $c > 1$ in case (iii) where c only depends on α and T.*

Proof. For case (i), each vertex has only one arc beginning at it and hence Γ is nice with multiplicity 1.

Next consider case (ii). At this time, for any $r_1 \neq r_2 \in \{0, \cdots, d_0 - 1\}$ and $x \in \mathbb{Z}_d^n$ it holds $F_{T,\omega}(x + r_1\alpha) - F_{T,\omega}(x + r_2\alpha) = (r_1 - r_2)\alpha T \notin \langle\alpha\rangle$. In other words, each coset of $\langle\alpha\rangle$ has at most one common element with $F_{T,\omega}(x + \langle\alpha\rangle)$. By the definition of a TCP digraph, we derive again that Γ is nice with multiplicity 1.

Finally, we turn to case (iii). In this situation, there exist $r_1, r_2 \in \{0, \cdots, d_0 - 1\}$ which are not all zeros such that $r_1\alpha T = r_2\alpha$. We claim that $r_1 \neq 0$ as otherwise we will have $r_2\alpha = 0$ and then $r_2 = 0 = r_1$ follows, which is a contradiction. So we can set $r_0 = \min\{r_1 > 0 : r_1\alpha T = r_2\alpha \text{ for some integer } r_2\}$ and let $c = \frac{d_0}{r_0} > 1$. Observe that $x + \langle\alpha\rangle = \cup_{j=0}^{r_0-1} Y_j$, where $Y_j = \{x + (tr_0 + j)\alpha : t = 0, \cdots, c - 1\}$. Since $r_0\alpha T \in \langle\alpha\rangle$, we get $F_{T,\omega}(x + (tr_0 + j)\alpha) = xT + tr_0\alpha T + j\alpha T + \omega \in xT + j\alpha T + \omega + \langle\alpha\rangle$ and thus we know that all c elements of Y_j are mapped by $F_{T,\omega}$ into a single coset of $\langle\alpha\rangle$. To finish the proof, it remains to check that different Y_j correspond to different cosets of α. This is done by the following calculation: $F_{T,\omega}(x + j\alpha) - F_{T,\omega}(x + i\alpha) = (j - i)\alpha T \notin \langle\alpha\rangle$ for $0 \leq i \neq j \leq r_0 - 1$. □

Remark 2.3. *When Γ_1 and Γ_2 are two nice digraphs with the same multiplicity, to prove that a bijection $f : V(\Gamma_1) \to V(\Gamma_2)$ induces an isomorphism from Γ_1 to Γ_2, we need only verify $f \circ N_{\Gamma_1}$ and $N_{\Gamma_2} \circ f$ are equal as a map from $V(\Gamma_1)$ to the power set of $V(\Gamma_2)$. Lemma 2.2 together with this observation will be applied to affine TCP digraphs several times in this paper.*

Lemma 2.4. *For any $\omega \in \mathbb{Z}_d^n$ and*

$$Q_n = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in Mat_n(\mathbb{Z}_d),$$

we have $TCP(\mathbb{Z}_d^n, e_{n-1}, F_{Q_n,\omega}) \cong TCP(\mathbb{Z}_d^n, e_{n-1}, F_{P_n,0})$.

Proof. Let $\Gamma_1 = TCP(\mathbb{Z}_d^n, e_{n-1}, F_{Q_n,\omega})$ and $\Gamma_2 = TCP(\mathbb{Z}_d^n, e_{n-1}, F_{P_n,0})$. Each element x of $\mathbb{Z}_d^n$ can be uniquely expressed as $x = \sum_{i=0}^{n-1} x_i e_i$, $x_i \in \mathbb{Z}_d$. We construct recursively a set of maps $f_k : \mathbb{Z}_d^n \to \mathbb{Z}_d$, $0 \leq k \leq n - 1$, by putting

$$f_0(x) = x_0,\ f_k(x) = f_{k-1}(F_{Q_n,\omega}(x)),\ 1 \leq k \leq n - 1, \forall x \in \mathbb{Z}_d^n. \tag{1}$$

Then we define a map f on $\mathbb{Z}_d^n$ such that $f(x) = (f_0(x), f_1(x), \cdots, f_{n-1}(x))$. Let us show that f induces an isomorphism from Γ_1 to Γ_2.

We first prove that f is a bijection. Let $c_0 = 1$ and $c_k = \sum_{i=1}^{k} a_{n-i}c_{k-i}$ for $1 \leq k \leq n-1$. A direct calculation gives

$$Q_n^0(\cdot, 0) = (1, 0, \cdots, 0)^\top = (c_0, 0, \cdots, 0)^\top,$$

$$Q_n^k(\cdot, 0) = Q_n Q_n^{k-1}(\cdot, 0) = (c_k, c_{k-1}, \cdots, c_0, 0, \cdots, 0)^\top,\ 1 \leq k \leq n-1.$$

Note that $f_0(x) = x_0 = xQ_n^0(\cdot, 0)$, $f_1(x) = f_0(xQ_n + \omega Q_n^0) = xQ_n(\cdot, 0) + \omega Q_n^0(\cdot, 0)$. Let $k \geq 2$. Assume we already know that $f_{k-1}(x) = xQ_n^{k-1}(\cdot, 0) + \omega \sum_{i=0}^{k-2} Q_n^i(\cdot, 0)$. Then we have

$$f_k(x) = f_{k-1}(xQ_n + \omega) = (xQ_n + \omega)Q_n^{k-1}(\cdot, 0) + \omega \sum_{i=0}^{k-2} Q_n^i(\cdot, 0) = xQ_n^k(\cdot, 0) + \omega \sum_{i=0}^{k-1} Q_n^i(\cdot, 0).$$

Therefore, we deduce by induction that

$$\begin{aligned} f(x) &= x(Q_n^0(\cdot, 0), Q_n^1(\cdot, 0), \cdots, Q_n^{n-1}(\cdot, 0)) + \beta \\ &= xA + \beta, \end{aligned} \tag{2}$$

where $\beta = (0, \omega \sum_{i=0}^{0} Q_n^i(\cdot, 0), \cdots, \omega \sum_{i=0}^{n-2} Q_n^i(\cdot, 0))$ and

$$A = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ 0 & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_0 & c_1 \\ 0 & 0 & \cdots & 0 & c_0 \end{pmatrix}.$$

Since $c_0 = 1$, A is invertible, thereby proving that f is a bijection.

Next we demonstrate that the map $\bar{f} : V(\Gamma_1) \to V(\Gamma_2)$ defined by

$$\bar{f}(x + \langle e_{n-1} \rangle) = f(x) + \langle e_{n-1} \rangle,\ x \in \mathbb{Z}_d^n,$$

is a bijection. We already know that f is surjective and hence so does $\bar{f}$. We continue to prove that $\bar{f}$ is injective. Assume that $f(x) + \langle e_{n-1} \rangle = f(y) + \langle e_{n-1} \rangle$ for some $x, y \in \mathbb{Z}_d^n$. Then from Eq. (2), $(x - y)A \in \langle e_{n-1} \rangle$. Observe that $e_{n-1}A = e_{n-1}$ and A is invertible. Hence $x - y \in \langle e_{n-1} \rangle A^{-1} = \langle e_{n-1} \rangle$. Therefore, $\bar{f}$ is injective, as desired.

Now we are ready to verify that $\bar{f}$ is an isomorphism. By Lemma 2.2 and Remark 2.3 our task reduces to showing that for any $x \in \mathbb{Z}_d^n$,

$$\bar{f} \circ N_{\Gamma_1}(x + \langle e_{n-1} \rangle) = N_{\Gamma_2} \circ \bar{f}(x + \langle e_{n-1} \rangle). \tag{3}$$

Moreover, we can obtain from Eq. (1) that $(f \circ F_{Q_n,\omega})_k(x) = f_{k+1}(x) = (F_{P_n,0} \circ f)_k(x),\ 0 \leq k \leq n-2$, i.e., $(f \circ F_{Q_n,\omega})(x) + \langle e_{n-1}\rangle = (F_{P_n,0} \circ f)(x) + \langle e_{n-1}\rangle$. Hence

$$\begin{aligned}(\bar{f} \circ N_{\Gamma_1})(x + \langle e_{n-1}\rangle) &= \{(f \circ F_{Q_n,\omega})(x + re_{n-1}) + \langle e_{n-1}\rangle : r \in \mathbb{Z}_d\} \\ &= \{(F_{P_n,0} \circ f)(x + re_{n-1}) + \langle e_{n-1}\rangle : r \in \mathbb{Z}_d\} \\ &= \{F_{P_n,0}(f(x) + re_{n-1}A) + \langle e_{n-1}\rangle : r \in \mathbb{Z}_d\}.\end{aligned}$$

Note further that $e_{n-1}A = e_{n-1}$ and $(N_{\Gamma_2} \circ \bar{f})(x + \langle e_{n-1}\rangle) = \{F_{P_n,0}(f(x) + re_{n-1}) + \langle e_{n-1}\rangle : r \in \mathbb{Z}_d\}$. This then ensures the truth of Eq. (3) and the proof is complete. □

Theorem 2.5. *$TCP(\mathbb{Z}_d(\alpha;T), \alpha, F_{T,\omega}) \cong B(d_0, m_{T,\alpha} - 1)$ for $\omega \in \mathbb{Z}_d(\alpha;T)$ if and only if $\#\mathbb{Z}_d(\alpha;T) = d_0^{m_{T,\alpha}}$.*

Proof. The necessity is obvious and thus we turn to consider the sufficiency. Let us use the shorthand notation $\Gamma = TCP(\mathbb{Z}_d(\alpha;T), \alpha, F_{T,\omega})$.

First consider the case of $m_{T,\alpha} = 1$. We can easily check that Γ has only one vertex with d_0 loops and hence is just $B(d_0, 0)$. This is the result.

So we assume from now on that $m_{T,\alpha} > 1$. We first point out that $d_0 = \#\langle\alpha\rangle$ gives $\#\langle\alpha^i\rangle \mid d_0$ for $i = 1, \cdots, m_{T,\alpha} - 1$. Then, considering the definition of $m_{T,\alpha}$, $\#\mathbb{Z}_d(\alpha;T) = d_0^{m_{T,\alpha}}$ asserts that each element $x \in \mathbb{Z}_d(\alpha;T)$ can be uniquely written as a linear combination of $\alpha^{m_{T,\alpha}-1}, \cdots, \alpha^1, \alpha$ over $\mathbb{Z}_{d_0}$, say $x = \sum_{i=0}^{m_{T,\alpha}-1} x_i \alpha^{m_{T,\alpha}-1-i},\ x_i \in \mathbb{Z}_{d_0}$. Let $\{\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{m-1}\}$ be the standard basis of $\mathbb{Z}_{d_0}^{m_{T,\alpha}}$. It is evident that the map $\eta : \sum_{i=0}^{m_{T,\alpha}-1} x_i\varepsilon_i \mapsto \sum_{i=0}^{m_{T,\alpha}-1} x_i \alpha^{m_{T,\alpha}-1-i}$ is a $\mathbb{Z}_{d_0}$-isomorphism from the $\mathbb{Z}_{d_0}$-module $\mathbb{Z}_{d_0}^{m_{T,\alpha}}$ to the $\mathbb{Z}_{d_0}$-module $\mathbb{Z}_d(\alpha;T)$. Let D be the $m_{T,\alpha} \times m_{T,\alpha}$ matrix defined by $D(i,\cdot) = \alpha^{m_{T,\alpha}-1-i}$, $i = 0, \cdots, m_{T,\alpha} - 1$ and suppose that $\alpha^{m_{T,\alpha}} = \sum_{i=0}^{m_{T,\alpha}-1} a_i \alpha^i,\ a_i \in \mathbb{Z}_{d_0}$. Note that $DT = \tilde{T}D$, where

$$\tilde{T} = \begin{pmatrix} a_{m_{T,\alpha}-1} & a_{m_{T,\alpha}-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in Mat_{m_{T,\alpha}}(\mathbb{Z}_{d_0}).$$

The assumption $\omega \in \mathbb{Z}_d(\alpha;T)$ guarantees that $\omega = bD$ for some $b \in \mathbb{Z}_{d_0}^{m_{T,\alpha}}$. For the time being it is not hard to verify that the map $\bar{\eta} : x + \langle\varepsilon_{m_{T,\alpha}-1}\rangle \mapsto xD + \langle\alpha\rangle$ is an isomorphism from $TCP(\mathbb{Z}_{d_0}^{m_{T,\alpha}}, \varepsilon_{m_{T,\alpha}-1}, F_{\tilde{T},b})$ to Γ. Finally, making use of Lemmas 2.1 and 2.4 we get $\Gamma \cong B(d_0, m_{T,\alpha} - 1)$, as desired. □

Lemma 2.6. *If $\Gamma = TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is a primitive digraph, then α is a cyclic vector for T, namely $\mathbb{Z}_d^n = \mathbb{Z}_d(\alpha;T)$.*

Proof. Let $V_0 = \mathbb{Z}_d(\alpha;T)/\langle\alpha\rangle$ and $V_k = (\sum_{i=1}^{k}\omega^{i-1} + \mathbb{Z}_d(\alpha;T))/\langle\alpha\rangle$ for $k \geq 1$. From the definition of Γ we have $N_\Gamma(V_k) \subseteq V_{k+1}$ and $N_\Gamma^k(\langle\alpha\rangle) \in V_k$ for $k \geq 0$. Moreover, if $k' < k$ and $V_k \cap V_{k'} \neq \emptyset$, then we will find that $\sum_{i=k'+1}^{k}\omega^{i-1} \in \mathbb{Z}_d(\alpha;T)$ and consequently $V_k = V_{k'}$ follows. Since Γ is strongly connected, $V(\Gamma) = \cup_{k\geq 0}N_\Gamma^k(\langle\alpha\rangle)$ and thus we conclude that there is $k > 0$ such that $V_k \cap V_0 \neq \emptyset$, which amounts to saying that $V_0 = V_k$, as we note above. Let us assume that $s = \min\{k > 0 : V_0 = V_k\}$. To sum up, we have arrived at the fact that $V_0, \cdots, V_{s-1}$ must be mutually disjoint nonempty sets whose union is $V(\Gamma)$ and $N_\Gamma(V_i) \subseteq V_{i+1}$ for $i \in \mathbb{Z}_s$. This tells us that the length of any cycle of Γ is a multiple of s. This is possible only if $s = 1$ as Γ is primitive. Therefore, $\mathbb{Z}_d^n/\langle\alpha\rangle = V(\Gamma) = V_0 = \mathbb{Z}_d(\alpha;T)/\langle\alpha\rangle$, completing the proof. □

In order to establish the main result of this section, we need the following lemma, an extension of which can be found in [7].

Lemma 2.7. $m_{T,\alpha} \leq n$.

Proof. Let $f(x)$ be the characteristic polynomial of $T \in Mat_n(\mathbb{Z}_d)$. By the Cayley-Hamilton theorem, it holds $f(T) = 0$ and so $\alpha f(T) = 0$. But $f(x)$ is a degree n monic polynomial. Henceforth, we can derive from $\alpha f(T) = 0$ that $\alpha T^n = \alpha^n$ is a linear combination of $\alpha T^i = \alpha^i$, $i = 1, \cdots, n$. At this stage there is no difficulty to see from the definition of $m_{T,\alpha}$ that $m_{T,\alpha} \leq n$, as required. □

Theorem 2.8. *The following assertions are equivalent:*

(a) $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega}) \cong B(d, n-1)$;

(b) $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ *is primitive;*

(c) α *is a cyclic vector for* T.

Proof. (a) implies (b) since $B(d, n-1)$ is primitive. (b) implies (c) is straightforward from Lemma 2.6. Now we prove that (c) implies (a).

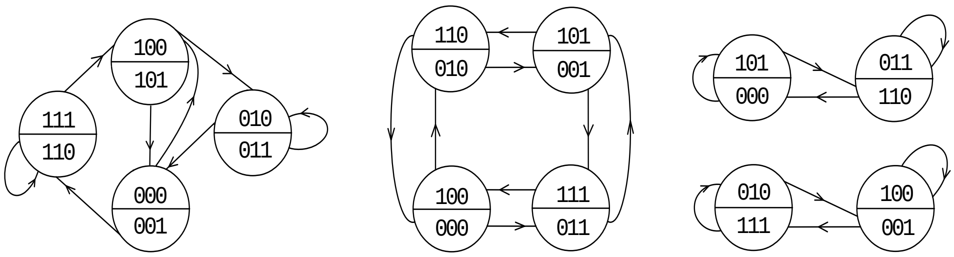
Suppose α is a cyclic vector for T, we get $d^n = \#\mathbb{Z}_d(\alpha;T) \leq d_0^{m_{T,\alpha}}$. But Lemma 2.7 asserts that $m_{T,\alpha} \leq n$ and so we have $d_0^{m_{T,\alpha}} \leq d_0^n \leq d^n$. Comparing these two inequalities, we find out that $\#\mathbb{Z}_d(\alpha;T) = d_0^{m_{T,\alpha}}$. Thus (a) comes from Theorem 2.5. □

Note that Theorem 2.8 settles Conjecture 1.2 positively and can be viewed as a generalization of parts (a) and (b) of Theorem 1.1. We mention that even for $T \in GL_n(\mathbb{Z})$ a direct generalization of parts (c) and (d) of Theorem 1.1 is not valid any more, as illustrated by the ensuing Example 2.10.

Example 2.9. *See Fig. 1(a).* $T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \alpha = (001), \omega = (100)$. $TCP(\mathbb{Z}_2^3, \alpha, F_{T,\omega}) \cong B(2,2)$.

Example 2.10. *See Fig. 1(b).* $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \alpha = (100), \omega = (011)$. $TCP(\mathbb{Z}_2^3, \alpha, F_{T,\omega})$ *is strongly connected but not isomorphic to* $B(2,2)$*. It is isomorphic to* $C_2 \otimes B(2,1)$.

Example 2.11. *See Fig. 1(c).* $T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \alpha = (101), \omega = (101)$. $TCP(\mathbb{Z}_2^3, \alpha, F_{T,\omega})$ *is not connected. Each component is isomorphic to* $B(2,1)$.



(a) (b) (c)

Fig. 1

3 Components of TCP digraphs

Throughout this section we follow the convention made in Section 2 and further assume that T **is invertible**. Observe that under this assumption $\Gamma = TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ becomes a balanced digraph. In the following we will write $[x] = x + \mathbb{Z}_d(\alpha; T)$.

Lemma 3.1. *For any* $x \in \mathbb{Z}_d^n$, $[x]/\langle\alpha\rangle$ *is contained in a single component of* Γ.

Proof. To accomplish the proof, it suffices to verify that for any $y \in [x]$, $y + \langle\alpha\rangle$ and $x + \langle\alpha\rangle$ are included in the same component of Γ. Suppose $y = x + \sum_{i=0}^{m_{T,\alpha}-1} b_i \alpha^i$ for some $b_i \in \mathbb{Z}_d$, $0 \le i \le m_{T,\alpha} - 1$. Let $z_{m_{T,\alpha}-1} = x^{-(m_{T,\alpha}-1)} - \sum_{i=1}^{m_{T,\alpha}-1} \omega^{-i}$ and let $z_i = x^{-i} - \sum_{k=1}^{i} \omega^{-k} + \sum_{k=1}^{m_{T,\alpha}-i-1} b_{i+k}\alpha^k$ for $1 \le i \le m_{T,\alpha} - 2$. Then we find that $y + \langle\alpha\rangle \in N_\Gamma(z_1 + \langle\alpha\rangle)$ and $z_i + \langle\alpha\rangle \in N_\Gamma(z_{i+1} + \langle\alpha\rangle)$ for $1 \le i \le m_{T,\alpha} - 2$, which tells us that $y + \langle\alpha\rangle \in N_\Gamma^{m_{T,\alpha}-1}(z_{m_{T,\alpha}-1} + \langle\alpha\rangle)$. But obviously $x + \langle\alpha\rangle \in N_\Gamma^{m_{T,\alpha}-1}(z_{m_{T,\alpha}-1} + \langle\alpha\rangle)$ as well and so we are done. ☐

Lemma 3.1 tells us that each component of Γ consists of several cosets of $[0]/\langle\alpha\rangle$. This then justifies our assumption in the next lemma.

Lemma 3.2. *If a component of Γ contains exactly t cosets of $[0]/\langle\alpha\rangle$, then it is a generalized t-cycle with stages $V_i = [x_i]/\langle\alpha\rangle$ for some $x_i \in \mathbb{Z}_d^n$, $i \in \mathbb{Z}_t$.*

Proof. The case of $t = 1$ is trivial. We now assume that a component Γ_1 of Γ contains $t > 1$ cosets, say $[x_i]/\langle\alpha\rangle$, $i \in \mathbb{Z}_t$. First note that for each x_i we have $N_\Gamma([x_i]/\langle\alpha\rangle) \subseteq [F_{T,\omega}(x_i)]/\langle\alpha\rangle$. Then we claim that $[F_{T,\omega}(x_i)] \neq [x_i]$, $i \in \mathbb{Z}_t$, since otherwise $[x_i]/\langle\alpha\rangle$ is itself a component of Γ, contradicting $t > 1$. Now we can reorder the t cosets by choosing an arbitrary x_0 and set $x_i = F_{T,\omega}(x_{i-1})$ so that $N_\Gamma([x_{i-1}]/\langle\alpha\rangle) \subseteq [x_i]/\langle\alpha\rangle$, $i = 1, \cdots, t-1$. Note that $N_\Gamma([x_{t-1}]/\langle\alpha\rangle) \subseteq [x_0]/\langle\alpha\rangle$ or else $N_\Gamma([x_{t-1}]/\langle\alpha\rangle)$ falls in another coset which will contradict the fact that Γ_1 is balanced and consists of t cosets. At this stage we can check that the map that sends $V_i = [x_i]/\langle\alpha\rangle$ into i for each $i \in \mathbb{Z}_t$ is a homomorphism from Γ_1 to C_t, thus completing the proof. □

If $xT - x + \omega \in \mathbb{Z}_d(\alpha; T)$, then $F_{T,\omega}([x]) \subseteq [xT + \omega] = [x]$ and $F_{T,xT-x+\omega}([0]) \subseteq [0]$. Hence the digraphs $H_1 = TCP([x], \alpha, F_{T,\omega})$ and $H_2 = TCP([0], \alpha, F_{T,xT-x+\omega})$ are well defined.

Lemma 3.3. *If $xT - x + \omega \in \mathbb{Z}_d(\alpha; T)$, then $H_1 \cong B(d_0, m_{T,\alpha} - 1)$ if and only if $\#\mathbb{Z}_d(\alpha; T) = d_0^{m_{T,\alpha}}$.*

Proof. The necessity is clear. Conversely, assume that $\#\mathbb{Z}_d(\alpha; T) = d_0^{m_{T,\alpha}}$. By Theorem 2.5 we conclude that $H_2 \cong B(d_0, m_{T,\alpha} - 1)$ and hence our goal becomes $H_1 \cong H_2$. If $m_{T,\alpha} = 1$, $H_2 \cong B(d_0, 0)$ and $H_1 = TCP(x + \langle\alpha\rangle, \alpha, F_{T,\omega})$ which is just $B(d_0, 0)$. Now we consider the case $m_{T,\alpha} > 1$. Let θ be the map from $[x]/\langle\alpha\rangle$ to $[0]/\langle\alpha\rangle$ which sends $x+y+\langle\alpha\rangle$ to $y+\langle\alpha\rangle$, $y \in [0]$. θ is obviously a bijection. Furthermore, $(\theta \circ N_{H_1})(x+y+\langle\alpha\rangle) = \theta(xT+yT+\omega+\langle\alpha T\rangle+\langle\alpha\rangle) = yT+xT-x+\omega+\langle\alpha T\rangle+\langle\alpha\rangle$, which is just $(N_{H_2} \circ \theta)(x+y+\langle\alpha\rangle)$. By Lemma 2.2 and Remark 2.3 this indicates that θ is an isomorphism from H_1 to H_2, as desired. □

Theorem 3.4. *If a component of $\Gamma = TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ contains exactly t cosets of $[0]/\langle\alpha\rangle$, then it is isomorphic to $C_t \otimes B(d_0, m_{T,\alpha} - 1)$ if and only if $\#\mathbb{Z}_d(\alpha; T) = d_0^{m_{T,\alpha}}$.*

Proof. The necessity can be established by comparing the number of vertices. Then, let us assume at this point that $\#\mathbb{Z}_d(\alpha; T) = d_0^{m_{T,\alpha}}$ and consider the reverse direction. Let Γ_1 be a component of Γ containing t cosets of $[0]/\langle\alpha\rangle$.

For the case $\alpha = 0$, by Lemma 3.2 the component Γ_1 is a cycle $C_t = C_t \otimes B(1, 0)$. Thus in the sequel we shall assume $\alpha \neq 0$.

If $t = 1$, then there is a coset $[x]/\langle\alpha\rangle$ such that $\Gamma_1 = TCP([x], \alpha, F_{T,\omega})$. By Lemma 3.3, Γ_1 is isomorphic to $B(d_0, m_{T,\alpha} - 1) = C_1 \otimes B(d_0, m_{T,\alpha} - 1)$.

Now consider the case $t > 1$. Recall from Theorem 2.5 that $TCP([0], \alpha, F_{T,0}) \cong B(d_0, m_{T,\alpha} - 1)$. Thus to prove $\Gamma_1 \cong C_t \otimes B(d_0, m_{T,\alpha} - 1)$ reduces to proving that $\Gamma_1 \cong \Gamma_2 \doteq C_t \otimes TCP([0], \alpha, F_{T,0})$. From Lemma 3.2, Γ_1 is a generalized t-cycle with stages $V_i = [x_i]/\langle\alpha\rangle$, $i \in \mathbb{Z}_t$, where

$$x_{i+1} = F_{T,\omega}(x_i),\ i = 0, 1, \cdots, t-2. \tag{4}$$

Our strategy is to find $x'_i, i \in \mathbb{Z}_t$, which satisfy

$$[x'_i] = [x_i],\ i \in \mathbb{Z}_t, \tag{5}$$

$$F_{T,\omega}(x'_i) \in x'_{i+1} + \langle\alpha\rangle,\ i = 0, 1, \cdots, t-2, \tag{6}$$

and

$$F_{T,\omega}(x'_{t-1}) \in x'_0 + \langle\alpha\rangle. \tag{7}$$

Having gotten such a set of x'_is, we can verify that the map $\psi : x'_i + y + \langle\alpha\rangle \mapsto (i, y + \langle\alpha\rangle)$, $y \in [0]$, is an isomorphism from Γ_1 to Γ_2 as follows. It is obvious that ψ is a bijection. Thus, by Lemma 2.2 and Remark 2.3, what we want becomes $\psi \circ N_{\Gamma_1} = N_{\Gamma_2} \circ \psi$. But for any $i \in \mathbb{Z}_t$, we have $(\psi \circ N_{\Gamma_1})(x'_i + y + \langle\alpha\rangle) = \psi(x'_i T + \omega + yT + \langle\alpha T\rangle + \langle\alpha\rangle) = \psi(x'_{i+1} + yT + \langle\alpha T\rangle + \langle\alpha\rangle) = (i+1, yT + \langle\alpha T\rangle + \langle\alpha\rangle) = N_{\Gamma_2}((i, y + \langle\alpha\rangle)) = (N_{\Gamma_2} \circ \psi)(x'_i + y + \langle\alpha\rangle)$. This is the result.

The rest of our proof is to prove the existence of $x'_i, i \in \mathbb{Z}_t$, which satisfy relations (5), (6) and (7). Since $F_{T,\omega}(x_{t-1}) \in [x_0]$, we may assume that

$$F_{T,\omega}(x_{t-1}) = x_0 + \sum_{k=0}^{m_{T,\alpha}-1} r_k \alpha^k,\ r_k \in \mathbb{Z}_{d_0}. \tag{8}$$

There are two cases.

Case 1. $t \geq m_{T,\alpha} - 1$. We choose

$$x'_0 = x_0 + s_0\alpha,\ x'_i = F_{T,\omega}(x'_{i-1}) + s_i\alpha,\ i = 1, \cdots, t-1, \tag{9}$$

where $s_i \in \mathbb{Z}_{d_0}$ are to be determined. Note that the form of our x'_is guarantees the truth of Eq. (5) and (6) already. It remains to find $s_0, s_1, \cdots, s_{t-1}$ such that (7) holds. Indeed, we show that there is a unique set of numbers $s_0, s_1, \cdots, s_{t-1}$, which fulfil (7).

Combining Eq. (4) and Eq. (9) we deduce that $x'_i = x_i + \sum_{j=0}^{i} s_j \alpha^{i-j}$. Substituting $x'_{t-1} = x_{t-1} + \sum_{j=0}^{t-1} s_j \alpha^{t-1-j}$ into (7) yields

$$F_{T,\omega}(x_{t-1}) + \sum_{j=0}^{t-1} s_j \alpha^{t-j} \in x'_0 + \langle\alpha\rangle = x_0 + \langle\alpha\rangle. \tag{10}$$

Formula (10) along with Eq. (8) implies $\sum_{k=0}^{m_{T,\alpha}-1} r_k\alpha^k + \sum_{j=0}^{t-1} s_j\alpha^{t-j} \in \langle\alpha\rangle$, which amounts to saying that

$$\sum_{k=1}^{m_{T,\alpha}-1} r_k\alpha^k + \sum_{j=0}^{t-1} s_j\alpha^{t-j} = 0. \tag{11}$$

It follows immediately that Eq. (11) is satisfied by taking

$$s_j = \begin{cases} 0 & 0 \le j \le t - m_{T,\alpha}, \\ -r_{t-j} & t - m_{T,\alpha} + 1 \le j \le t-1. \end{cases}$$

Thus the corresponding x_i's will be the solution to (7), as desired.

Case 2. $t < m_{T,\alpha} - 1$. We choose x_i's satisfying (5) and (6) by putting

$$x_0' = x_0 + \sum_{i=0}^{m_{T,\alpha}-t-1} l_i\alpha^i \tag{12}$$

and $x_i' = F_{T,\omega}(x_{i-1}') + s_i\alpha$, $1 \le i \le t-1$, where $s_1, \cdots, s_{t-1}, l_0, \cdots, l_{m_{T,\alpha}-t-1} \in \mathbb{Z}_{d_0}$ are to be determined to guarantee (7). Notice that $x_i' = x_i + \sum_{j=1}^{i} s_j\alpha^{i-j} + \sum_{k=0}^{m_{T,\alpha}-t-1} l_k\alpha^{i+k}$. We substitute $x_{t-1}' = x_{t-1} + \sum_{j=1}^{t-1} s_j\alpha^{t-1-j} + \sum_{i=0}^{m_{T,\alpha}-t-1} l_i\alpha^{t-1+i}$ and Eq. (12) into (7) to get

$$F_{T,\omega}(x_{t-1}) + \sum_{j=1}^{t-1} s_j\alpha^{t-j} + \sum_{i=0}^{m_{T,\alpha}-t-1} l_i\alpha^{t+i} \in x_0 + \sum_{i=0}^{m_{T,\alpha}-t-1} l_i\alpha^i + \langle\alpha\rangle,$$

which together with Eq. (8) implies

$$\sum_{k=1}^{m_{T,\alpha}-1} r_k\alpha^k + \sum_{j=1}^{t-1} s_j\alpha^{t-j} + \sum_{i=0}^{m_{T,\alpha}-t-1} l_i\alpha^{t+i} = \sum_{i=1}^{m_{T,\alpha}-t-1} l_i\alpha^i. \tag{13}$$

Equating coefficients of α^i, $i = 1, \cdots, m_{T,\alpha}-1$, we obtain a system of $m_{T,\alpha}-1$ linear equations in $m_{T,\alpha}-1$ unknowns $s_1, \cdots, s_{t-1}, l_0, \cdots, l_{m_{T,\alpha}-t-1}$. This system of linear equations can be written as

$$A(l_{m_{T,\alpha}-t-1}, \cdots, l_0, s_1, \cdots, s_{t-1})^\top = (-r_{m_{T,\alpha}-1}, \cdots, -r_2, -r_1)^\top, \tag{14}$$

where A is the matrix of order $m_{T,\alpha} - 1$ whose (i,j) term is

$$A(i,j) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = t + j, \\ 0 & \text{otherwise.} \end{cases}$$

Since A is invertible, from Eq. (14) we can uniquely determine the $m_{T,\alpha} - 1$ unknowns $s_1, \cdots, s_{t-1}, l_0, \cdots, l_{m_{T,\alpha}-t-1} \in \mathbb{Z}_{d_0}$. Clearly the corresponding x_i's validate (7), as desired. □

The following immediate consequence can be seen as an extension of the equivalence of statements (c) and (d) in Theorem 1.1.

Corollary 3.5. *The digraph $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is strongly connected if and only if it is connected.*

It is notable that all above results in this section are stated under the implicit assumption that T is invertible. The example below shows that Corollary 3.5 is false when T is not invertible.

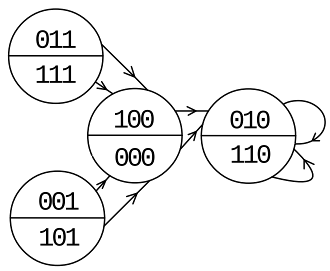


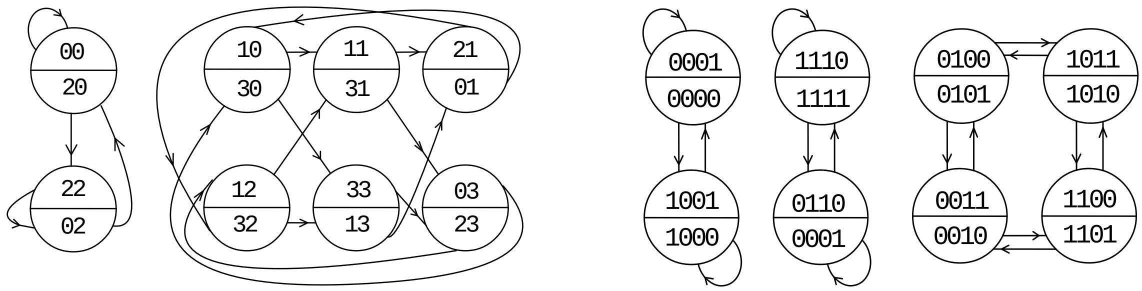
Fig. 2

Example 3.6. *See Fig. 2. Let* $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \alpha = (100), \omega = (010)$. $TCP(\mathbb{Z}_2^3, \alpha, F_{T,\omega})$ *is connected but not strongly connected.*

Notice that the component of Γ containing a vertex $x + \langle\alpha\rangle$ is just the orbit of $x + \langle\alpha\rangle$ under the action of $F_{T,\omega}$ and the size of an orbit is just $d_0^{m_{T,\alpha}-1}$ times the number of cosets of $x + \mathbb{Z}_d(\alpha; T)$ contained in the corresponding component. In particular, if $\omega \in \mathbb{Z}_d(\alpha; T)$, then $\mathbb{Z}_d(\alpha; T)/\langle\alpha\rangle$ is an orbit under the action of $F_{T,\omega}$ and hence it follows from Theorem 2.5 that the digraph Γ has a component isomorphic to $B(d_0, m_{T,\alpha} - 1)$. We observe that generally different components of Γ might have different orbit sizes as illustrated in Examples 3.7 and 3.8.

Example 3.7. *See Fig. 3(a). Let* $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\alpha = (20)$, $\omega = (00)$. $TCP(\mathbb{Z}_4^2, \alpha, F_{T,\omega})$ *has two components: one is isomorphic to* $B(2, 1)$, *the other is isomorphic to* $C_3 \otimes B(2, 1)$.

Example 3.8. *See Fig. 3(b).* $T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $\alpha = (0001)$, $\omega = (1001)$. $TCP(\mathbb{Z}_2^4, \alpha, F_{T,\omega})$ *has three components: two of them are isomorphic to* $B(2, 1)$ *and the third is isomorphic to* $C_2 \otimes B(2, 1)$.



(a) (b)

Fig. 3

Before leaving this section, we point out that for an arbitrary positive integer d, $\mathbb{Z}_d(\alpha; T)$ is not necessarily free over $\mathbb{Z}_d$ and it might occur $\#\mathbb{Z}_d(\alpha; T) < d_0^{m_{T,\alpha}}$. Nevertheless, when d is a prime, $\mathbb{Z}_d^n$ is a linear space over the field $\mathbb{Z}_d$ and $\mathbb{Z}_d(\alpha; T)$ is an $m_{T,\alpha}$-dimensional subspace of $\mathbb{Z}_d^n$. This means that the assumption of $\#\mathbb{Z}_d(\alpha; T) = d_0^{m_{T,\alpha}}$ made in Lemmas 2.4, 3.3 and Theorems 2.5, 3.4 is automatically satisfied by now. That is, the structure of $B(d, n)$ seems to be more universal when d is a prime. We mention that when d is a prime there is a characterization for $B(d, n)$ in terms of its unique path property; see [22, Theorem 6.1].

Example 3.9. $d = 3$, $n = 3$. $T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\alpha = (001)$. *Then* $d_0 = d = 3$, $m_{T,\alpha} = 2$. $\mathbb{Z}_3(\alpha; T)$ *is a free 2-dimensional* $\mathbb{Z}_3$*-module.*

Example 3.10. $d = 4$, $n = 3$. $T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\alpha = (002)$. *Then* $d_0 = 2$, $m_{T,\alpha} = 1$. $\mathbb{Z}_4(\alpha; T)$ *is not free over* $\mathbb{Z}_4$ *while it is isomorphic to the free module* $\mathbb{Z}_2^1$ *as a* $\mathbb{Z}_2$*-module.*

Example 3.11. $d = 8$, $n = 3$. $T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\alpha = (002)$. *Then* $d_0 = 4$, $m_{T,\alpha} = 2$. $Z_8(\alpha; T)$ *is not free over either* $\mathbb{Z}_8$ *or* $\mathbb{Z}_4$. $\#\mathbb{Z}_8(\alpha; T) = 8 < d_0^{m_{T,\alpha}}$.

4 $A(g, \pi, j)$

We prove in this section that the digraphs $A(g, \pi, j)$ introduced in [6] turns out to be a class of TCP digraphs. We also establish some necessary and sufficient conditions for those digraphs to be isomorphic to the De Bruijn digraphs or the affine TCP digraphs discussed in the foregoing two sections.

For any positive integer m, the symmetric group on $\mathbb{Z}_m$ is denoted S_m. Let $g \in S_n$ and $\pi \in S_d$. Permutations on $\mathbb{Z}_d^n$ induced by g and π, which are also written g and π, respectively, are defined by $g((x_0, x_1, \cdots, x_{n-1})) = (x_{g(0)}, x_{g(1)}, \cdots, x_{g(n-1)})$ and $\pi((x_0, x_1, \cdots, x_{n-1})) = (\pi(x_0), \pi(x_1), \cdots, \pi(x_{n-1}))$. Note that this usage of g and π will be fixed in this section. For a fixed $j \in \mathbb{Z}_n$, the digraph $\Gamma = A(g, \pi, j)$ is defined by $V(\Gamma) = \mathbb{Z}_d^n$ and $xy \in E(\Gamma)$ if and only if $y \in \pi(g(x)) + \langle e_j \rangle$ [6]. Let σ be the permutation on $\mathbb{Z}_n$ such that $\sigma(i) = i + 1$ and ϵ the identity permutation of S_d. Then the digraph $A(\sigma, \epsilon, n-1)$ is just the De Bruijn digraph $B(d, n)$.

In the following, let $\{\varepsilon_i : i \in \mathbb{Z}_n\}$ and $\{e_k : k \in \mathbb{Z}_{n+1}\}$ represent the standard bases of $\mathbb{Z}_d^n$ and $\mathbb{Z}_d^{n+1}$, respectively. Given $g \in S_n$, it corresponds a permutation matrix of order n, say T_0, such that $g(x) = xT_0$ for every $x = (x_0, x_1, \cdots, x_{n-1}) \in \mathbb{Z}_d^n$. In other words, the (i, k) term of T_0 is $T_0(i, k) = \delta_{i,g(k)}$, where δ is the Kronecker Delta. Let $T = \begin{pmatrix} T_0 & 0 \\ \varepsilon_j & 1 \end{pmatrix}$. We simply write $\bar{x}$ for the vertex $(x, 1) + \langle e_n \rangle$.

Lemma 4.1. *The digraph $TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$ admits no multiple arcs.*

Proof. Observe that $e_n T = (\varepsilon_j, 1) = e_j + e_n$ and that $\pi(\alpha + \langle e_k \rangle) = \pi(\alpha) + \langle e_k \rangle$ for arbitrary $\alpha \in \mathbb{Z}_d^{n+1}$ and $k \in \mathbb{Z}_{n+1}$. We obtain that in $\Gamma = TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$,

$$\begin{aligned} N_\Gamma(\bar{x}) &= \pi(\bar{x}T + \langle e_n \rangle) \\ &= \pi((xT_0 + \varepsilon_j, 1) + \langle e_j + e_n \rangle) + \langle e_n \rangle \\ &= \pi(xT_0, 1) + \langle e_j \rangle + \langle e_n \rangle \\ &= (\pi(xT_0), 1) + \langle e_j \rangle + \langle e_n \rangle. \end{aligned} \tag{15}$$

This ensures that Γ has no multiple arcs since $\{e_j, e_n\}$ is linearly independent over $\mathbb{Z}_d$. $\square$

The following proposition reveals the algebraic nature of $A(g, \pi, j)$.

Lemma 4.2. $TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0}) \cong A(g, \pi, j)$.

Proof. We write $\Gamma_1 = TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$ and $\Gamma_2 = A(g, \pi, j)$. Let φ be a map from $V(\Gamma_1)$ to $V(\Gamma_2)$ defined by $\varphi(\bar{x}) = x$, $x \in \mathbb{Z}_d^n$. Obviously, φ is a bijection. Considering that there are no multiple arcs both in Γ_2 by its definition and in Γ_1 by Lemma 4.1, it is enough to prove $N_{\Gamma_2} \circ \varphi = \varphi \circ N_{\Gamma_1}$. From Eq. (15), $(\varphi \circ N_{\Gamma_1})(\bar{x}) = \pi(xT_0) + \langle \varepsilon_j \rangle$, which is just $(N_{\Gamma_2} \circ \varphi)(\bar{x})$. This ends the proof. $\square$

Note that by putting together Theorem 3.4 and Lemma 4.2 we arrive at the conclusion that each component of $A(g, \pi, j)$ is the conjunction of a De Bruijn digraph with a cycle when π is the identity permutation.

Since $\pi \circ F_{T,0}$ is not necessarily an affine transformation, we have to prepare some further results in order to apply the results established in the earlier two sections to the analysis of $TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$.

Suppose there are t orbits of $\mathbb{Z}_n$ under the action of g. We choose a set of representatives of these orbits as $j_0 = j,\ j_1, \cdots, j_{t-1}$. We denote the orbit containing j_k by $Orb(j_k)$ whose cardinality is $\gamma_k,\ 0 \le k \le t-1$. For each $i \in \mathbb{Z}_n$, if $i \in Orb(j_k)$, we write $\tau(i) = min\{l \ge 1 : g^l(j_k) = i\}$. Notice that $\tau(j_k) = \gamma_k$. The notation $\boldsymbol{o}(\pi)$ represents the order of π in S_d. The next lemma exposes the relationship between the digraph $TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$ and the affine TCP digraphs we discussed in Sections 2 and 3.

Lemma 4.3. *$TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0}) \cong TCP(\mathbb{Z}_d^{n+1}, e_n, F_{T,0})$ if and only if*

$$t = 1 \text{ or } \boldsymbol{o}(\pi) \mid \gcd(\gamma_1, \cdots, \gamma_{t-1}). \tag{16}$$

Proof. Let $\Gamma_1 = TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$ and $\Gamma_2 = TCP(\mathbb{Z}_d^{n+1}, e_n, F_{T,0})$.

Assume that relation (16) holds, we claim that the map

$$f : (x_0, x_1, \cdots, x_{n-1}, 1) + \langle e_n \rangle \mapsto (\pi^{\tau(0)-1}(x_0), \pi^{\tau(1)-1}(x_1), \cdots, \pi^{\tau(n-1)-1}(x_{n-1}), 1) + \langle e_n \rangle$$

is an isomorphism from Γ_1 to Γ_2. f is bijective since $\tau(i)$ is unique for each $i \in \mathbb{Z}_n$ and π is a permutation. Now by Lemma 4.1 and Remark 2.3 we need only prove $N_{\Gamma_2} \circ f = f \circ N_{\Gamma_1}$. For any $x = (x_0, x_1, \cdots, x_{n-1}) \in \mathbb{Z}_d^n$, we have

$$(N_{\Gamma_2} \circ f)(\bar{x}) = (\pi^{\tau(g(0))-1}(x_{g(0)}), \cdots, \pi^{\tau(g(n-1))-1}(x_{g(n-1)}), 1) + \langle e_j \rangle + \langle e_n \rangle$$

and

$$\begin{aligned}(f \circ N_{\Gamma_1})(\bar{x}) &= f((\pi(x_{g(0)}), \pi(x_{g(1)}), \cdots, \pi(x_{g(n-1)}), 1) + \langle e_j \rangle + \langle e_n \rangle) \\ &= (\pi^{\tau(0)}(x_{g(0)}), \pi^{\tau(1)}(x_{g(1)}), \cdots, \pi^{\tau(n-1)}(x_{g(n-1)}), 1) + \langle e_j \rangle + \langle e_n \rangle.\end{aligned}$$

If $t = 1$, then $\tau(i) = \tau(g(i)) - 1$ for all $i \in \mathbb{Z}_n$ except j. Hence we have $(N_{\Gamma_2} \circ f)(\bar{x}) = (f \circ N_{\Gamma_1})(\bar{x})$. If $t > 1$ and $\boldsymbol{o}(\pi) \mid \gcd(\gamma_1, \cdots, \gamma_{t-1})$, then $\tau(i) \equiv \tau(g(i)) - 1 \pmod{\boldsymbol{o}(\pi)}$ since $\tau(i) \equiv \tau(g(i)) - 1 \pmod{\gamma_k}$ for all $i \in Orb(j_k)$, $k = 1, \cdots, t-1$. Note that $\tau(i) = \tau(g(i)) - 1$ for all $i \in Orb(j)$ except j. Then $\pi^{\tau(i)} = \pi^{\tau(g(i))-1}$ for all $i \in \mathbb{Z}_n$ except j. Hence we still have $N_{\Gamma_2} \circ f = f \circ N_{\Gamma_1}$.

Conversely, suppose $\Gamma_1 \cong \Gamma_2$ and $t > 1$. Assume that f is an isomorphism from Γ_2 to Γ_1. Let $f_i(\bar{x}) = f(\bar{x}) \cdot e_i^\top,\ i = 0, 1, \cdots, n$. Given $x = (x_0, x_1, \cdots, x_{n-1})$ with $x_i,\ i \in \mathbb{Z}_n \backslash Orb(j_k),\ k = 1, \cdots, t-1$, being equal, then $N_{\Gamma_2}^{\gamma_k}(\bar{x}) = \bar{x} + \langle e_n \rangle + \sum_{i=0}^{\gamma_k} \langle e_{g^{-i}(j)} \rangle$. This says that $\bar{x} \in N_{\Gamma_2}^{\gamma_k}(\bar{x})$. Hence $f(\bar{x}) \in N_{\Gamma_1}^{\gamma_k}(f(\bar{x}))$ and then $\pi^{\gamma_k}(f_{g^{\gamma_k}(i)}(\bar{x})) = f_i(\bar{x}),\ i \in Orb(j_k)$. Since $g^{\gamma_k}(j_k) = j_k$,

$$\pi^{\gamma_k}(f_{j_k}(\bar{x})) = f_{j_k}(\bar{x}). \tag{17}$$

Note that x_{j_k} in $\bar{x}$ is an arbitrary number in $\mathbb{Z}_d$ and then so is $f_{j_k}(\bar{x})$. Thus from Eq. (17), we have $\boldsymbol{o}(\pi) \mid \gamma_k$ for $k = 1, \cdots, t-1$. □

Lemma 4.4. *g is a cyclic permutation if and only if e_n is a cyclic vector for T.*

Proof. If we can deduce for any $k > 0$ that

$$e_n^k = e_n + \sum_{t=0}^{k-1} e_{g^{-t}(j)}, \tag{18}$$

then we see immediately that $\mathbb{Z}_d(e_n, T) = \mathbb{Z}_d^n$ if and only if g is a cyclic permutation in S_n and hence the lemma follows. We induct on k to prove Eq. (18). Note that for $k = 1$, Eq. (18) holds as $e_n^1 = e_nT = e_n + e_j$. Now suppose $k \geq 2$ and the Eq. (18) holds for smaller values of k. Note that $e_n^k = e_n^{k-1}T$ and that $e_iT = e_{g^{-1}(i)}$, $i \in \mathbb{Z}_{n+1}$. By the inductive hypothesis, $e_n^k = (e_n + \sum_{t=0}^{k-2} e_{g^{-t}(j)})T = e_n + e_j + \sum_{t=0}^{k-2} e_{g^{-t-1}(j)} = e_n + \sum_{t=0}^{k-1} e_{g^{-t}(j)}$. This ends the proof. □

Lemma 4.5. *If e_n is a cyclic vector for T, then*

$$TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0}) \cong TCP(\mathbb{Z}_d^{n+1}, e_n, F_{T,0}) \cong B(d, n).$$

Proof. By means of Theorem 2.8, the fact that e_n is a cyclic vector for T implies the second equality. In addition, Lemma 4.4 tells us that g is a cyclic permutation and so relation (16) holds. Therefore the first equality comes from Lemma 4.3, as expected. □

Lemma 4.6. *If $\Gamma = TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$ is a primitive digraph, then e_n is a cyclic vector for T.*

Proof. Let us write $V_k = (\pi^k(0), \cdots, \pi^k(0), 1) + \langle e_n \rangle + \sum_{i \in Orb(j)} \langle e_i \rangle$, $k \in \mathbb{Z}_t$, where t is the size of the orbit of 0 under the action of π.

Observe that $N_\Gamma(V_k) \subseteq V_{k+1}$, $k \in \mathbb{Z}_t$. Hence $V(\Gamma) = \cup_{i=0}^{t-1} V_i$ because Γ is strongly connected. Moreover, $V_0, \cdots, V_{t-1}$ are mutually disjoint by the definition of t. This fact together with $N_\Gamma(V_k) \subseteq V_{k+1}$ indicates that the length of any cycle of Γ is a multiple of t. It follows that $t = 1$ since Γ is primitive. This means $V_0 = V(\Gamma)$, i.e., $\mathbb{Z}_d^{n+1} = \langle e_n \rangle + \sum_{i \in Orb(j)} \langle e_i \rangle = \mathbb{Z}_d(e_n, T)$. Thus we are finished. □

Lemmas 4.5 and 4.6 together lead to another generalization of parts (a) and (b) of Theorem 1.1.

Theorem 4.7. *$TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0}) \cong B(d, n)$ if and only if any of the following statements holds:*

(a) The digraph $TCP(\mathbb{Z}_d^{n+1}, e_n, \pi \circ F_{T,0})$ is primitive;

(b) e_n is a cyclic vector for T.

Combining Lemmas 4.2, 4.4 and Theorem 4.7, we obtain

Corollary 4.8. *[6] $A(g,\pi,j) \cong B(d,n)$ if and only if g is a cyclic permutation in S_n.*

In general, for any $\alpha \in \mathbb{Z}_d^{n+1}$ and $T \in Mat_{n+1}(\mathbb{Z}_d)$, even if α is a cyclic vector for T, the digraph $TCP(\mathbb{Z}_d^{n+1}, \alpha, \pi \circ F_{T,0})$ may not be a De Bruijn digraph. See the following examples.

Example 4.9. *Here we write $\bar{x}$ for the vertex $x + \langle\alpha\rangle$. Let $d = 4$, $n = 2$, $\alpha = (120)$, $\omega = (301)$, $\pi = (0132)$ and $T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in GL_3(\mathbb{Z}_4)$. We have $N(\overline{0}) = \{\overline{(002)}, \overline{(013)}, \overline{(023)}, \overline{(032)}\}$ and $N(\overline{(001)}) = \{\overline{(002)}, \overline{(003)}, \overline{(032)}, \overline{(033)}\}$. Then $N(\overline{0}) \cap N(\overline{(001)}) \neq \phi$ but $N(\overline{0}) \neq N(\overline{(301)})$. Hence the digraph $TCP(\mathbb{Z}_4^3, \alpha, \pi \circ F_{T,\omega})$ is not a line digraph and thus cannot be isomorphic to any De Bruijn digraph.*

Example 4.10. *Let $d = 5$, $n = 1$, $\alpha = (11)$, $\omega = 0$, $\pi = (134)$ and $T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \in GL_2(\mathbb{Z}_5)$. Note that $\pi((11)T) = (42) = \pi((22)T) + \alpha$, $\pi((44)T) = (24) = \pi((33)T) + \alpha$ and $\pi(0T) = 0$, which means that there is a unique loop incident on $\overline{0}$ but there are multiple arcs from $\overline{0}$ to $\overline{(42)}$. Hence the digraph $TCP(\mathbb{Z}_5^2, \alpha, \pi \circ F_{T,\omega})$ is not isomorphic to any De Bruijn digraph.*

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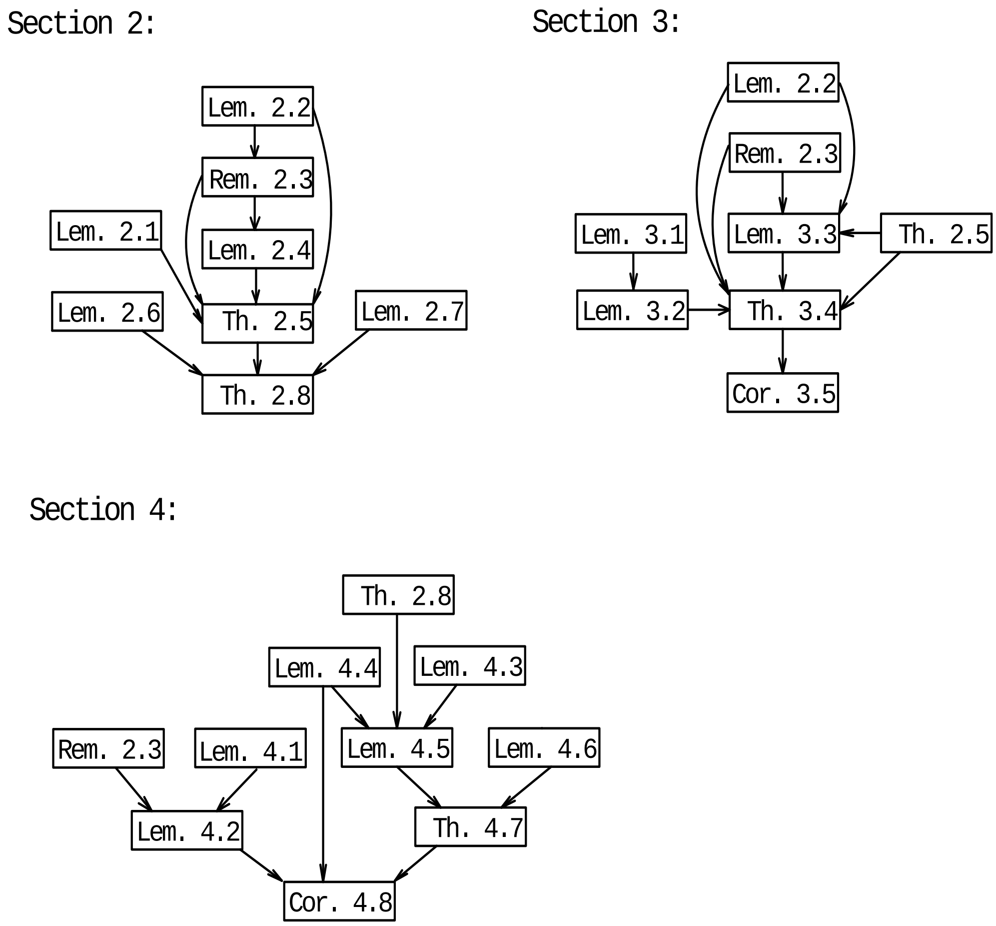

Figure 4: Outline