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Lit-only σ-game on pseudo-trees ☆

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ABSTRACT

A configuration of a graph is an assignment of one of two states, ON or OFF, to each vertex of it. A regular move at a vertex changes the states of the neighbors of that vertex. A valid move is a regular move at an ON vertex. A pseudo-tree is a graph obtained from a tree by attaching zero or more loops. The following result is proved in this note: given any starting configuration $\mathbf{x}$ of a pseudo-tree, if there is a sequence of regular moves which brings $\mathbf{x}$ to another configuration in which there are ℓ ON vertices then there must exist a sequence of valid moves which takes $\mathbf{x}$ to a configuration with at most $\ell+2$ ON vertices. We provide an example to show that the upper bound $\ell+2$ is sharp. Some related problems and conjectures are also reported.

1. σ-game and lit-only σ-game

We consider graphs without multiple edges but which may have loops. That is, for any graph G with *vertex set* $V(G)$, its *edge set* $E(G)$ is a subset of $\binom{V(G)}{2} \cup V(G)$. We say that there is a *loop* at a vertex v of G provided $\{v\} \in E(G)$ and we often record an edge $\{u, v\}$ as uv when $u \neq v$. Two vertices u and v, possibly equal, are *adjacent* in G if $\{u, v\} \in E(G)$. The *neighbors* of v in G, denoted $N_G(v)$, is the set of vertices adjacent to v in G. For any $v \in V(G)$, $\chi_v \in \mathbb{F}_2^{V(G)}$ stands for the binary function for which $\chi_v(u) = 1$ if $u = v$ and $\chi_v(u) = 0$ otherwise. For any $U \subseteq V(G)$, we set χ_U to be $\sum_{v\in U} \chi_v \in \mathbb{F}_2^{V(G)}$. The elements of $\mathbb{F}_2^{V(G)}$ will often be regarded as binary row vectors in an obvious way. Let us call each element of $\mathbb{F}_2^{V(G)}$ a *configuration* of G. We can think of a configuration $\mathbf{x}$ as an assignment of one of two states, ON or OFF, to the vertices of G such that v is ON if $\mathbf{x}(v) = 1$ and v is OFF if $\mathbf{x}(v) = 0$. The *light number* (*Hamming weight*) of a configuration $\mathbf{x}$, written as $|\mathbf{x}|$, refers to the number of vertices which are assigned the ON state by $\mathbf{x}$.

A *regular move* at a vertex v on a graph G transforms a configuration $\mathbf{x}$ to $\mathbf{x} + \chi_{N_G(v)}$ and we write $\mathbf{x} \to_G \mathbf{y}$ to designate that we can make successive regular moves to go from $\mathbf{x}$ to $\mathbf{y}$. If we make regular moves at all the vertices of G, then we go from $\mathbf{x}$ to $\sigma(\mathbf{x})$ that satisfies

$$\sigma(\mathbf{x})(v) = \mathbf{x}(v) + \sum_{u\in N_G(v)} \mathbf{x}(u)$$

for all $v \in V(G)$. The transition rule σ gives rise to the so-called σ-automata [10] which further motivates the introduction of the σ-game on a graph [12] as a solitaire combinatorial game. In the *σ-game on G*, we are given a configuration $\mathbf{x}$ of

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G and aim to determine all those $\mathbf{y}$ satisfying $\mathbf{x} \to_G \mathbf{y}$ and especially among these those with the smallest light number. Correspondingly, for any $\mathbf{x} \in \mathbb{F}_2^{V(G)}$, we put its *minimum light number* for the σ-game on G to be

$$ML_G(\mathbf{x}) = \min_{\mathbf{x} \to_G \mathbf{y}} |\mathbf{y}|.$$

The *minimum light number for the σ-game on G*, denoted $ML(G)$, is the worst result a smart player can encounter, namely

$$ML(G) = \max_{\mathbf{x} \in \mathbb{F}_2^{V(G)}} ML_G(\mathbf{x}).$$

For any $v \in V(G)$, let

$$P_{v,G} = I + \chi_v^\top \chi_{N_G(v)} \tag{1}$$

where I is the $|V(G)| \times |V(G)|$ identity matrix. A *lit-only move* at a vertex v on a graph G transforms a configuration $\mathbf{x}$ to

$$\mathbf{x}P_{v,G} = \mathbf{x} + \mathbf{x}(v)\chi_{N_G(v)} \tag{2}$$

and we write $\mathbf{x} \xrightarrow{*}_G \mathbf{y}$ to mean that we can make successive lit-only moves to go from $\mathbf{x}$ to $\mathbf{y}$. Given a configuration $\mathbf{x}$, the lit-only move at v is called *invalid* when v is OFF in $\mathbf{x}$ as it simply keeps the configuration unchanged and is called *valid* when v is ON in $\mathbf{x}$. If we restrict the moves of the σ-game at ON vertices only, then we come to the *lit-only σ-game*; in other words, we are replacing the regular moves of the σ-game by lit-only moves to get a new combinatorial game. For any $\mathbf{x} \in \mathbb{F}_2^{V(G)}$, we set its *minimum light number* for the lit-only σ-game on G, also called its *lit-only minimum light number*, to be

$$ML^*_G(\mathbf{x}) = \min_{\mathbf{x} \xrightarrow{*}_G \mathbf{y}} |\mathbf{y}|.$$

We use the notation $ML^*(G)$ for the *minimum light number for the lit-only σ-game on G* which is defined to be

$$\max_{\mathbf{x} \in \mathbb{F}_2^{V(G)}} ML^*_G(\mathbf{x}).$$

An interesting interpretation of the lit-only σ-game is proposed by Fraenkel [4]: an ON vertex is just a vertex occupied by a virus and making a valid move at a vertex stands for the event that the virus there replicates itself a copy for each of the neighbors of that vertex for the next moment and that copy will either occupy the corresponding neighbor if there is no virus there at present or fight with the virus at that neighbor now and kill each other so that no virus can live at the next moment in that vertex. Another interpretation of the game which better explains its name can be stated basically following Eriksson et al. [3]: in every room of a museum there is a button and pressing that button toggles the light ON/OFF in all adjacent rooms and the janitor can find the location of the button only when the lamp in that room is lit. Finally, as hinted by Eqs. (1) and (2), the lit-only σ-game is naturally associated with certain groups generated by transvections and hence is also studied (implicitly) in some algebra settings [8,9].

The σ-game is invertible, namely $\mathbf{x} \to_G \mathbf{y}$ if and only if $\mathbf{y} \to_G \mathbf{x}$, and the order in which we execute the moves is irrelevant. In fact, $\mathbf{x} \to_G \mathbf{y}$ if and only if $\mathbf{x} - \mathbf{y}$ lies in the abelian group generated by $\{\chi_{N_G(v)} : v \in V(G)\}$. To study the σ-game is just to study the action of this abelian group on $\mathbb{F}_2^{V(G)}$.

On the contrary, the lit-only σ-game may be unilateral, i.e., $\mathbf{x} \xrightarrow{*}_G \mathbf{y}$ does not imply $\mathbf{y} \xrightarrow{*}_G \mathbf{x}$, and the order of moves is significant. Let $\mathbb{H}$ be the matrix multiplicative semigroup generated by $\{P_{v,G} : v \in V(G)\}$, which is rarely abelian. To study the lit-only σ-game on G is the same as to study the action of $\mathbb{H}$ on $\mathbb{F}_2^{V(G)}$. Note that the existence of loops causes the non-invertibility of the lit-only σ-game. As a trivial example of non-invertibility, considering the graph G with $E(G) = V(G) \cup \binom{V(G)}{2}$, we easily find that $\mathbf{1} = \chi_{V(G)} \xrightarrow{*}_G \mathbf{0}$ and that $\mathbf{0}$ cannot go anywhere else in the lit-only σ-game. Also note that if G has no loops then $\mathbb{H}$ becomes a group; accordingly, the lit-only σ-game on G is invertible and for any $\mathbf{x} \in \mathbb{F}_2^{V(G)}$ the set $\{\mathbf{y} \in \mathbb{F}_2^{V(G)} : \mathbf{x} \xrightarrow{*}_G \mathbf{y}\}$ forms an orbit under the group action of $\mathbb{H}$.

On most graphs the lit-only σ-game looks harder to understand than the σ-game. We try to compare the difference of the reachability relationship between the lit-only σ-game and the σ-game on the same graph and wish to reduce the study of the former to a study of the latter in some sense. This note is one link in a series of papers which show that the difference is surprisingly small for several graph classes. The basic approach and several technical lemmas developed in this paper will also be helpful in our subsequent work.

The rest of the paper is organized as follows. In Section 2, we present our main result, the subject of which will be pseudo-trees, along with related background facts and conjectures on the influences of the lit-only restriction. In Section 3 we explain the main idea of our approach to compare the σ-game and lit-only σ-game. In Section 4, we prepare various technical lemmas and then prove our main result.

2. Influences of the lit-only restriction

To understand the influences of the lit-only restriction, the following may be a basic question to answer.

Problem 1. Suppose G is a graph, $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^{V(G)}$, and $\mathbf{x} \rightarrow_G \mathbf{y}$. When can we conclude that $\mathbf{x} \overset{*}{\rightarrow}_G \mathbf{y}$? How large can $ML_G^*(\mathbf{x}) - ML_G(\mathbf{x})$ be? How large is $ML^*(G) - ML(G)$?

It is obvious that

$$ML_G(\mathbf{x}) \leq ML_G^*(\mathbf{x}) \quad \text{and} \quad ML(G) \leq ML^*(G). \tag{3}$$

Set

$$\mathcal{D}(G) = \max_{\mathbf{x} \in \mathbb{F}_2^{V(G)}} (ML_G^*(\mathbf{x}) - ML_G(\mathbf{x})).$$

When the adjacency matrix of the graph G is nonsingular over $\mathbb{F}_2$, we have $ML(G) = 0$ and so $\mathcal{D}(G)$ is nothing but $ML^*(G)$. The next observation suggests a way to bound $ML^*(G) - ML(G)$ from above.

Theorem 2. *The inequality $ML^*(G) \leq ML(G) + \mathcal{D}(G)$ holds for any graph G.*

Proof. Assuming that $\mathbf{y}$ and $\mathbf{z}$ are two configurations of G such that $ML^*(G) = ML_G^*(\mathbf{z})$ and $ML(G) = ML_G(\mathbf{y})$, we have $ML^*(G) - ML(G) = ML_G^*(\mathbf{z}) - ML_G(\mathbf{y}) \leq ML_G^*(\mathbf{z}) - ML_G(\mathbf{z}) \leq \mathcal{D}(G)$. □

The example below says that $ML^*(G) - ML(G)$ and hence $\mathcal{D}(G)$ can be very large. Note that this example refutes an earlier conjecture [5, Conjecture 4] that $ML^*(G) - ML(G) \leq \lceil \frac{n}{6} \rceil$ for any graph G of order n and without isolated vertices.

Example 3. Let m be a positive integer and let $G = K_{2m}$ be the complete graph without loops on $2m$ vertices. The adjacency matrix of G is

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}.$$

Since $\det(A) = 1 - n = 1$, we have $ML(G) = 0$. For any $\mathbf{x} \in \mathbb{F}_2^{V(G)} \setminus \{\mathbf{0}\}$, after a valid move at any ON vertex, the final configuration $\mathbf{y}$ satisfies $|\mathbf{y}| = n + 1 - |\mathbf{x}|$. Hence, $ML^*(\mathbf{x}) = \min(|\mathbf{x}|, n + 1 - |\mathbf{x}|)$ for any $\mathbf{x} \in \mathbb{F}_2^{V(G)}$ and $ML^*(G) = m = \frac{n}{2}$ follows.

Conjecture 4. *$ML^*(G) - ML(G) \leq \frac{1}{2}|V(G)|$ for any graph G.*

For the purpose of bounding the minimum light number for the lit-only σ-game by taking advantage of the possibly existing knowledge on the σ-game, it would be good to know that the lit-only restriction does not make a big difference under some conditions. The next example indicates one such important case.

Example 5. Let G be a graph with loops at all vertices. As a consequence of [6, Theorem 3], we know that $ML_G^*(\mathbf{x}) = ML_G(\mathbf{x})$ is valid for any configuration $\mathbf{x}$ of G and hence $ML^*(G) = ML(G)$. Further note that we now have infinitely many graphs for which both equalities in (3) hold.

Besides the above result for graphs with a loop at each vertex, the former results for trees also demonstrate that the lit-only restriction does not matter too much in some circumstances. To state these results, some more concepts are necessary. The *degree* of a vertex v in a graph G is defined to be the number of edges in $E(G) \setminus V(G)$ that contain v and we will use the notation $\deg_G(v)$ for it. A vertex of degree no greater than one is said to be a *leaf*. The *shadow graph* of a graph G, which we denote by $\mathcal{S}(G)$, is the loopless graph with vertex set $V(G)$ and edge set $E(G) \setminus V(G)$. A graph is a *pseudo-tree* if its shadow graph is a tree.

Theorem 6 ([13,14])**.** *Let G be a pseudo-tree with ℓ leaves. If $\ell \geq 2$, then $ML(G) \leq \lfloor \ell/2 \rfloor$ and $ML^*(\mathcal{S}(G)) \leq \lceil \ell/2 \rceil$. Both equalities can be attained.*

Note that in Theorem 6 we do not directly compare the difference between the minimum light numbers of the σ-game and the lit-only σ-game, which is an object of interest posed in both [5, Question 3] and [6, Section 1.3].

Conjecture 7. *$ML^*(G) - ML(G) \in \{0, 1\}$ for any pseudo-tree G.*

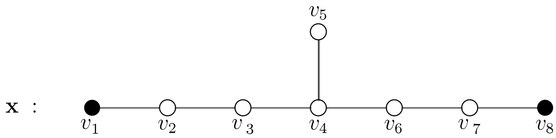


Fig. 1. $ML^*(\mathbf{x}) - ML(\mathbf{x}) = 1$.

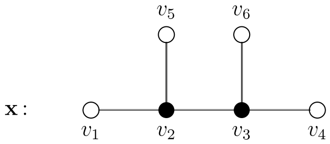


Fig. 2. $ML^*(\mathbf{x}) - ML(\mathbf{x}) = 2$.

Let $\mathbf{x}$ be any configuration of a pseudo-tree G. According to Example 5, $ML^*_G(\mathbf{x}) - ML_G(\mathbf{x}) = 0$ if G has a loop at every vertex. The next two examples show that it is possible for $ML^*_G(\mathbf{x}) - ML_G(\mathbf{x})$ to take value 1 or 2 as well. To illustrate a configuration, we will draw the underlying graph and use a bullet to indicate an ON vertex and a circle for an OFF vertex.

Example 8. Let $\mathbf{x}$ be the configuration depicted in Fig. 1. An exhaustive computer search gives that $ML(\mathbf{x}) = 1$ and $ML^*(\mathbf{x}) = 2$.

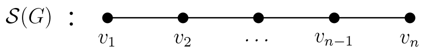
Example 9. Let $\mathbf{x}$ be the configuration depicted in Fig. 2. An exhaustive computer search shows that $ML(\mathbf{x}) = 0$ and $ML^*(\mathbf{x}) = 2$.

Relating the lit-only σ-game to the so-called Reeder's puzzle [11] as well as using some algebraic results of Reeder [11], Huang proves the following interesting result.

Theorem 10 *([7, Theorem 5.5]). Suppose G is a tree with a perfect matching but not a path. Then $ML^*_G(\mathbf{x}) = 1$ and $ML_G(\mathbf{x}) = 0$ for any $\mathbf{x} \in \mathbb{F}_2^{V(G)} \setminus \{\mathbf{0}\}$.*

Problem 11. Can we find a combinatorial proof of Theorem 10 without using the algebraic result of Reeder? Can we reach a proof of some algebraic results of Reeder [11] by playing the lit-only σ-game?

Theorem 12. *Let G be obtained from an n-path $v_1v_2\cdots v_n$ by adding zero or more loops. Then any configuration $\mathbf{x}$ of G can be transformed to a configuration $\mathbf{y}$ with $|\mathbf{y}| \le 1$ by a series of valid moves inside $V(G) \setminus \{v_n\}$. Therefore $\mathcal{D}(G) \le 1$.*



Proof. Let $t_{\mathbf{y}} = 0$ for $\mathbf{y} = \mathbf{0}$ and $t_{\mathbf{y}} = \max\{t : \mathbf{y}(v_t) = 1\}$ for any $\mathbf{y} \in \mathbb{F}_2^{V(G)} \setminus \{\mathbf{0}\}$. Let $\mathcal{C}$ be the set of configurations which can be reached from $\mathbf{x}$ by applying a series of valid moves inside $V(G) \setminus \{v_n\}$. Choose a configuration $\mathbf{y}$ from $\mathcal{C}$ whose $t_{\mathbf{y}}$ is as small as possible. It suffices to deduce that $|\mathbf{y}| \le 1$. Assuming otherwise that $|\mathbf{y}| > 1$, then there is $t < t_{\mathbf{y}}$ such that $\mathbf{y}(v_t) = 1$ and $\mathbf{y}(v_q) = 0$ for any q satisfying $t_{\mathbf{y}} > q > t$. Now a series of valid moves at $v_t, v_{t+1}, \ldots, v_{t_{\mathbf{y}}-1}$ transforms $\mathbf{y}$ into another member $\mathbf{y}'$ of $\mathcal{C}$ with $t_{\mathbf{y}'} < t_{\mathbf{y}}$, yielding a contradiction. □

Problem 13. Besides paths and trees with perfect matchings, which kind of trees G satisfy $ML^*(G) = 1$? Can we characterize those pseudo-trees G with $ML^*(G) = 0$?

Regarding Problem 13, we mention that Amin and Slater [1] characterize those G with $ML(G) = 0$ among the graphs obtained from trees by attaching a loop to each vertex.

In view of Theorem 2, what comes next may be viewed as a partial support to Conjecture 7; this is our main result and will be proved in Section 4. We remind the reader that the sharpness of Theorem 14 does not contradict Conjecture 7.

Theorem 14. *$\mathcal{D}(G) \le 2$ for any pseudo-tree G and the bound is sharp.*

The proof of Theorem 14 mainly relies on Lemmas 17 and 22 (See Section 4). Lemma 22 says that $\mathcal{D}(G)$ is small provided G has a certain special local structure and Lemma 17 guarantees that G does have such a special structure when it is a pseudo-tree. We have developed results like Theorem 14 for several other graph classes in much the same vein and will report them in the follow-up papers. These results lead to the following conjecture.

Conjecture 15. *If G is a connected graph, then $\mathcal{D}(G) \le \max_{v \in V(G)} \deg_G(v) - 1$.*

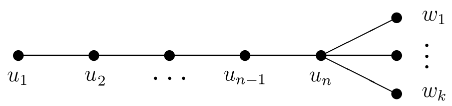


Fig. 3. The rake $P_{n,k}$ with k teeth and an n-handle.

3. A strategy to play the lit-only σ-game

We explain here a simple strategy to play the lit-only σ-game that allows us make good use of any existing strategy to play the σ-game. In practical applications, we utilize variants of this basic strategy.

For any graph G and $S \subseteq V(G)$, the subgraph of G induced by S is denoted $G[S]$. Choose an ordering τ of $V(G)$, say $v_1, v_2, \dots, v_n$, such that $G[\{v_i, v_{i+1}, \dots, v_n\}]$ is connected for any $1 \le i \le n$. Such an ordering τ is called a *connected vertex ordering*. For example, if G is connected and the ordering τ satisfies

$$\text{distance}(v_1, v_n) \ge \text{distance}(v_2, v_n) \ge \cdots \ge \text{distance}(v_n, v_n),$$

then τ is a connected vertex ordering of G.

Given any connected vertex ordering τ of G and any $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^{V(G)}$ with $\mathbf{x} \to_G \mathbf{y}$, we adopt the following *basic strategy* to play the lit-only σ-game on G with the initial configuration $\mathbf{x}$:

Step 1. Let $k = 1$ and find $\mathbf{w} \in \mathbb{F}_2^{V(G)}$ such that $\mathbf{y} = \mathbf{x} + \sum_{i=1}^n \mathbf{w}(v_i)\chi_{N_G(v_i)}$.
Step 2. If $k > n$ or all of $v_k, v_{k+1}, \dots, v_n$ are OFF, then exit the process with code k. Otherwise we make use of zero or more lit-only moves inside $\{v_{k+1}, \dots, v_n\}$ to turn v_k ON, and make a possible move at v_k so that the total number of valid moves at v_k since we begin the game have the same parity as $\mathbf{w}(v_k)$.
Step 3. Let $k = k + 1$ and go to Step 2.

If the above process ends with code $k = K$, then the final configuration is

$$\mathbf{z} = \mathbf{x} + \sum_{i=1}^{n} m_i \chi_{N_G(v_i)} = \mathbf{y} + \sum_{i=K}^{n} (m_i + \mathbf{w}(v_i))\chi_{N_G(v_i)}, \tag{4}$$

where m_i is the total number of valid moves made at v_i during all the process, and

$$\mathbf{z}(v_K) = \mathbf{z}(v_{K+1}) = \cdots = \mathbf{z}(v_n) = 0. \tag{5}$$

For any $k \in \{1, \dots, n\}$, let $\partial_G(\tau_k)$ be the set of those vertices from $\{v_1, \dots, v_{k-1}\}$ which are adjacent to at least one vertex in $\{v_k, \dots, v_n\}$. Observe that $\partial_G(\tau_1) = \emptyset$. By Eqs. (4) and (5), if $\mathbf{z}(v_i) = 1$, then either $\mathbf{y}(v_i) = 1$ or $v_i \in \partial_G(\tau_K)$. This implies that

$$|\mathbf{z}| \le |\mathbf{y}| + |\partial_G(\tau_K)|.$$

For any connected graph G, we define $\mathscr{B}(G)$ to be

$$\min_{\tau} \max_{1 \le k \le |V(G)|} |\partial_G(\tau_k)|$$

where τ runs through all connected vertex orderings of G. For instance, $\mathscr{B}(G) = k - 2$ when G is the k-star (see Section 4) and $\mathscr{B}(G) = 1$ when G is a path of positive length. Our basic strategy together with its analysis leads to the following result.

Theorem 16. *$\mathscr{D}(G) \le \mathscr{B}(G)$ for any connected graph G.*

Note that the parameter $\mathscr{B}(G)$ may have some connection with the well-studied vertex isoperimetric number [2]. We refer to [15] for some known connection between the lit-only σ-game and isoperimetric problems.

4. Proof of Theorem 14

For any two positive integers n and k, let $P_{n,k}$ be the graph with vertex set $\{u_1, u_2, \dots, u_n, w_1, \dots, w_k\}$ and edge set $\{u_1u_2, u_2u_3, \dots, u_{n-1}u_n, u_nw_1, \dots, u_nw_k\}$; see Fig. 3. We refer to $P_{n,k}$ as the *rake with k teeth* $w_1, \dots, w_k$ and an *n-handle* $u_1, \dots, u_n$. We call the vertex u_1 the *top* of the rake. When $k = 1$, $P_{n,k}$ is just an $(n+1)$-path one of whose two leaves is specified as the top. The rake $P_{1,k}$ is also called a *k-star*. We say that v is a *branch vertex* of a graph G if $\deg_G(v) \ge 3$.

We now present the first key fact for proving Theorem 14.

Lemma 17. *Let G be a tree with $n \ge 2$ vertices. Then, either* (a) *G is a star, or* (b) *G has a vertex v such that $G - v$ contains two connected components U and W such that both $G[\{v\} \cup U]$ and $G[\{v\} \cup W]$ are rakes with v being the common top and $|W| \ge 2$.*

Proof. *Case* 1: G has no branch vertex and hence is a path $v_1v_2 \cdots v_n$. If $n = 2$, then G is a 1-star. If $n = 3$, then G is a 2-star. If $n \ge 4$, we can take $v = v_2$, $U = \{v_1\}$, $W = \{v_3, \dots, v_n\}$, and thus (b) holds.

Case 2: G has exactly one branch vertex v. Let W be a connected component of $G - v$ with the largest size, U be one of the remaining connected components of $G - v$. If $|W| = 1$, then (a) holds; otherwise, (b) holds.

Case 3: G has at least two branch vertices. Choose a path P in G which contains as many branch vertices as possible. Let $v_1, v_2, \ldots, v_t$ be all the branch vertices on P in that order when you traverse from one endpoint of P to the other endpoint. Note that $t \geq 2$ and each connected component of $G - v_1$ contains no branch vertices unless it contains v_2.

Subcase 3.1: One of the connected components of $G - v_1$ which does not contain v_2 has size at least two. We will come to (b) in this case by letting $v = v_1$, W be the aforementioned connected component, U be any of the remaining connected components which does not contain v_2.

Subcase 3.2: Each connected component of $G - v_1$ which does not contain v_2 has size one. Take any connected component X of $G - v_2$ which contains neither v_1 nor v_3 and let Y be the connected component of $G - v_2$ that contains v_1. If X contains a branch vertex $\overline{v}$ of G, by replacing the path P with the path containing $\overline{v}, v_2, \ldots, v_t$, our analysis in Case 3.1 says that it is enough to consider the case that each connected component of $G - \overline{v}$ which does not contain v_2 has size one and so we can take $v = v_2$, $U = X$ and $W = Y$ to arrive at (b). If X does not contain any branch vertex of G, we still take $v = v_2$, $U = X$ and $W = Y$ and get to (b) again, as desired. □

The next few results will culminate in Lemma 22, another main ingredient of our proof of Theorem 14.

Lemma 18. *Let G be a connected graph and $\mathbf{x} \in \mathbb{F}_2^{V(G)} \setminus \{\mathbf{0}\}$. Suppose a and b are two vertices of G satisfying $N_G(a) \neq N_G(b)$. Then, there is $\mathbf{y} \in \mathbb{F}_2^{V(G)}$ such that $\mathbf{y}(a) \neq \mathbf{y}(b)$ and $\mathbf{x} \xrightarrow{*}_G \mathbf{y}$.*

(N_G(a)\cup N_G(b)\cup \{a,b\} \setminus \{c\})

Proof. Without loss of generality, assume that there is $v_1 \in N_G(a) \setminus N_G(b)$. Let $v_1 v_2 \cdots v_t$ be th [illegible] $\mathbf{x}(v_t) = 1$. After the sequence of valid moves at $v_t, v_{t-1}, \ldots, v_2$ in that order, v_1 is ON. We no [illegible] required configuration $\mathbf{y}$ or can make a valid move at v_1 to reach such a configuration $\mathbf{y}$. □

Lemma 19. *Let G be a graph, $a, b \in V(G)$, $ab \notin E(G)$, $c \in N_G(a) \cap N_G(b)$. Take S such that $c \in S \subseteq V(G) \setminus (N_G(a) \cup N_G(b) \cup \{a, b\})$ and $G[S]$ is connected. Assume that $\mathbf{x}$ is a configuration of G satisfying $\mathbf{x}(a) \neq \mathbf{x}(b)$. Then for any finite set $T \subseteq S$, there exists $R \subseteq \{a, b\}$ such that $\mathbf{x} \xrightarrow{*}_G \mathbf{z}$ where $\mathbf{z} = \mathbf{x} + \sum_{v \in T \cup R} \chi_{N_G(v)}$ and $\mathbf{z}(a) \neq \mathbf{z}(b)$.*

Proof. Let r be the maximum distance in $G[S]$ between c and a vertex in T; we set r to be -1 when $T = \emptyset$. As the claim is trivial for $r = -1$, to prove the claim by induction on r and the number of vertices of T which has distance r to c in $G[S]$, we only need to verify the following: let d be a vertex in T whose distance to c in the graph $G[S]$ is $r \geq 0$ and let S' be the set of those vertices in S which have a distance less than r to c in $G[S]$, then we can find $U \subseteq S' \cup \{a, b\}$ such that $\mathbf{x} \xrightarrow{*}_G \mathbf{w}$ where $\mathbf{w} = \mathbf{x} + \chi_{N_G(d)} + \sum_{v \in U} \chi_{N_G(v)}$ and $\mathbf{w}(a) \neq \mathbf{w}(b)$. To establish this claim, we choose a shortest path in $G[S]$ which connects c and d, say $v_1 v_2 \cdots v_{r+1}$ where $v_1 = c$ and $v_{r+1} = d$. Let us refer to the only ON vertex among $\{a, b\}$ in the configuration $\mathbf{x}$ as v_0 and let t be the largest integer no greater than $r + 1$ such that $\mathbf{x}(v_t) = 1$. What is left to do is to distinguish two cases.

Case 1: Either $t > 0$ or there is no loop at v_0 in G. It is easy to check that the valid moves at $v_t, v_{t+1}, \ldots, v_{r+1}$ in that order transforms $\mathbf{x}$ to $\mathbf{w}$.

Case 2: There is a loop at v_0 in G and $t = 0$. The sequence of valid moves at $v_0, v_1, \ldots, v_{r+1}, v_0$ successively is what we want. □

Lemma 20. *Suppose $\mathcal{S}(G) = P_{n,k}$ is as depicted in Fig. 3. For any configurations $\mathbf{x}$ and $\mathbf{y}$ of G, if there is a sequence of regular moves inside $V(G) \setminus \{u_1\}$ which brings $\mathbf{x}$ to $\mathbf{y}$, then there exists $\mathbf{z} \in \mathbb{F}_2^{V(G)}$ satisfying $|\mathbf{z}| \leq |\mathbf{y}| + 1$ and a series of valid moves inside $V(G) \setminus \{u_1\}$ which transforms $\mathbf{x}$ to $\mathbf{z}$.*

Proof. We play the lit-only σ-game using the idea of our basic strategy. That is, we first choose an ordering τ of $V(G) \setminus \{u_1\}$, say $u_2, \ldots, u_n, w'_1, \ldots, w'_k$ where $w'_1, \ldots, w'_k$ is some permutation of $w_1, \ldots, w_k$ to be specified later. Let us execute the basic strategy on $G - u_1$ for approaching $\mathbf{y}$ from $\mathbf{x}$, even though the ordering τ is not connected when $k > 1$. If the process ends at some handle vertex, the analysis in Section 3 shows that the resulting configuration could be taken as the required $\mathbf{z}$. If the process does not stop at u_n, then we need to specify the permutation $w'_1, \ldots, w'_k$ to continue playing the lit-only σ-game. Let W be those teeth which are ON currently and let U be those teeth at which we should make regular moves to turn the current configuration into $\mathbf{y}$. We now fix the ordering τ so that the vertices in $U \cap W$ appear earlier than the teeth outside of $U \cap W$. It is not hard to find that the configuration which we arrive at when terminating the execution of our basic strategy is a configuration $\mathbf{z}$ which we are seeking for. □

Remark 21. A slight modification of the above proof of Lemma 20 shows that $\mathcal{D}(G) \leq 1$ if $\mathcal{S}(G)$ is a rake.

For any $\mathbf{x} \in \mathbb{F}_2^{V(G)}$ and any $U \subseteq V(G)$, $\mathbf{x}_U$ is the restriction of $\mathbf{x}$ on U.

Lemma 22. *Let G be a connected graph, $c \in V(G)$, and $\mathbf{x} \in \mathbb{F}_2^{V(G)}$. Suppose that U and W are two components of $G - c$. Further assume that the shadow graphs of $G[\{c\} \cup U]$ and $G[\{c\} \cup W]$ are both rakes with c being its top. If there are $u \in U$ and $w \in W$ such that either $N_G(u) \neq N_G(w)$ or $\mathbf{x}(u) \neq \mathbf{x}(w)$, then $ML^*_G(\mathbf{x}) - ML_G(\mathbf{x}) \leq 2$. Furthermore, $ML^*_G(\mathbf{x}) - ML_G(\mathbf{x}) \leq 1$ if $|U| = |W| = 1$.*

Proof. The claim is trivially true when $\mathbf{x} = \mathbf{0}$ and so we only need to consider the case that $\mathbf{x} \neq \mathbf{0}$. Let $\{a\} = N_G(c) \cap U$ and $\{b\} = N_G(c) \cap W$. Take $\mathbf{y}$ such that

$$\mathbf{x} \to_G \mathbf{y} \quad \text{and} \quad |\mathbf{y}| = ML_G(\mathbf{x}). \tag{6}$$

In the light of Lemma 18, we can assume that $\mathbf{x}(a) \neq \mathbf{x}(b)$. By Lemma 19, there are then $U' \subseteq U$ and $W' \subseteq W$ such that

$$\mathbf{x} \xrightarrow{*}_G \mathbf{z} = \mathbf{y} + \sum_{v \in U' \cup W'} \chi_{N_G(v)} \quad \text{and} \quad \mathbf{z}(a) \neq \mathbf{z}(b).$$

Since $\mathbf{z}(a) \neq \mathbf{z}(b)$, we can assume without loss of generality that $\mathbf{z}(a) = 1$ and $\mathbf{z}(b) = 0$. By a possible valid move at a, we transform $\mathbf{z}$ to the configuration

$$\widetilde{\mathbf{z}} = \mathbf{y} + \sum_{v \in U'' \cup W'} \chi_{N_G(v)} \tag{7}$$

where $U'' = U' \setminus \{a\} \subseteq U \setminus \{a\}$. Applying Lemma 20 on $\widetilde{\mathbf{z}}_{\{c\} \cup W}$, the graph $G[\{c\} \cup W]$ and the set of regular moves at all vertices in W', we deduce from Eq. (7) that there is a configuration $\mathbf{z}'$ of G which fulfils the following:

$$\mathbf{x} \xrightarrow{*}_G \widetilde{\mathbf{z}} \xrightarrow{*}_G \mathbf{z}', \tag{8}$$

$$|\mathbf{z}'_{W \cup \{c\}}| \leq |\mathbf{y}_{W \cup \{c\}}| + 1, \tag{9}$$

$$\mathbf{z}'_{V(G) \setminus (U \cup W \cup \{c\})} = \mathbf{y}_{V(G) \setminus (U \cup W \cup \{c\})}, \tag{10}$$

$$\mathbf{z}'_U = \mathbf{y}_U + \sum_{v \in U''} \chi_{N_G(v)}. \tag{11}$$

From Eqs. (9) and (10) we obtain

$$|\mathbf{z}'_{V(G) \setminus U}| \leq |\mathbf{y}_{V(G) \setminus U}| + 1. \tag{12}$$

Case 1: $|U| = 1$. In this case, we have $U'' = \emptyset$ and hence Eq. (11) states that $\mathbf{z}'_U = \mathbf{y}_U$. This along with Eq. (12) guarantees that $|\mathbf{z}'| \leq |\mathbf{y}| + 1$ and so Eqs. (6) and (8) lead to $ML^*_G(\mathbf{x}) - ML_G(\mathbf{x}) \leq 1$.

Case 2: $|U| > 1$ and so $\mathcal{S}(G[U])$ is a rake with a being its top. Observing Eq. (11) and appealing to Lemma 20 for $\mathbf{z}'_U$, the graph $G[U]$ and the set of regular moves at all vertices of $U'' \subseteq U \setminus \{a\}$, we can obtain a $\mathbf{z}''$ such that $\mathbf{z}' \xrightarrow{*}_G \mathbf{z}''$, $\mathbf{z}''_{V(G) \setminus U} = \mathbf{z}'_{V(G) \setminus U}$ and $|\mathbf{z}''_U| \leq |\mathbf{y}_U| + 1$. Combining this with Eqs. (6), (8) and (12), we find that $ML^*_G(\mathbf{x}) \leq |\mathbf{z}''| = |\mathbf{z}''_{V(G) \setminus U}| + |\mathbf{z}''_U| \leq |\mathbf{z}'_{V(G) \setminus U}| + |\mathbf{y}_U| + 1 \leq |\mathbf{y}| + 2 = ML_G(\mathbf{x}) + 2$, as claimed. □

Remark 23. Mimicking the above proof of Lemma 22, we can prove the following extra claim where all undefined parameters are as described in Lemma 22: putting $\mathbf{x}'$ to be the restriction of $\mathbf{x}$ on $V(G) \setminus (U \cup W)$, we have

$$ML^*_G(\mathbf{x}) \leq ML_{G-(U \cup W)}(\mathbf{x}') + 2$$

provided either $N_G(u) \neq N_G(w)$ or $\mathbf{x}(u) \neq \mathbf{x}(w)$.

Remark 24. If we replace the condition $|U| = |W| = 1$ by $\min(|U|, |W|) = 1$ in Lemma 22, it is generally not true that $ML^*_G(\mathbf{x}) - ML_G(\mathbf{x}) \leq 1$, as can be seen from Example 9.

Proof of Theorem 14. Take any configuration $\mathbf{x}$ of G. If $\mathcal{S}(G)$ is a star, G has no loop at any tooth, and all its teeth are assigned the same state by $\mathbf{x}$, it is straightforward that $ML^*_G(\mathbf{x}) \leq 1$. In the remaining case, the inequality $ML^*_G(\mathbf{x}) - ML_G(\mathbf{x}) \leq 2$ is an immediate consequence of Lemmas 17 and 22. The tightness of the bound is shown in Example 9. □

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