Discrete Applied Mathematics 116 (2002) 289–296

DISCRETE APPLIED MATHEMATICS

Note

The underlying line digraph structure of some (0, 1)-matrix equations

Joan Gimbert[a,*], Yaokun Wu[b,1]

[a] Departament de Matemàtica, Universitat de Lleida, Jaume II, 69, 25005 Lleida, Catalonia, Spain
[b] Department of Applied Mathematics, Shanghai Jiao Tong University, 1954 Huashan Road, Shanghai 200030, People's Republic of China

Received 9 February 2000; received in revised form 12 December 2000; accepted 16 January 2001

Abstract

From the theory of Hoffman polynomial, it is known that the adjacency matrix A of a strongly connected regular digraph of order n satisfies certain polynomial equation $A^lP(A)=J_n$, where l is a nonnegative integer, $P(x)$ is a polynomial with rational coefficients, and J_n is the $n\times n$ matrix of all ones. In this paper we present some sufficient conditions, in terms of the coefficients of $P(x)$, to ensure that all (0, 1)-matrices satisfying such an equation with $l>0$ have an underlying line digraph structure, that is to say, for any solution A there exists a (0, 1)-matrix C satisfying $P(C)=J_{n/d^l}$ and the associated (d-regular) digraph of A, $\Gamma(A)$, is the lth iterated line digraph of $\Gamma(C)$. As a result, we simplify the study of some digraph classes with order functions asymptotically attaining the Moore bound. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Line digraph; (0, 1)-matrix equation; Moore bound

1. Introduction

The line digraph iteration technique, introduced by Fiol et al. [5,6], is a useful tool to construct large digraphs from suitable small digraphs. In fact, for any connected regular digraph G of degree $d\geqslant 2$, which is necessarily strongly connected since it is eulerian, the sequence of its iterated line digraphs, $\{L^l G\}_l$, constitutes a family of d-regular digraphs whose diameters increase linearly while their orders do it exponentially.

Besides, since the adjacency matrix A of G satisfies certain polynomial equation $P(A)=J_n$, where $P(x)$ is a polynomial with rational coefficients, n is the order of G, and J_n is the $n\times n$ matrix of all ones (see [10]), it turns out that the adjacency matrix A_{L^l} of $L^l G$ verifies that $A_{L^l}^lP(A_{L^l})=J_{nd^l}$ (see Section 2). Some well-known

* Corresponding author.
E-mail addresses: joangim@eup.udl.es (J. Gimbert), ykwu@mail1.sjtu.edu.cn (Y. Wu).
[1] Supported by the NNSFC Grant No. 19971056 and the RFDP Grant No. 2000024837.

PII: S0166-218X(01)00186-X

families of dense digraphs, that is to say digraphs with a relatively large order in comparison with the corresponding Moore bound (see [5]), can be seen as iterated line digraphs of a subfamily of basic digraphs indexed by their degrees. For example, the family of De Bruijn digraphs $\{B(d,k)\}_{d,k}$, defined in [4], is just $\{L^{k-1}B(d,1)\}_{d,k}$ and the family of Kautz ones $\{K(d,k)\}_{d,k}$, proposed in [12], is $\{L^{k-1}K(d,1)\}_{d,k}$, where $B(d,1)$ and $K(d,1)$ are the complete digraph K_d^* of degree d with loops and the complete digraph K_{d+1} of degree d without loops, respectively. As a result, since the adjacency matrices A of the digraphs $B(d,1)$ $(K(d,1))$ verify the common type of equations $A = J_d$ $(I + A = J_{d+1})$, we know that there corresponds a common type of matrix equations $A^k = J_{d^k}$ $(A^{k-1} + A^k = J_{d^k+d^{k-1}})$ for the family of digraphs $\{B(d,k)\}_{d,k}$ $(\{K(d,k)\}_{d,k})$. This observation also holds for some subfamilies of Imase-Itoh digraphs, which were introduced in [11] as a generalization of Kautz digraphs. For example, one can check that the adjacency matrices A of the subfamily of Imase-Itoh digraphs $\{G_I(d,1+d^k)\}_{d,k}$ of degree d and order $n = 1 + d^k$, for any odd integer $k > 0$, satisfy the common type of equations $I + A^k = J_n$. It then follows that the adjacency matrix of the iterated line digraph $L^l G_I(d,1+d^k)$, which is isomorphic to $G_I(d, d^l + d^{l+k})$, is a solution to the $(0,1)$-matrix equation

$$A^l + A^{l+k} = J_{nd^l}. \tag{1}$$

An interesting result, which was found by Wu and Li in [18], is that for any solution A of (1) there must exist a $(0,1)$-matrix C with $I + C^k = J_n$ and $\Gamma(A) = L^l\,\Gamma(C)$. Here we adopt the notation $\Gamma(A)$ for the associated digraph of a nonnegative integer matrix A. This result reveals the hidden underlying line digraph structure of the class of (large) digraphs specified by Eq. (1) and enables the reduction to the study of the (basic) class of digraphs satisfying a simpler type of equations $I + A^k = J_n$, which has already been considered in [14,19]. This also suggests a more general problem about when the line digraph structure will appear as an inherent property for any digraph whose adjacency matrix A satisfies a given polynomial equation $A^l P(A) = J_{nd^l}$, where l is a positive integer and $P(x)$ is a polynomial. As we will see in Section 4 and as the preceeding examples have shown, some type of matrix equations naturally correspond to digraph classes with large order. So the study of our problem may also help to understand the contribution of the line digraph construction in the extremal behavior of the order functions of some classes of dense digraphs.

Partially solving the above problem, we present in this paper some constraints on the coefficients of $P(x)$ which will force the iterated line digraph structure and hence allow the reduction from $A^l P(A) = J_{nd^l}$ to $P(A) = J_n$ just as in the special case $P(x) = 1 + x^k$ (see Section 3). As a consequence, we obtain another proof of Bosák's theorem [2] on the existence of $[k-h,k]$-*digraphs*, $0 \leqslant h \leqslant k$, namely, the digraphs with unique walks of length not smaller than $k-h$ and not greater than k between all pairs of (not necessarily distinct) vertices. Moreover, we simplify the study of some $(0,1)$-matrix equations associated with asymptotically optimal families of digraphs, which are digraphs with orders relatively close to the Moore bound (see Section 4).

2. Preliminary results on regular line digraphs

The following characterization of regular line digraphs (see [8]) is based on Heuchenne's condition [9], which states that a $(0,1)$-matrix A is the adjacency matrix of a line digraph G if and only if the rows (columns) of A are mutually orthogonal or identical. In particular, $\Gamma(A)$ is a line digraph if and only if its converse digraph $\Gamma(A^\top)$ is a line digraph.

Lemma 1. *A regular digraph of degree $d \geqslant 1$ and order n is a line digraph if and only if the rank of its adjacency matrix is equal to n/d.*

Thus, when the degree d divides the order n, the condition of $\Gamma(A)$ being a line digraph can be seen as an "extremal property", since rank A is at least n/d. This remark also holds if $\Gamma(A)$ is just out-regular (in-regular) and its minimum in-degree (out-degree) is at least one. Since it can be easily understood that an extremal digraph with respect to its order function will naturally behave extremely in some other respect, our result in this paper about the iterated line digraph structure of some classes of dense digraphs will be of no surprise after this observation.

From the known relation between the minimum polynomials of a regular digraph G of degree $d > 1$ and its line digraph LG, $\mu_{LG}(x) = x\mu_G(x)$ (see [1]), we derive the next result

Lemma 2. *Let G be a connected regular digraph of degree d and order n. Let A and A_L be the adjacency matrices of G and LG, respectively. Then,*

$$P(A) = J_n \Leftrightarrow A_L P(A_L) = J_{nd}, \tag{2}$$

where $P(x)$ is a polynomial.

Proof. If $d = 1$, then G and LG are both the cycle digraph on n vertices which implies that A and A_L are permutation similar. In addition, A_L is a permutation matrix and hence it holds $A_L^{-1} = A_L^\top$ and $A_L^\top J_n = J_n$. Thus,

$$A_L P(A_L) = J_n \Leftrightarrow P(A_L) = A_L^\top J_n = J_n \Leftrightarrow P(A) = J_n.$$

Assume now G has degree $d > 1$. Recall that the Hoffman polynomial $P_G^{\mathrm{H}}(x)$ of a strongly connected d-regular digraph G is the unique polynomial of minimum degree such that $P_G^{\mathrm{H}}(A) = J_n$, which implies that $P_G^{\mathrm{H}}(x)(x-d)$ annihilates the adjacency matrix A of G. Since a connected regular digraph must be strongly connected, we have from [10] that

$$P_G^{\mathrm{H}}(x) = c\frac{\mu_G(x)}{x-d} \quad \text{and} \quad P_{LG}^{\mathrm{H}}(x) = c_L\frac{\mu_{LG}(x)}{x-d},$$

where c, c_L are rational numbers such that $P_G^{\mathrm{H}}(d) = n$ and $P_{LG}^{\mathrm{H}}(d) = dn$. Making use of the identity $\mu_{LG}(x) = x\mu_G(x)$ mentioned before, we can deduce from these two relations that

$$P^{H}_{L\,G}(x) = xP^{\mathrm{H}}_{G}(x).$$

Therefore, taking into account that

$$P(A) = J_n \Leftrightarrow \mu_G(x) \mid (P(x) - P^{\mathrm{H}}_{G}(x)),$$

we arrive at

$$\begin{aligned} P(A) = J_n &\Leftrightarrow \mu_{L\,G}(x) \mid (xP(x) - P^{\mathrm{H}}_{L\,G}(x)) \\ &\Leftrightarrow A_L P(A_L) = J_{nd}. \quad \square \end{aligned}$$

Finally, let us present a result which somehow describes the "hereditary character" of the regular line digraph structure.

Lemma 3. *Let A be a d-regular $(0,1)$-matrix of order n ($AJ_n = J_nA = dJ_n$, $d \geqslant 1$) such that there exists an integer $l \geqslant 1$ satisfying the following conditions:*
(i) A^{l+1} is a $(0,1)$-matrix;
(ii) $\Gamma(A^l)$ is a line digraph.
Then, $\Gamma(A)$ is a line digraph.

Proof. Observe first that A^e is a d^e-regular $(0,1)$-matrix for $e = 0, 1, \ldots, l+1$, since A is a d-regular matrix and A^{l+1} is a $(0,1)$-matrix. We will see that $\Gamma(A^{l-1})$ is a line digraph. Then, using an inductive argument, we can deduce that $\Gamma(A^e)$ is a line digraph for each $e \leqslant l$, and hence complete the proof.

To show that $\Gamma(A^{l-1})$ is a line digraph we follow the proof of [18, Theorem 4.1]. We let the row (column) indices of A run from 1 to n and set

$$P(i,e) := \{s \ : \ 1 \leqslant s \leqslant n \text{ and } (A^e)_{is} = 1\},$$

for each pair $(i,e) \in \{1,\ldots,n\} \times \{1,\ldots,l+1\}$. By abusing notations, we simply identify a vertex in $\Gamma(A)$ with its corresponding row (column) index of A. Notice that $P(i,e)$, thus, represents the set of vertices of $\Gamma(A)$ which are reachable from vertex i in exactly e steps, and as A^e is a d^e-regular $(0,1)$-matrix, $P(i,e)$ contains exactly d^e vertices.

Clearly, according to Heuchenne's characterization of line digraph, our task is to show that any two nonorthogonal rows of A^{l-1} must be identical, or equivalently, $P(i, l-1) = P(j, l-1)$ whenever $P(i, l-1) \cap P(j, l-1) \neq \emptyset$. We will do it in the sequel under the assumption that there is an $s \in P(i, l-1) \cap P(j, l-1)$. In graphical terms, our assumption just means that $\Gamma(A)$ has two walks of length $l-1$ ending at the same vertex s and starting at the vertices i and j, respectively. Note here that we have

$$P(i,l) \cap P(j,l) \neq \emptyset \tag{3}$$

too since it includes all the out-neighbors of s in $\Gamma(A)$. Because of $J_nA = dJ_n \geqslant J_n$, which means that each vertex of $\Gamma(A)$ has at least one in-neighbour, we can take

$q, r \in \{1, \ldots, n\}$ such that $A_{qi} = A_{rj} = 1$. Then, $s \in P(q, l) \cap P(r, l)$, and consequently, $P(q, l) = P(r, l)$, since $\Gamma(A^l)$ is a line digraph. Therefore,

$$P(j, l-1) \subseteq P(r, l) = P(q, l) = P(i, l-1) \cup \left(\bigcup_{t=2}^{d} P(i_t, l-1) \right), \tag{4}$$

where we have assumed that $P(q, 1) = \{i, i_2, \ldots, i_d\}$ since $AJ_n = dJ_n$.

Note that if we can derive now

$$P(j, l-1) \cap \left(\bigcup_{t=2}^{d} P(i_t, l-1) \right) = \emptyset, \tag{5}$$

then the required result $P(i, l-1) = P(j, l-1)$ will follow from (4), since $|P(i, l-1)| = |P(j, l-1)| = d^{l-1}$. Let us consider the auxiliary set $P(q, l+1)$, which can be expressed as

$$P(q, l+1) = P(i, l) \cup \left(\bigcup_{t=2}^{d} P(i_t, l) \right) = P(j, l) \cup \left(\bigcup_{t=2}^{d} P(i_t, l) \right).$$

The second equality in the above formula comes from $P(i, l) = P(j, l)$, which is due to (3) together with the fact that $\Gamma(A^l)$ is a line digraph. Taking into account $|P(q, l+1)| = d^{l+1}$, it follows that $P(j, l)$ and $\bigcup_{t=2}^{d} P(i_t, l)$ must be disjoint, because otherwise

$$|P(q, l+1)| < |P(j, l)| + \sum_{t=2}^{d} |P(i_t, l)| = d^l + (d-1)d^l = d^{l+1},$$

which is a contradiction. Thus (5) holds. □

We remark that instead of the regularity condition the previous lemma can also apply in the (more general) case $AJ_n = dJ_n$ $(A^\top J_n = dJ_n)$ and $A^\top J_n \geqslant J_n$ $(AJ_n \geqslant J_n)$, that is to say when $\Gamma(A)$ is a d-out-regular (d-in-regular) digraph with minimum in-degree (out-degree) at least one. We also point out that the condition that A^{l+1} is a $(0, 1)$-matrix cannot be removed. To illustrate it, consider any solutions A to $A^2 = J_{d^2}$ such that $\Gamma(A)$ is not isomorphic to the De Bruijn digraph $B(d, 2)$ (see, for instance, [13,15]).

3. Main result

In general, it is not true that all $(0, 1)$-matrices which are solutions to a polynomial equation $A^l P(A) = J_n$, where $l \geqslant 1$, correspond to line digraphs. An example is the equation $A(A^2 - I) = J_n$ which is satisfied by the family of cycle-prefix digraphs of diameter three (see [16]) and also by the family of Imase-Itoh digraphs of degree d and order $n = d^3 - d$ (see [11]), but only the second ones being line digraphs. However, by restricting the coefficients of $P(x)$ in some way, we will really see the emergence of the underlying line digraph structure in a wide range. Now, we are ready to give our main result, whose proof generalizes the corresponding argument for the special case $P(x) = 1 + x^k$ given by Wu and Li [18].

Theorem 1. *Let A be a* $(0,1)$*-matrix satisfying* $A^l P(A) = J_n$, where $l \geqslant 1$ *is an integer and* $P(x) = 1 + c_1 x + \cdots + c_r x^r$ *is a polynomial of degree* $r \geqslant 1$ *with nonnegative coefficients (which implies in fact* $0 \leqslant c_i \leqslant 1$*) such that* $c_j = 1$, *for at least one index* $j \geqslant 1$. *Then, there exists a* $(0,1)$*-matrix C satisfying* $P(C) = J_{n/d^l}$ and $\Gamma(A) = L^l\,\Gamma(C)$, where $AJ_n = A^\top J_n = dJ_n$.

Proof. We will show that any $(0,1)$-matrix A as described in the theorem must fulfil the conditions of Lemma 3 and, consequently, $\Gamma(A)$ is a line digraph. Actually, after doing so, we can get that $\Gamma(A) = L\,\Gamma(C_1)$ for some digraph $\Gamma(C_1)$, which is necessarily d-regular, and then, as a consequence of Lemma 2, we will have that C_1 satisfies the same type of equation $C_1^{l-1} P(C_1) = J_{n/d}$ just as A. Continuing this way, we shall finally get the required $(0,1)$-matrix $C = C_l$ such that $\Gamma(A) = L^l\,\Gamma(C_l)$ and $P(C_l) = J_{n/d^l}$.

From $A^l P(A) = J_n$, it follows that A is an irreducible d-regular matrix, where $d^l P(d) = n$. Moreover, since $A^{l+j} \leqslant A^l P(A)$, where $c_j = 1$, it follows that A^i is a d^i-regular $(0,1)$-matrix, for each $i = 0, \ldots, l+j$. In particular, A^{l+1} is a $(0,1)$-matrix, which is the first condition in Lemma 3.

Now, it remains to prove that $\Gamma(A^l)$ is a line digraph (second condition of Lemma 3). By virtue of Lemma 1, we need only establish that rank $A^l = n/d^l$. But, from the equation $A^l P(A) = J_n$, it turns out that the minimum polynomial of A, $\mu_A(x)$, divides $(x-d)x^l P(x)$, since $(A - dI)A^l P(A) = (A - dI)J_n = 0$. In particular, the multiplicity of the factor x in $\mu_A(x)$, say e, is at most l. So, $\ker A^e = \ker A^l$ and, consequently, dim ker $A^l = m_A(0)$, where $m_A(0)$ denotes the (algebraic) multiplicity of 0 as an eigenvalue of A. Hence, rank $A^l = n - m_A(0)$. Besides, we know that the remaining eigenvalues of A, apart from the degree d which has multiplicity 1 (by the Perron-Frobenius's Theorem), must be zeros of $P(x)$. In order to compute the sum of their multiplicities we will consider the trace of $P(A) - I$. It is clear that $\mathrm{tr}(P(A) - I) \geqslant 0$, since the coefficients of $P(x) - 1$ are nonnegative. Further, if $\mathrm{tr}(P(A) - I)$ were positive, then $A^l P(A) = A^l(I + (P(A) - I))$ would have an entry greater than one, which is impossible. Therefore, $\mathrm{tr}(P(A) - I) = 0$. Expressing such a condition in terms of the eigenvalues of A and taking into account that $P(0) = 1$, we obtain

$$\begin{aligned} \mathrm{tr}(P(A) - I) &= P(d) - 1 + (P(0) - 1)m_A(0) + \sum_{\lambda \neq 0, d} (P(\lambda) - 1)m_A(\lambda) \\ &= P(d) - (n - m_A(0)) = 0. \end{aligned}$$

Hence, rank $A^l = P(d) = n/d^l$, as desired. □

We remark that there are strongly connected regular digraphs whose Hoffman polynomial are of the form $x^l P(x)$, where $l \geqslant 1$ and $P(x)$ have nonnegative coefficients, which are not line digraphs. As an example, note that the Hoffman polynomials of the almost Moore digraph G_2^2 of degree and diameter two, defined in [6], is $\frac{1}{19}x(3 + 3x + 4x^2 + 2x^3 + x^4)$.

4. Final comments

The previous theorem can be used to provide an alternative proof of the existence result [2] for $[k-h,k]$-digraphs, $0 \leqslant h \leqslant k$, as follows. Since the adjacency matrix A of a $[k-h,k]$-digraph G of degree d satisfies the equation

$$A^{k-h} + \cdots + A^k = J_n,$$

Theorem 1 tells us there exists a $(0,1)$-matrix C such that $I + C + \cdots + C^h = J_{n/d^{k-h}}$ and $G = L^{k-h}\Gamma(C)$, provided that $0 < h$. Notice that $\Gamma(C)$ is a Moore digraph of degree d and diameter h, which does only exist in the case $h = 1$, if $d > 1$ (see [3,17]). This then leads us to conclude that a $[k-h,k]$-digraphs of degree $d > 1$ can only exist when $h = 0, 1$. Of course, the characterization of Kautz digraphs, $L^{k-1}K_{d+1}$, as the unique $[k-1,k]$-digraphs, given by Wu and Li [18], can also be derived from Theorem 1. Nevertheless, we cannot reduce the equation $A^k = J$, which has deserved much attention (see, for instance, [7,20]).

We define a family of connected d-regular digraphs of a fixed diameter $k \geqslant 2$, $\{G(d,k)\}_{d \geqslant 2}$, as being *asymptotically optimal* if its order function $n(d,k)$ satisfies

$$\lim_{d\to\infty} \frac{n(d,k)}{M_{\mathscr{D}}(d,k)} = 1,$$

where $M_{\mathscr{D}}(d,k) = 1 + d + \cdots + d^k$ is the (unattainable) Moore bound. Suppose that the adjacency matrix A of each digraph $G(d,k)$ satisfies the same type of polynomial equations $P_k(A) = J_n$, where $n{:=}n(d,k) = P_k(d)$. Then $P_k(x)$ is a monic polynomial of degree k, since $\lim_{d\to\infty} P_k(d)/(d^k + \cdots + d + 1) = 1$. But, the minimum polynomial of A, $\mu_A(x) \in \mathrm{Z}[x]$, is a monic polynomial of degree at least $k+1$; on the other hand, $\mu_A(x) \mid (x-d)P_k(x)$ since $(x-d)P_k(x)$ annihilates A. This in turn gives $\mu_A(x) = (x-d)P_k(x)$ and thus $P_k(x)$ has integer coefficients too. Furthermore, when $P_k(x)$ has nonnegative coefficients, Theorem 1 allows us to reduce the study to the case $P_k(A) = I + A^{a_1} + \cdots + A^{a_r} = J$, where $1 \leqslant a_1 < \cdots < a_r = k$. Since the problem of the existence of solutions in both extreme situations ($P_k(A) = I + A + \cdots + A^k = J$ and $P_k(A) = I + A^k = J$) has already been solved (see [17,14], respectively), it may be worth studying the intermediate cases.

Acknowledgements

The authors thank the referees for their comments which have improved the style of the paper.

References

[1] C. Balbuena, X. Marcote, I. Pelayo, The Jordan normal form of a line digraph, submitted for publication.
[2] J. Bosák, Directed graphs and matrix equations, Math. Slovaca 28 (1978) 189–202.

[3] W.G. Bridges, S. Toueg, On the impossibility of directed Moore graphs, J. Combin. Theory Ser. B 29 (1980) 339–341.
[4] N.G. de Bruijn, A combinatorial problem, Koninklijke Nederlandse Academie van Wetenschappen Proc. A 49 (1946) 758–764.
[5] M.A. Fiol, I. Alegre, J.L.A. Yebra, Line digraphs iterations and the (d,k) problem for directed graphs, Proceedings of the 10th International Symposium Computation Archies, Stockholm, 1983, pp. 174–177.
[6] M.A. Fiol, J.L.A. Yebra, I. Alegre, Line digraphs iterations and the (d,k) digraph problem, IEEE Trans. Comput. C-32 (1984) 400–403.
[7] M.A. Fiol, I. Alegre, J.L.A. Yebra, J. Fàbrega, Digraphs with walks of equal length between vertices, In: Y. Alavi et al. (Eds.), Graph Theory and its Applications to Algorithms and Computer Science, Wiley, New York, 1985, pp. 702–713.
[8] J. Gimbert, On digraphs with unique walks of closed lengths between vertices, Austral. J. Combin. 20 (1999) 77–90.
[9] C. Heuchenne, Sur une certaine correspondance entre graphes, Bull. Soc. Roy. Sci. Liège 33 (1964) 743–753.
[10] A.J. Hoffman, M.H. McAndrew, The polynomial of a directed graph, Proc. Amer. Math. Soc. 16 (1965) 303–309.
[11] M. Imase, M. Itoh, A design for directed graphs with minimum diameter, IEEE Trans. Comp. 32 (1983) 782–784.
[12] W.H. Kautz, Design of optimal interconnection networks for multiprocessors, Architecture and Design of Digital Computers, NATO Advanced Summer Institute, 1969, pp. 249–272.
[13] D.E. Knuth, Notes on central grupoids, J. Combin. Theory 8 (1970) 376–390.
[14] C.W.H. Lam, J.H. van Lint, Directed graphs with unique paths of fixed length, J. Combin. Theory Ser. B 24 (1978) 331–337.
[15] N.S. Mendelsohn, Directed graphs with the unique path property, Combinatorial Theory and its Applications II (Proc. Colloq. Balatonfürer, 1969), North-Holland Amsterdam, 1970, pp. 783–799.
[16] M. Mitjana, F. Comellas, The spectra of cycle prefix digraphs, submitted for publication.
[17] J. Plesník, Š. Znám, Strongly geodetic directed graphs Acta F. R. N. Univ. Comen.-Mathematica (XXIX) 1974.
[18] Y. Wu, Q. Li, On the matrix equation $A^l + A^{l+k} = J_n$, Linear Algebra Appl. 277 (1998) 41–48.
[19] Y. Wu, Q. Li, X. Huang, On the matrix equation $A^k = J - I$, Linear Algebra Appl. 295 (1999) 249–260.
[20] Y. Wu, R. Jia, Q. Li, g-circulant solutions to the $(0,1)$ matrix equation $A^m = J_n$, submitted for publication.