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Two minimal forbidden subgraphs for double competition graphs of posets of dimension at most two[☆]

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ABSTRACT

Let S be any set of points in the Euclidean plane $\mathbb{R}^2$. For any $p = (x, y) \in S$, put $SW(p) = \{(x', y') \in S : x' < x \text{ and } y' < y\}$ and $NE(p) = \{(x', y') \in S : x' > x \text{ and } y' > y\}$. Let G_S be the graph with vertex set S and edge set $\{pq : NE(p) \cap NE(q) \neq \emptyset \text{ and } SW(p) \cap SW(q) \neq \emptyset\}$. We prove that the graph H with $V(H) = \{u, v, z, w, p, p_1, p_2, p_3\}$ and $E(H) = \{uv, vz, zw, wu, p_1p_3, p_2p_3, pu, pv, pz, pw, pp_1, pp_2, pp_3\}$ and the graph H' obtained from H by removing the edge pp_3 are both minimal forbidden subgraphs for the class of graphs of the form G_S.

1. Introduction

We start with some notation and terminology. For a few undefined terms in the paper, we will follow common practice.

A *graph* G is a pair $(V(G), E(G))$ of sets where $V(G) \neq \emptyset$ and $E(G) \subseteq \binom{V(G)}{2}$. For any $\{u, v\} \in E(G)$, we often denote it by uv and say that u and v are *adjacent* in G. A vertex of G is *isolated* if it appears in no members of $E(G)$. If G' is a graph with $V(G') = \{w_1, w_2, \ldots, w_t\} \subseteq V(G)$ and $E(G') = E(G) \cap \binom{V(G')}{2}$, we say that G' is an *induced subgraph* of G and often denote it by $G[w_1, w_2, \ldots, w_t]$. A *proper induced subgraph* of G is one of its induced subgraph other than itself. We often write $G - v$ for the subgraph $G[V(G) \setminus \{v\}]$.

In what follows, H stands for the graph with $V(H) = \{u, v, z, w, p, p_1, p_2, p_3\}$ and $E(H) = \{uv, vz, zw, wu, p_1p_3, p_2p_3, pu, pv, pz, pw, pp_1, pp_2, pp_3\}$ and H' denotes the graph obtained from H by deleting the edge pp_3. For an illustration, see Fig. 1. For any $n \geq 3$, we use the notation $C_n = (v_1v_2 \cdots v_n)$ for the cycle of length n, namely the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. The path of length n, denoted P_n, is the graph obtained from C_{n+1} by deleting one edge. The *complete graph* K_n is the graph on n vertices and $E(G) = \binom{V(G)}{2}$. For a graph G and a class of graphs $\mathscr{C}$, we say that G is a *forbidden subgraph* for $\mathscr{C}$ provided no member of $\mathscr{C}$ has G as an induced subgraph and we call G a *minimal forbidden subgraph* for $\mathscr{C}$ if it is a forbidden subgraph for $\mathscr{C}$ but none of its proper induced subgraphs are.

Let S be a set of points of $\mathbb{R}^2$. For any $p \in S$, put $SW(p) = \{(x', y') \in S : x' < x \text{ and } y' < y\}$ and $NE(p) = \{(x', y') \in S : x' > x \text{ and } y' > y\}$. Let G_S designate the graph with vertex set S and edge set $\{pq : NE(p) \cap NE(q) \neq \emptyset \text{ and } SW(p) \cap SW(q) \neq \emptyset\}$. Since the set S determines a poset of dimension at most two [1,2,10], we call G_S the *double competition graph of a poset of dimension at most two* [5–7,9,11,12]. We mention that the *dimension* of any poset $P = (V, <)$ is the minimum positive

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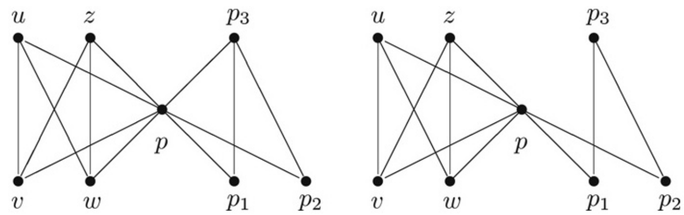

Fig. 1. The graphs H and H'.

integer n for which there is a mapping from V to $\mathbb{R}^n$ such that $p < q$ in P if and only if all components of $f(q) - f(p)$ are positive.

Let $\mathscr{D} = \{G_S : S \subseteq \mathbb{R}^2\}$. The graph class $\mathscr{D}$ was first introduced and studied by Kim, Kim and Rho [7]. Developing the work in [7], Wu and Lu [12] showed some necessary conditions for a graph to lie in $\mathscr{D}$, including that $\mathscr{D}$ is a proper subclass of trapezoid graphs. This paper aims to further understand the structure of $\mathscr{D}$. We list our main results below which combine to assert that both H and H' are minimal forbidden subgraphs for $\mathscr{D}$.

Theorem 1. *H and H' are both forbidden subgraphs for $\mathscr{D}$.*

Theorem 2. *No proper induced subgraphs of either H or H' can be forbidden subgraphs for $\mathscr{D}$.*

2. Proof of Theorem 1

In this section we fix the notation S to refer to a nonempty subset of $\mathbb{R}^2$. For any point $p = (x, y) \in \mathbb{R}^2$, we write $X(p)$ and $Y(p)$ for x and y, respectively. Let

$$SE(p) = \{p' \in S : p' \neq p, X(p') \geq X(p), Y(p') \leq Y(p)\},$$
$$NW(p) = \{p' \in S : p' \neq p, X(p') \leq X(p), Y(p') \geq Y(p)\},$$

and

$$\mathscr{U}_S = \{\overrightarrow{pq} : p \in S, q \in SW(p)\} = \{\overrightarrow{pq} : q \in S, p \in NE(q)\},$$
$$\mathscr{V}_S = \{\overrightarrow{pq} : p \in S, q \in SE(p)\} = \{\overrightarrow{pq} : q \in S, p \in NW(q)\}.$$

For any $p, p' \in S$, an observation which will be used often implicitly is:

$$SW(P) \cap SW(p') = SW((\min\{X(p), X(p')\}, \min\{Y(p), Y(p')\})),$$
$$NE(P) \cap NE(p') = NE((\max\{X(p), X(p')\}, \max\{Y(p), Y(p')\})).$$

In particular, if $X(p) \leq X(p')$ and $Y(p) \leq Y(p')$, which is surely the case when $\overrightarrow{p'p} \in \mathscr{U}_S$, we have

$$SW(p) \subseteq SW(p') \quad \text{and} \quad NE(p') \subseteq NE(p). \tag{1}$$

We are ready to go into a discussion of the point distributions which are enforced by the appearance of several small induced subgraphs of G_S. The proof of Theorem 1 will follow quickly from these discussions.

2.1. $2P_1$

Lemma 3. *Let a, b, c be three points of S such that $ab \in E(G_S)$, $ac, bc \notin E(G_S)$ and c is not isolated in G_S. Then we have either $c \in NW(a) \cap NW(b)$ or $c \in SE(a) \cap SE(b)$.*

Proof. Without loss of generality, there are two cases to consider, $\overrightarrow{ab} \in \mathscr{U}_S$ or $\overrightarrow{ab} \in \mathscr{V}_S$.

Case 1: $\overrightarrow{ab} \in \mathscr{U}_S$. In light of the observation in Eq. (1) and the fact that $ac, bc \notin E(G_S)$ and that none of a, b, c is isolated, we conclude that $c \notin NE(b) \cup SW(a)$ and hence $c \in (NW(a) \cap NW(b)) \cup (SE(a) \cap SE(b))$ follows.

Case 2: $\overrightarrow{ab} \in \mathscr{V}_S$. By observation in Eq. (1) and the fact that $ac, bc \notin E(G_S)$ and that none of a, b, c is isolated, we have $c \notin NE(a) \cup SW(a) \cup NE(b) \cup SW(b)$. Hence, to finish the proof we need to exclude the possibility that it holds both $X(a) \leq X(c) < X(b)$ and $Y(b) \leq Y(c) < Y(a)$. But for such a case, we will have $SW(c) \supseteq SW(a) \cap SW(b)$ and $NE(c) \supseteq NE(a) \cap NE(b)$. We thus deduce from $ab \in E(G_S)$ that $ac, bc \in E(G_S)$, a contradiction. □

Lemma 4. *Suppose that $p, p', r, r' \in S$ are four vertices such that $G_S[p, p', r, r']$ has only two edges pp' and rr'. Then it holds either $r, r' \in NW(p) \cap NW(p')$ or $r, r' \in SE(p) \cap SE(p')$.*

Proof. Putting $a = p, b = p', c = r$ or r' in Lemma 3, we get that $r, r' \in (NW(a) \cap NW(b)) \cup (SE(a) \cap SE(b))$. Note that for any point $t \in NW(a) \cap NW(b)$ and any point $t' \in SE(a) \cap SE(b)$, it holds that $a, b \notin NW(t) \cup SE(t') = (NW(t) \cap NW(t')) \cup (SE(t) \cap SE(t'))$. Now taking $a = r, b = r', c = p$ or p' in Lemma 3, we arrive at the required claim. □

2.2. P_2

Lemma 5. *Suppose $p_1, p_2, p \in S$ are three points such that $G_S[p_1, p_2, p]$ contains just two edges p_1p and p_2p. Then neither $\{\overrightarrow{pp_1}, \overrightarrow{pp_2}\} \subseteq \mathscr{V}_S$ nor $\{\overrightarrow{p_1p}, \overrightarrow{p_2p}\} \subseteq \mathscr{V}_S$ can happen.*

Proof. Our strategy is to deduce from $\{\overrightarrow{pp_1}, \overrightarrow{pp_2}\} \subseteq \mathscr{V}_S$ and $\{\overrightarrow{p_1p}, \overrightarrow{p_2p}\} \subseteq \mathscr{V}_S$, respectively, that $p_1p_2 \in E(G_S)$, yielding a contradiction.

Suppose $\{\overrightarrow{pp_1}, \overrightarrow{pp_2}\} \subseteq \mathscr{V}_S$. Then we know that there is $s \in S$ such that $X(s) > \max\{X(p_1), X(p_2)\}$ and $Y(s) > Y(p) \geq \max\{Y(p_1), Y(p_2)\}$ and there is $s' \in S$ such that $Y(s') < \min\{Y(p_1), Y(p_2)\}$ and $X(s') < X(p) \leq \min\{X(p_1), X(p_2)\}$. This implies that $s \in NE(p_1) \cap NE(p_2)$ and $s' \in SW(p_1) \cap SW(p_2)$. Consequently, $p_1p_2 \in E(G_S)$, as desired.

Suppose $\{\overrightarrow{p_1p}, \overrightarrow{p_2p}\} \subseteq \mathscr{V}_S$. Then we know that there is $s \in S$ such that $Y(s) > \max\{Y(p_1), Y(p_2)\}$ and $X(s) > X(p) \geq \max\{X(p_1), X(p_2)\}$ and there is $s' \in S$ such that $X(s') < \min\{X(p_1), X(p_2)\}$ and $Y(s') < Y(p) \leq \min\{Y(p_1), Y(p_2)\}$. This implies that $s \in NE(p_1) \cap NE(p_2)$ and $s' \in SW(p_1) \cap SW(p_2)$. Consequently, $p_1p_2 \in E(G_S)$, as wanted. □

2.3. $K_5 - 2P_1$

Lemma 6 ([7, Theorem 8][12, Lemma 38]). *Suppose that $(uvzw) = G_S[u, v, z, w]$. Then after a suitable relabeling of the four vertices $u, v, z, w \in \mathbb{R}^2$, we must have*

$$\overrightarrow{uv}, \overrightarrow{zw} \in \mathscr{U}_S \quad \text{and} \quad \overrightarrow{uz}, \overrightarrow{vw} \in \mathscr{V}_S. \tag{2}$$

Lemma 7. *Let $(uvzw)$ be an induced cycle of G_S such that Eq. (2) holds. Then*

(1) *$SW(v), SW(w), NE(u)$ and $NE(z)$ are all nonempty;*
(2) *$SW(v) \cap SW(w) = \emptyset$ and $NE(u) \cap NE(z) = \emptyset$.*

Proof. (1) This is obviously true as none of u, v, z, w is isolated in G_S.

(2) Since $vw \notin E(G_S)$, to show that $SW(v) \cap SW(w) = \emptyset$, it suffices to demonstrate that $NE(v) \cap NE(w) \neq \emptyset$. But Eqs. (1) and (2) guarantee that $NE(v) \supseteq NE(u)$ while $uw \in E(G_S)$ tells us that $NE(u) \cap NE(w) \neq \emptyset$. It thus follows $NE(v) \cap NE(w) \supseteq NE(u) \cap NE(w) \neq \emptyset$, as was to be shown.

Since $uz \notin E(G_S)$, to show that $NE(u) \cap NE(z) = \emptyset$, it suffices to demonstrate that $SW(u) \cap SW(z) \neq \emptyset$. But Eqs. (1) and (2) guarantee that $SW(z) \supseteq SW(w)$ while $uw \in E(G_S)$ tells us that $SW(u) \cap SW(w) \neq \emptyset$. It hence follows $SW(u) \cap SW(z) \supseteq SW(u) \cap SW(w) \neq \emptyset$, finishing the proof. □

Let $(uvzw)$ be an induced cycle of G_S for which Eq. (2) holds. Eq. (2) along with Lemma 7 allows us to pick x_1, x_2, y_1, y_2 such that

$$\begin{aligned} x_1 &= \min\{X(q) : q \in SW(w)\}, \qquad x_2 = \max\{X(q) : q \in NE(u)\}, \\ y_1 &= \min\{Y(q) : q \in SW(v)\}, \qquad y_2 = \max\{Y(q) : q \in NE(z)\}. \end{aligned} \tag{3}$$

An immediate consequence of Eq. (3) is that

$$SW((x_1, Y(w))) = SW((X(v), y_1)) = NE((x_2, Y(u))) = NE((X(z), y_2)) = \emptyset. \tag{4}$$

We further define $\mathfrak{R}_{u,v,z,w}(S)$ to be

$$\{p \in S : x_1 < X(p) < x_2 \text{ and } y_1 < Y(p) < y_2\}$$

and call it *the rectangle determined by* $(uvzw)$. For an illustration, see Fig. 2.

It is easy to see that for any $p \in \mathfrak{R}_{u,v,z,w}(S)$ we have

$$X(v) < X(p) < X(z) \quad \text{and} \quad Y(w) < Y(p) < Y(u). \tag{5}$$

Lemma 8. *Suppose $(uvzw)$ is an induced subgraph of G_S such that Eq. (2) holds. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be as specified in Eq. (3). Then, we have the followings:*

(1) *If $X(p) \leq x_1$, then $pw \notin E(G_S)$.*
(2) *If $Y(p) \leq y_1$, then $pv \notin E(G_S)$.*
(3) *If $X(p) \geq x_2$, then $pu \notin E(G_S)$.*
(4) *If $Y(p) \geq y_2$, then $pz \notin E(G_S)$.*

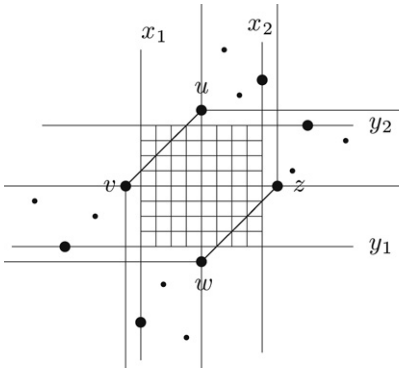

Fig. 2. The shaded part is the rectangle $\mathfrak{R}_{u,v,z,w}(S)$ determined by $(uvzw)$.

Proof. (1) If $X(p) \le x_1$, then

$$SW(p) \cap SW(w) = SW((\min\{X(p), X(w)\}, \min\{Y(p), Y(w)\})) \subseteq SW((x_1, Y(w))).$$

By Eq. (4), we get $SW(p) \cap SW(w) = \emptyset$ and hence $pw \notin E(G_S)$ follows.
(2) If $Y(p) \le y_1$, then

$$SW(p) \cap SW(v) = SW((\min\{X(p), X(v)\}, \min\{Y(p), Y(v)\})) \subseteq SW((X(v), y_1)).$$

By Eq. (4), we obtain $SW(p) \cap SW(v) = \emptyset$ and hence $pv \notin E(G_S)$ follows.
(3) If $X(p) \ge x_2$, then

$$NE(p) \cap NE(u) = NE((\max\{X(p), X(u)\}, \max\{Y(p), Y(u)\})) \subseteq NE((x_2, Y(u))).$$

By Eq. (4), we have $NE(p) \cap NE(u) = \emptyset$ and thus yield $pu \notin E(G_S)$.
(4) If $Y(p) \ge y_2$, then

$$NE(p) \cap NE(z) = NE((\max\{X(p), X(z)\}, \max\{Y(p), Y(z)\})) \subseteq NE((X(z), y_2)).$$

It follows from Eq. (4) that $NE(p) \cap NE(z) = \emptyset$. This implies that $pz \notin E(G_S)$ and then we are done. □

Lemma 9. *Let $S' = \{u, v, z, w, p\} \subset S$. If $E(G_S[S']) = \binom{S'}{2} \setminus \{uz, vw\}$, then, up to vertex relabeling, Eq. (2) holds and $p \in \mathfrak{R}_{u,v,z,w}(S)$.*

Proof. Lemmas 6 and 8 conspire to give this result. □

2.4. Final proof

Proof of Theorem 1. Let H^* denote the graph H or H'. We assume that H^* is an induced subgraph of G_S and try to derive a contradiction. Note that for our arguments below to hold whether or not $pp_3 \in E(H^*)$ does not matter. Applying Lemma 9 to the subgraph $G_S[u, v, z, w, p]$, we can without loss of generality assume that u, v, z, w satisfy Eq. (2) and know that

$$p \in \mathfrak{R}_{u,v,z,w}(S). \quad (6)$$

We proceed to make use of Lemma 4 on $G_S[z, w, p_1, p_3]$, $G_S[z, w, p_2, p_3]$, $G_S[u, v, p_1, p_3]$, $G_S[u, v, p_2, p_3]$, $G_S[u, w, p_1, p_3]$, $G_S[u, w, p_2, p_3]$, $G_S[z, v, p_1, p_3]$ and $G_S[z, v, p_2, p_3]$. According to Eq. (2), what we obtain is the fact that it holds either

$$p_1, p_2, p_3 \in SE(z) \cap SE(w), \quad (7)$$

or

$$p_1, p_2, p_3 \in NW(u) \cap NW(v). \quad (8)$$

We next turn to $G_S[p, p_1, p_2]$ and conclude from Lemma 5 that neither $\{\overrightarrow{pp_1}, \overrightarrow{pp_2}\} \subseteq \mathscr{V}_S$ nor $\{\overrightarrow{p_1p}, \overrightarrow{p_2p}\} \subseteq \mathscr{V}_S$ can happen. But Eqs. (5) and (6) together with Eq. (7) give $\{\overrightarrow{pp_1}, \overrightarrow{pp_2}\} \subseteq \mathscr{V}_S$ while Eqs. (5) and (6) together with Eq. (8) bring to us $\{\overrightarrow{p_1p}, \overrightarrow{p_2p}\} \subseteq \mathscr{V}_S$. This violates our former conclusion and then ends the proof. □

3. Proof of Theorem 2

A graph G is an *interval graph* if each $v \in V(G)$ can be assigned an interval I_v on the real line such that $uv \in E(G)$ if and only if $I_u \cap I_v \ne \emptyset$. An *asteroidal triple* in a graph is a triple of (independent) vertices such that between any two there exists a path avoiding the (closed) neighborhood of the third.

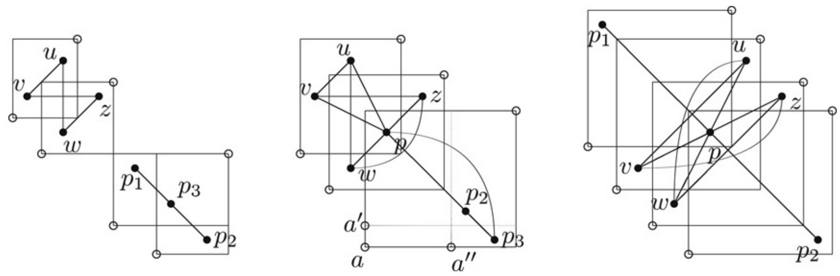

Fig. 3. $H^* - p$, $H^* - p_1$ and $H^* - p_3$ as induced subgraphs of G_S.

Lemma 10 (*Lekkerkerker and Boland [8]*). *A graph is an interval graph if and only if it has no asteroidal triple and it does not contain any cycle of length greater than or equal to four as an induced subgraph.*

Lemma 11 (*Kim, Kim and Rho [7, Theorem 10]*). *Every interval graph is an induced subgraph of* G_S *for some* $S \subseteq \mathbb{R}^2$.

Proof of Theorem 2. Let H^* stand for either H or H'. Our task is to show that for any $b \in V(H^*)$, $H^* - b$ can be an induced subgraph of G_S for some $S \subseteq \mathbb{R}^2$.

Suppose $b \in \{u, v, z, w\}$. Using Lemma 10, we can check that $H^* - b$ is an interval graph. Henceforth, the result follows from Lemma 11.

Observe that $H^* - p_1$ and $H^* - p_2$ are isomorphic. So it remains to consider $b \in \{p, p_1, p_3\}$. We give the distributions of $S \subseteq \mathbb{R}^2$ in Fig. 3 for which G_S has $H^* - b$ as induced subgraphs for $b = p, p_1, p_3$ from left to right, respectively. Note that we use a bullet for the location of any point from $\{u, v, z, w, p, p_1, p_2, p_3\}$ and a circle for each remaining point in S. Also note that for the middle picture, we should remove the point a to kill the edge pp_3 when $H^* = H'$. In all these cases, each point indicated by a circle is an isolated vertex in G_S. $\square$

4. Concluding remarks

The *incomparability graph* of a poset $P = (V, <)$ is the graph with vertex set V and edge set $\{uv \in \binom{V}{2}$: neither $u < v$ nor $v < u$ holds $\}$. Due to the earlier work on the graph class $\mathscr{D}$ [7,12], to show that $G \notin \mathscr{D}$, we can proceed as follows: if G is not an incomparability graph, then $G \notin \mathscr{D}$; if G is an incomparability graph of a poset P, then construct two associated graphs $\mathfrak{D}_P$ and $\mathfrak{B}_P$ and if either of them is not bipartite we also conclude that $G \notin \mathscr{D}$. It has been shown that G is the incomparability graph of a poset P with a bipartite $\mathfrak{D}_P$ if and only if G is a trapezoid graph [3,4] while for any incomparability graph G of a poset P, $\mathfrak{B}_P$ is totally determined by G and does not depend on the choice of P [12]. We can find posets P and P' with respective incomparability graphs H and H' such that $\mathfrak{B}_P, \mathfrak{D}_P, \mathfrak{B}_{P'}, \mathfrak{D}_{P'}$ are all bipartite. This means that our Theorem 1 does detect a new obstruction for a graph to lie in $\mathscr{D}$. It is known that the set of forbidden subgraphs for the double competition graphs of posets of dimension one is just the set of all graphs other than the disjoint unions of complete graphs [12, Example 3]. It is not clear whether or not a good forbidden subgraph characterization for $\mathscr{D}$ will be possible. We also wonder whether H and/or H' could be an induced subgraph of the double competition graph of a poset of dimension 3.

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