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Recognizing edge clique graphs among interval graphs and probe interval graphs☆

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Abstract

The edge clique graph of a graph H is the one having the edge set of H as vertex set, two vertices being adjacent if and only if the corresponding edges belong to a common complete subgraph of H. We characterize the graph classes {edge clique graphs} $\cap$ {interval graphs} as well as {edge clique graphs} $\cap$ {probe interval graphs}, which leads to polynomial time recognition algorithms for them. This work generalizes corresponding results in [M.R. Cerioli, J.L. Szwarcfiter, Edge clique graphs and some classes of chordal graphs, Discrete Math. 242 (2002) 31–39].

Keywords: Edge clique graphs; Interval graphs; Probe interval graphs

1. Edge clique graph

We consider only finite undirected graphs without parallel edges or loops. Let G be a graph. A *clique* of G is a subset of $V(G)$ which induces a complete subgraph in G. We denote by $\mathcal{C}(G)$ the set of all maximal cliques of G. For $S \subseteq V(G)$, let $G(S)$ denote the subgraph of G induced by S.

The *edge clique graph* of a graph G, denoted $K_e(G)$, is the one whose vertices are the edges of G and two vertices are adjacent if and only as edges in G their endpoints all belong to a common clique of G. The construction of edge clique graphs is first implicitly used by Kou, Stockmeyer and Wong in 1978 [1], while this concept is first formally introduced by Albertson and Collins in 1984 [2]. Many results and applications of edge clique graphs can be found in [1–12].

The following is a very useful basic result on edge clique graphs.

Proposition 1 (*Albertson and Collins [2]*). *Let H be a graph. There exists a one-to-one correspondence between nontrivial maximal cliques (intersection of nontrivial maximal cliques) of H and maximal cliques (intersection of maximal cliques) of $K_e(H)$. Moreover, if C is a nontrivial maximal clique (intersection of maximal cliques) of H,*

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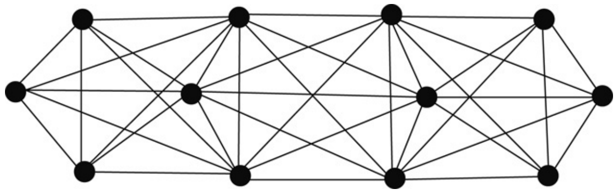

Fig. 1. A graph which satisfies A1 but is not an edge clique graph.

then the corresponding clique of $K_e(H)$ is formed by the vertices which correspond to the edges of H with both endpoints in C.

A *triangular number* is a number of the form $\frac{n(n-1)}{2}$ for some nonnegative integer n. A *triangular clique* is a clique whose size is a triangular number. By Proposition 1 we get

Proposition 2 (*Chartrand et al. [7]*). *Each edge clique graph must satisfy:*

(A1) *any intersection of a set of maximal cliques is a triangular clique.*

A *starlike threshold graph* [6] is a graph which admits an ordering of its maximal cliques as $C_1, \ldots, C_s, C$ so that $C_i \setminus C$ are pairwise disjoint, $C \cap C_i \subseteq C \cap C_{i+1}$ and all vertices coming from the same $C_i \setminus C$ have the same set of closed neighborhood.

Example 3. Cerioli and Szwarcfiter [6] show that for a starlike-threshold graph G, (A1) is a necessary and sufficient condition for it to be an edge clique graph. In general, Chartrand et al. [7] point out that there exists a graph which fulfils (A1) but is not an edge clique graph; see the graph depicted in Fig. 1.

Two characterizations of edge clique graphs have been presented in [5,7], respectively. But there is not yet any polynomial time recognition algorithm for edge clique graphs. As a generalization of the observations made in Example 3, we will introduce interval graphs and probe interval graphs in the next section and then give in Section 3 a polynomial time recognition algorithm for edge clique graphs among these two classes of graphs.

2. Interval graph and probe interval graph

A graph G is an *interval graph* [13,14] if there is a surjective map f from $V(G)$ to a collection S of closed intervals of the real line such that any two different vertices u and v of G are adjacent if and only if $f(u)$ and $f(v)$ have a nonempty intersection. In this case, we also call G the *intersection graph* of (S, f), or simply S. Benzer [15] and Hajös [16] independently initiated the study of interval graphs. Since then, interval graphs have become one of the most useful mathematical structures for modelling real world problems [13, p. 181].

Let $\pi = (C_1, \ldots, C_s)$ be an ordering of maximal cliques of G. We call the linear ordering π a *consecutive clique arrangement* of G, or a *consecutive ordering* of $\mathcal{C}(G)$, provided for every vertex the maximal cliques containing it occur consecutively in π.

Theorem 4 (*Gilmore and Hoffman [17]*). *A graph G is an interval graph if and only if $\mathcal{C}(G)$ has a consecutive ordering.*

Let G be a graph and $V(G)$ be a disjoint union of P and N. Let S be a set of closed intervals of the real line and f a surjective mapping from $V(G)$ to S. We say that (S, f) is a *probe interval representation* of G with respect to (P, N) provided $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \emptyset$ and at least one of u, v lies in P. The graph G is a *probe interval graph* with respect to (P, N) whenever it has a probe interval representation with respect to (P, N). Probe interval graphs are introduced in physical mapping and sequencing of DNA and have received a wide study [18–23]. Especially, we mention that a polynomial time recognition algorithm for probe interval graphs can be found in [18].

3. Main result

To establish our main characterization results, we have to prepare some results on the so-called inverse problem [6, p. 32] for the edge clique graph operator and interval graphs.

Lemma 5. *A graph is both an interval graph and an edge clique graph if and only if it is the edge clique graph of an interval graph.*

Proof. The backward direction is straightforward from Proposition 1 and Theorem 4. So we turn to the forward implication.

Suppose $G = Ke(H)$ is an interval graph. Then, Theorem 4 says that $\mathcal{C}(G)$ has a consecutive ordering $\pi = (C_1, \ldots, C_s)$. Without loss of generality, we may assume that H has no isolated vertex, namely all its maximal cliques are nontrivial. By Proposition 1, each $C_i \in \mathcal{C}(G)$ corresponds to a $Q_i \in \mathcal{C}(H)$ and these Q_i's enumerate all elements of $\mathcal{C}(H)$. If H is itself not an interval graph, then we can locate a vertex $v \in V(H)$ such that the maximal cliques containing v appear in $t_v = t > 1$ segments of consecutive cliques, say cliques among $\cup_{i=1}^{t} S_i$, where $S_i = \{Q_{k_i}, \ldots, Q_{k_i+\ell_i}\}$, $1 \le k_1 < k_1+\ell_1+1 < k_2 < k_2+\ell_2+1 < \cdots < k_t < k_t+\ell_t+1 \le s+1$. It is not hard to see that we can choose t new vertices $v_1, \ldots, v_t$ and construct a new graph H' such that $V(H') = (V(H) \setminus \{v\}) \cup \{v_1, \ldots, v_t\}$ and $E(H') = (E(H) \setminus \{vw : vw \in E(H)\}) \cup (\cup_{i=1}^{t}\{v_i w : \ vw \in E(H), v, w \in \cup_{Q \in S_i} Q\})$.

Under the most obvious correspondence between $E(H)$ and $E(H')$, we may identify $V(Ke(H))$ with $V(Ke(H'))$. We proceed to show that $E(Ke(H)) = E(Ke(H'))$, which will lead to the conclusion that the above vertex splitting operation does not affect the edge clique graph, that is, $Ke(H') = Ke(H) = G$. This will follow from the fact that

$$\mathcal{C}(H') = \{Q_i' : i = 1, \ldots, s\}, \tag{1}$$

where $Q_i' = (Q_i \setminus \{v\}) \cup \{v_j\}$ if Q_i lies in S_j and $Q_i' = Q_i$ otherwise. To establish Eq. (1), it suffices to check that there is no clique in G which contains vertices v, u, w whenever v, u appear in a clique in S_i and v, w appear in a clique in S_j for $1 \le i < j \le t$. This must be true, because, as π is a consecutive clique arrangement, only those C_m with $m < k_i + \ell_i + 1$ can contain both v and u and only those C_m with $m > k_i + \ell_i + 1$ can contain both v and w.

Clearly, there is one less vertex w with $t_w > 1$ after replacing H by H' and the ordering $(Q_1, \ldots, Q_s)$ by the ordering $(Q_1', \ldots, Q_s')$. Therefore, according to Theorem 4, by continuing this splitting vertex process we will finally come to an interval graph whose edge clique graph is G, finishing the proof. □

An interval graph is said to be *good* provided the size of the intersection of any two different maximal cliques of it does not equal to one.

Lemma 6. *A graph is the edge clique graph of an interval graph if and only if it is the edge clique graph of a good interval graph.*

Proof. Take an interval graph H. Without any loss of generality, let us assume that S is a set of intervals whose endpoints are pairwise distinct and H is the intersection graph of S. Let $a_1 < a_2 < \cdots < a_{2|S|}$ be the set of endpoints of intervals in S. Call a_i a left point if it is the left endpoint of some interval in S and a right point otherwise. A segment $[a_i, a_{i+1}]$ is nice whenever a_i is a left point and a_{i+1} a right point. Due to the Helly property for interval graphs [14, Exercise 2.1.72], we know that the set of nice segments are in one to one correspondence with the set of maximal cliques of H, a nice segment $[a_i, a_{i+1}]$ corresponding to the clique consisting of all vertices whose intervals contain $[a_i, a_{i+1}]$.

The fact that H is not good means that there is an interval $[b, c] \in S$ and two nice segments $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$, $i + 1 < j$, such that $[b, c]$ is the unique element from S that covers both $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$. Clearly, there is $i + 1 \le k \le j - 1$ satisfying a_k is right and a_{k+1} is left. Replacing $[b, c]$ by two intervals $[b, \frac{2a_k+a_{k+1}}{3}]$ and $[\frac{a_k+2a_{k+1}}{3}, c]$ we get a new set of intervals whose intersection graph shares the same edge clique graph with H. Proceeding with this interval splitting process whenever possible, we will terminate at a good interval graph whose edge clique graph is $Ke(H)$, as wanted. □

Define on the set of triangular numbers a function θ as follows. Let $\theta(0) = 0$ and $\theta(m) = n > 1$ for any positive triangular number $m = \frac{n(n-1)}{2}$. We come to a necessary condition for an interval graph to be an edge clique graph.

Lemma 7. *Let G be an interval graph with a consecutive ordering $\pi = (C_1, \ldots, C_s)$ of $\mathcal{C}(G)$ and put $\alpha_{i,j} = \theta(|C_i \cap C_j|)$ for $s \ge j \ge i \ge 1$. If G is an edge clique graph, then for any $1 \le i \le j \le k \le \ell \le s$ we have*

(A2) $\alpha_{i,k} + \alpha_{j,\ell} \le \alpha_{i,\ell} + \alpha_{j,k}$.

Proof. By Lemmas 5 and 6 we may assume that $G = Ke(H)$ for some good interval graph H, and by Proposition 2 the four terms in Condition (A2) are all well-defined. We only need to show that condition (A2) holds for G. For any $C \in \mathcal{C}(G)$, let C^H be the corresponding element of $\mathcal{C}(H)$, as mentioned in Proposition 1. Since H is good, it holds for any $p \le q$ that $\alpha_{p,q} = |C_p^H \cap C_q^H|$ and hence (A2) becomes

$$|C_i^H \cap C_k^H| + |C_j^H \cap C_\ell^H| \le |C_i^H \cap C_\ell^H| + |C_j^H \cap C_k^H|. \tag{2}$$

Furthermore, from the proofs of Lemmas 5 and 6, we know that we can require that $\pi^H = (C_1^H, \ldots, C_s^H)$ is a consecutive ordering of $\mathcal{C}(H)$. For $v \in V(H)$ and a clique Q of H, let $I_v(Q) = 1$ if $v \in Q$ and let $I_v(Q) = 0$ otherwise. By now, to prove (A2), namely Eq. (2), it is enough to show that if v occurs consecutively in an interval, say $\mathcal{F}$, among the ordering of four cliques (A, B, C, D), then

$$I_v(A \cap C) + I_v(B \cap D) \le I_v(A \cap D) + I_v(B \cap C). \tag{3}$$

Note that, when $\mathcal{F} = (A, B, C)$ or (B, C, D), Eq. (3) is just $1 = 1$; when $\mathcal{F} = (A, B, C, D)$, Eq. (3) becomes $2 = 2$; when $\mathcal{F} = (B, C)$, Eq. (3) turns out to be $0 \le 1$; and in all other cases, Eq. (3) is simply the trivial relation $0 = 0$. This completes the proof of the lemma. □

From Theorem 4, we can easily check that the graphs considered in Example 3 are all interval graphs. The ensuing theorem says that (A1) together with (A2) is a necessary and sufficient condition for an interval graph to be an edge clique graph, hence providing an easy understanding of the observations in Example 3.

Theorem 8. *Let G be an interval graph with a consecutive ordering $\pi = (C_1, \ldots, C_s)$ of $\mathcal{C}(G)$. Then G is an edge clique graph if and only if, keeping the notation of Lemma 7, it satisfies the following conditions:*

(A1′) *the intersection of any two not necessarily distinct maximal cliques is a triangular clique;*
(A2′) *if $j < \ell$, $C_j \cap C_\ell \ne \emptyset$, then $\alpha_{j-1,\ell-1} + \alpha_{j,\ell} \le \alpha_{j-1,\ell} + \alpha_{j,\ell-1}$.*

(Note that without condition (A1′) *even the notation $\alpha_{i,j}$ will make no sense.)*

Proof. The necessity part follows from Proposition 2 and Lemma 7.

For the reverse direction, we carry out a proof by induction on s. The assertion is trivially true when $s = 1$. Consider now the case of $s > 1$ under the assumption that the result holds for smaller s. Let G' be the graph with vertex set $\cup_{i=1}^{s-1} C_i$ and $uv \in E(G')$ if and only if $u, v \in C_i$ for some $i = 1, \ldots, s-1$. Since π is a consecutive ordering of $\mathcal{C}(G)$, we can find that $\pi' = (C_1, \ldots, C_{s-1})$ is a consecutive ordering of $\mathcal{C}(G')$. Thus, by induction hypothesis, there is a graph H' such that $G' = Ke(H')$ and the maximal cliques of H' are C_i^H, $i = 1, \ldots, s-1$, where C_i^H corresponds to $C_i \in \mathcal{C}(G)$ in the sense of Proposition 1. From the proof of Lemma 5, we assume that $(C_1^H, \ldots, C_{s-1}^H)$ is a consecutive ordering of $\mathcal{C}(H')$.

Let $i = \min\{t : C_t \cap C_s \ne \emptyset\}$. From (A2′) and that π is consecutive, we deduce that

$$\alpha_{j,s} - \alpha_{j-1,s} \le \alpha_{j,s-1} - \alpha_{j-1,s-1}, \quad j = i, i+1, \ldots, s-1. \tag{4}$$

Consequently, we can take $\alpha_{j,s} - \alpha_{j-1,s}$ vertices from $(C_j^H \setminus C_{j-1}^H) \cap C_{s-1}^H$, $j = i, i+1, \ldots, s-1$. Denote the union of these vertices by O. Let H be the graph obtained from H' by adding a set N of $\alpha_{s,s} - \alpha_{s-1,s} > 0$ new vertices and adding all edges among these vertices and all edges between these vertices and those in O. Because of Eq. (4), we conclude that $\mathcal{C}(H) = \{C_1^H, \ldots, C_{s-1}^H, C_s^H\}$, where $C_s^H = O \cup N$. In addition, owing to the fact that $(C_1^H, \ldots, C_{s-1}^H)$ is a consecutive ordering of $\mathcal{C}(H')$, we can check that $|C_s^H \cap C_t^H| = \alpha_{t,s}$ for any $t < s$ and then verify that $G = Ke(H)$, ending the proof. □

Theorem 9. *Let G be an edge clique graph. Then G is a probe interval graph if and only if G is an interval graph.*

Proof. It is trivial that every interval graph is a probe interval graph.

To go the other way, let the given edge clique graph G be a probe interval graph with respect to (P, N). There is no loss of generality in assuming that G has no isolated vertices. In view of Proposition 2, this excludes the possibility that there exist $C_1, C_2 \in \mathcal{C}(G)$ such that $|C_1 \setminus C_2| = 1$. Henceforth, with a little thought, it is not hard to argue that

(B) for any $C_1, C_2 \in \mathcal{C}(G)$ which contain $v_1, v_2 \in N$, respectively, $C_1 \cap P$ and $C_2 \cap P$ are different members of $\mathcal{C}(G(P))$.

Take a probe interval representation (S, f) of G with respect to (P, N). We may assume that the endpoints of the intervals in S are pairwise distinct. Let $a_1 < a_2 < \cdots < a_{2|P|}$ be the set of endpoints of those intervals corresponding to the vertices of P. Since $G(P)$ is an interval graph, as shown in the proof of Lemma 6 we know that the set of nice segments with respect to the interval representation $\{f(v) : v \in P\}$ of $G(P)$ are in one to one correspondence with $\mathcal{C}(G(P))$, a nice segment $[a_i, a_{i+1}]$ corresponding to the clique consisting of all vertices from P whose intervals contain $[a_i, a_{i+1}]$.

For any $v \in N$, define $\mathcal{S}_v$ to be the set of nice segments with respect to the interval representation $\{f(u) : u \in P\}$ which have nonempty intersection with $f(v)$. Let $T_v = \cup_{I \in \mathcal{S}_v} I$, $L_v = \min\{x \in T_v\}$ and $R_v = \max\{x \in T_v\}$. By the preceding claim (B), it follows that for $u \neq v \in N$, $T_u \cap T_v = \emptyset$. But both $f(u)$ and $f(v)$ are intervals, which implies that $[L_v, R_v] \cap [L_u, R_u] = \emptyset$.

We are ready to establish an interval representation for G. We just associate with each $v \in P$ the interval $f(v)$ and associate with each $v \in P$ the interval $[L_v, R_v]$. It is a simple matter that the existence of these intervals gives rise to what we want. □

For any graph G, there is an $O(|V(G)|^2)$ time algorithm which either produces a consecutive ordering of $\mathcal{C}(G)$, hence indicating that G is an interval graph, or else outputs the answer that no such ordering exists [13, p. 175] [24]. Note that an interval graph G has at most $|V(G)|$ maximal cliques [25]. So, utilizing any consecutive ordering π of $\mathcal{C}(G)$ we can check whether or not G satisfies conditions (A1′) and (A2′) within $O(|V(G)|^3)$ time. To sum up, Theorems 8 and 9 allow us to assert that $O(|V(G)|^3)$ time is enough to test whether a given interval graph or a given probe interval graph is an edge clique graph.

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