Attractors for a second order nonautonomous lattice dynamical system with nonlinear damping

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Abstract

In this Letter, the existence of compact pullback attractors is considered for a second order nonautonomous lattice dynamical system with nonlinear damping arising from spatial discretization of wave equations in $\mathbb{R}^k$. And the finite-dimensional approximations of the attractors are studied. Finally, an upper bound of fractal dimension of the attractors is obtained.

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1. Introduction

Lattice dynamical systems (LDSes) are infinite systems of ordinary differential equations (lattice ODEs) or of difference equations [2]. Lattice systems arise in many applications, for example, in chemical reaction theory [11,16], image processing and pattern recognition [10,15,19], material science [6,13], biology [5,17,18], electrical engineering, laser systems [12], etc. They possess their own form, but in some cases, they arise as spatial discretizations of partial differential equations.

Let $k \in \mathbb{N}$ be a fixed positive integer. Denote by

$$\ell^2 = \left\{ u \mid u = (u_i)_{i_1,i_2,\ldots,i_k} \in \mathbb{Z}^k, u_i \in \mathbb{R}, \sum_{i \in \mathbb{Z}^k} |u_i|^2 < +\infty \right\},$$

where $\mathbb{Z}$ is the set of integers. Define a linear operator $A$ acting on $\ell^2$ in the following way: for any $u = (u_i)_{i_1,i_2,\ldots,i_k} \in \ell^2$, $i = (i_1,i_2,\ldots,i_k) \in \mathbb{Z}^k$,

$$(Au)_{i_1,i_2,\ldots,i_k} = 2ku_{i_1-1,i_2,\ldots,i_k} - u_{i_1,i_2-1,\ldots,i_k} - u_{i_1,i_2,\ldots,i_k-1} - \cdots - u_{i_1,i_2,\ldots,i_k+1} - u_{i_1+1,i_2,\ldots,i_k} - u_{i_1,i_2+1,\ldots,i_k} - \cdots - u_{i_1,i_2,\ldots,i_k+1}.$$

In this Letter, we will consider the following second order damped nonautonomous lattice dynamical system:

$$\begin{cases}
\ddot{u} + h'(\dot{u}) + Au + \lambda u + \tilde{g}(u,t) = \tilde{q}(t), & t \geq \tau, \quad \tau \in \mathbb{R}, \\
u(\tau) = (u_{i_0})_{i_1,i_2,\ldots,i_k} = u_0, & \dot{u}(\tau) = (u_{i_0})_{i_1,i_2,\ldots,i_k} = u_{i_0}.
\end{cases}$$

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where $\lambda > 0$, $u = (u_i)_{i \in \mathbb{Z}^d}$, $\dot{u} = (\dot{u}_i)_{i \in \mathbb{Z}^d}$ and $\ddot{u} = (\ddot{u}_i)_{i \in \mathbb{Z}^d}$ denote the first and the second order time derivatives respectively, and $h(\dot{u}) = (h(\dot{u}_i))_{i \in \mathbb{Z}^d}$, $\ddot{q}(u, t) = (g(q_i(t)))_{i \in \mathbb{Z}^d} \in \mathcal{C}_b(\mathbb{R}, \ell^2)$, the space of bounded continuous functions (operators) from $\mathbb{R}$ into $\ell^2$.

The equation in (1) can be regarded as a discrete analogue of the following nonautonomous wave equation in $\mathbb{R}^k$

$$\partial_t u + h(\partial_t u) - \Delta u + \lambda u + g(u, t) = q(x, t)$$

(2)

which arises in various areas of mathematical physics, where $\Delta$ is the Laplace operator.

Recently, various properties of the solutions to lattice dynamical systems have been studied by many authors (see [1,3,4] and the references therein). Bates et al. in [4] proved the existence of a global attractor for the first-order autonomous lattice dynamical systems and investigated the approximation of the attractor by the corresponding ones of finite-dimensional ordinary differential equations. Zhou [21–23] applied these results to study the first order and the second order dissipative autonomous lattice dynamical systems and investigated the approximation of the attractor by the corresponding ones of finite-dimensional ordinary differential equations.

Definition 2. A family $\mathcal{A} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ of nonempty subsets of $\mathbb{R}$ is called a pullback absorbent family if for any $\tau \in \mathbb{R}$ and $u_0 \in \mathcal{A}(\tau)$, there exists a unique weak solution $u(t) = u(t; \tau, u_0)$ satisfying

$$\left\{ \begin{array}{l}
\frac{du}{dt} = F(u, t), \quad t > \tau, \\
u(\tau) = u_0
\end{array} \right.$$  

(3)

and $u \in C([\tau, T]; \mathcal{H}), F(u(t), t) \in L^1(\tau, T; \mathcal{H}^*)$, for any fixed $T > \tau$. Define the solution operator $U(t, \tau) (t \geq \tau)$ as $U(t, \tau)u_0 = u(t; \tau, u_0)$, $\tau \leq t$, $u_0 \in \mathcal{A}(\tau)$. We know that the map family $\{U(t, \tau)\}_{t \geq \tau}$ is a process determined by (3). In this Letter, we shall always denote $\text{dist}$ the Hausdorff semidistance of sets as follows:

$$\text{dist}(B, C) = \sup_{x \in B} \inf_{y \in C} \|x - y\|$$.  

for any $B, C \subset \mathcal{H}$.

Denote $\mathcal{P}(\mathcal{H})$ the family of all nonempty subsets of $\mathcal{H}$, and $\mathfrak{T}$ the class of all families $\mathcal{D} = \{D(t), t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{H})$. For a given nonempty subclass $\mathcal{D} \subset \mathfrak{T}$, we introduce:

Definition 1. The process $U(t, \tau)$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any fixed $t$, and $\mathcal{D} = \{D(t), t \in \mathbb{R}\} \in \mathcal{D}$, and any sequences $t_n \to +\infty$, $x_n \in D(-t_n)$, the sequence $\{U(t_n, -t_n)x_n\}_{n \geq 1}$ possesses a convergent subsequence.

Definition 2. A family $\mathcal{B} = \{B(t), t \in \mathbb{R}\} \subset \mathfrak{T}$ is said to be pullback $\mathcal{D}$-absorbing if for any $\mathcal{D} = \{D(t), t \in \mathbb{R}\} \in \mathcal{D}$, there exists $T_0(\mathcal{D})$ such that

$$U(t, \tau)D(t) \subset B(t), \quad \text{for all } \tau \leq t - T_0(\mathcal{D}).$$

Definition 3. A family $\mathcal{C} = \{C(t), t \in \mathbb{R}\} \subset \mathfrak{T}$ is said to be pullback $\mathcal{D}$-attracting if for all $\mathcal{D} = \{D(t), t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \text{dist}(U(t, \tau)D(t), C(t)) = 0.$$
\textbf{Definition 4.} Let $\hat{A} = \{A(t)\}_{t \in \mathbb{R} \in \mathcal{R}}$ be a time dependent family of compact sets. $\hat{A}$ is said to be a (global) pullback $\mathcal{D}$-attractor ($A(t)$ is a pullback attractor at the time $t$) if it satisfies

1. $\hat{A}$ is pullback $\mathcal{D}$-attracting,
2. $\hat{A}$ is invariant, i.e.

$$U(t, \tau)A(\tau) = A(t), \quad \text{for all } t \geq \tau.$$ 

\textbf{Theorem 1.} Suppose the process $U(t, \tau)$ determined by (3) is pullback $\mathcal{D}$-asymptotically compact, and there exists $\hat{B} = \{B(t), \quad t \in \mathbb{R}\}_{t \in \mathcal{D}}$ which is pullback $\mathcal{D}$-absorbing. Then, the family $\hat{A}$ is defined by

$$A(t) = \bigcup_{s \geq 0} \bigcup_{\tau \geq s} U(t, -\tau)B(-\tau)$$

is a global pullback $\mathcal{D}$-attractor which is minimal in the sense that if $\hat{C} \in \mathcal{R}$ is a family such that $C(t)$ is closed and $\lim_{t \to -\infty} \text{dist}(U(t, \tau)B(\tau), C(t)) = 0$, then $\hat{A}(t) \subset C(t)$ here.

As the proof of this theorem repeats that of Theorem 7 in [8], with slight modifications, we will omit it here.

For any $u = (u_i)_{i \in \mathbb{Z}_k} \in \ell^2$, $i = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}_k^2$, we shall always denote by $\|i\| = \max_{1 \leq j \leq k} |i_j|$ in the following discussion. For any $u = (u_i)_{i \in \mathbb{Z}_k} \in \ell^2, i = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}_k$, define the operators $B_j, \tilde{B}_j$ and $A_j, j \in \{1, 2, \ldots, k\}$ from $\ell^2$ to itself as follows:

$$\ell_j - \hat{B}_ju_i = u(i_1, i_2, \ldots, i_k) - u(i_1, i_2, \ldots, i_k), \quad \ell_j - \hat{B}_ju_i = u(i_1, i_2, \ldots, i_k) - u(i_1, i_2, \ldots, i_k),$$

$$\ell_j - \hat{B}_ju_i = 2u(i_1, i_2, \ldots, i_k) - u(i_1, i_2, \ldots, i_k) - u(i_1, i_2, \ldots, i_k). \quad (4)$$

Obviously, we have

$$A = A_1 + A_2 + \cdots + A_k, \quad A_j = B_j \tilde{B}_j = \tilde{B}_j B_j, \quad j = 1, 2, \ldots, k. \quad (5)$$

For any $u = (u_i)_{i \in \mathbb{Z}_k}, v = (v_i)_{i \in \mathbb{Z}_k} \in \ell^2$, we define their inner products and norms as follows

$$(u, v) = \sum_{i \in \mathbb{Z}_k} u_i v_i, \quad \|u\| = \left( \sum_{i \in \mathbb{Z}_k} |u_i|^2 \right)^{1/2},$$

$$(u, v)_k = \sum_{j=1}^k (B_j u, B_j v) + \lambda(u, v), \quad \|u\|_k = \sum_{j=1}^k \|B_j u\|^2 + \|u\|.$$

It is obvious that for any $u = (u_i)_{i \in \mathbb{Z}_k} \in \ell^2$,

$$\|B_j u\|^2 \leq 4\|u\|^2, \quad \lambda(u, v)_k \leq \|u\|_k^2 \leq (4k + \lambda)\|u\|^2.$$ \quad (7)

So, the norms $\| \cdot \|$ and $\| \cdot \|_k$ are equivalent to each other. By using (5), we obtain for any $u = (u_i)_{i \in \mathbb{Z}_k}, v = (v_i)_{i \in \mathbb{Z}_k} \in \ell^2$

$$(Au, v) = \sum_{j=1}^k (B_j u, B_j v) = \sum_{j=1}^k (\tilde{B}_j u, \tilde{B}_j v). \quad (8)$$

Denote by $\ell^2, \ell^2_k$ the spaces with the inner products and norms in (6), respectively,

$$\ell^2 = \ell^2(\ell, \ell, \| \cdot \|), \quad \ell^2_k = \ell^2(\ell, \ell, \| \cdot \|_k)$$

then $\ell^2$ is a Hilbert space. By (7), $\ell^2_k$ is also a Hilbert space.

Let $E = \ell^2_k \times \ell^2$ endowed with the inner product and norm as: for any $\varphi_j = (u^{(j)}_i, v^{(j)}_i) = ((u^{(j)}_i), (v^{(j)}_i))_{i \in \mathbb{Z}_k} \in E, j = 1, 2$,

$$(\varphi_1, \varphi_2)_E = (u^{(1)}_i, v^{(1)}_i) + (v^{(1)}_i, v^{(1)}_i), \quad \|\varphi\|_E^2 = \|u\|_k^2 + \|v\|^2, \quad \forall \varphi = (u, v) = ((u_i), (v_i))_{i \in \mathbb{Z}_k} \in E.$$

For convenience, we often denote $\|\varphi\|_E^2 = \sum_{i \in \mathbb{Z}_k} \|\varphi_i\|^2$ where $\varphi = (\varphi_i)_{i \in \mathbb{Z}_k} \in E$.

Let $G(s, t) = \int_0^t g(r, t)dr$. For the problem (1), we always assume that there exist four positive constants $c_1, c_2, \alpha_1$ and $\alpha_2$ such that

(H1) $g(0, 0) \equiv 0, \sg s, t \geq c_1 G(s, t) \geq 0, \forall s, t \in \mathbb{R}.$

(H2) $G'(s, t) \leq c_2 G(s, t), \forall s, t \in \mathbb{R}, (0 < c_2 \leq \frac{c_1}{2})$. \varepsilon is as (9) below).
Theorem 2. There exists a nondecreasing continuous function $K(r) : \mathbb{R}^+ \to \mathbb{R}^+$ with $K(0) = 0$ such that
\[
\sup_{t \in \mathbb{R}} \sup_{|s| \leq t} |g'_j(s, t)| \leq K(r),
\]
where $\mathbb{R}^+ = [0, +\infty)$.

(H4) $h(0) = 0$, $\alpha_1 \leq h'(s) \leq \alpha_2$, $\forall s \in \mathbb{R}$.

Then (1) can be written as
\[
\begin{aligned}
\dot{S}(t, \tau) &\in \mathbb{R}^2 \\
&= S(t, \tau)B + \frac{\varepsilon}{2} \left| \int_0^1 \dot{h}(v(t) - \varepsilon u(t)) - \varepsilon (v(t) - \varepsilon u(t)) + A_0 u(t) + \lambda u(t) \right| dt,
\end{aligned}
\]
where
\[
\begin{aligned}
H(\varphi) &= \left( \frac{\varepsilon}{2} |v(t) - \varepsilon u(t) - \varepsilon (v(t) - \varepsilon u(t)) + A_0 u(t) + \lambda u(t)| \right), \\
F(\varphi, t) &= \left( \frac{0}{-\bar{g}(u(t)) \bar{q}(t)} \right).
\end{aligned}
\]
From (H3) and (H4), it is easily checked that $-H(\varphi) + F(\varphi, t)$ is locally Lipschitz continuous in $\varphi$ from $E$ to $E$. By the standard theory of differential equations, we obtain the existence and uniqueness of a local solution to the system (10) with initial data $\varphi(\tau) = \varphi_0 \in E$.

Theorem 2. For any initial data $\varphi_0 = (u_0, v_0)^T \in E$, there exists a unique local solution $\varphi(t) = (u(t), v(t))^T$ of (10) with $\varphi(\tau) = \varphi_0$ such that $\varphi(t) \in C^1([\tau, \tau + T], E)$ for any $T > 0$. Furthermore, $\varphi(t) = \varphi(t, \varphi_0)$ is continuous on $\varphi_0$ in $E$.

In fact, it will be showed in Lemma 2 below that the local $\varphi(t)$ of (10) exists globally, that is, $\varphi(t) \in C^1([\tau, +\infty), E)$. It implies that the map
\[
S(t, \tau) : \varphi(\tau) = \varphi_0 \mapsto \varphi(t), \quad E \mapsto E
\]
(11)
generates a continuous process from $E$ to itself.

We present a positivity of the nonlinear operator $H(\varphi)$, which will play an important role later.

Lemma 1. For any $\varphi = (u, v)^T \in E$,
\[
(H(\varphi), \varphi)_E \geq \frac{\varepsilon}{2} \|v\|^2 + \frac{\alpha_1}{2} \|v\|^2.
\]

Proof. By definition, we have
\[
(H(\varphi), \varphi)_E = \varepsilon \|Bu\|^2 + \lambda \|u\|^2 + \varepsilon^2 (u, v) + (\bar{h}(v - \varepsilon u), v) - \varepsilon \|v\|^2
\]
and by using the mean value theorem and (H4),
\[
\epsilon^2 (u, v) + (\bar{h}(v - \varepsilon u), v) = \epsilon^2 (u, v) + \sum_{i \in \mathbb{Z}^2} \bar{h}'(\tau_i (v_i - \varepsilon u_i)) (v_i - \varepsilon u_i) v_i \geq \alpha_1 \|v\|^2 - \varepsilon (\alpha_2 - \epsilon) \|u\| \cdot \|v\|, \quad \tau_i \in (0, 1)
\]
follows
\[
(H(\varphi), \varphi)_E - \frac{\varepsilon}{2} \|v\|^2 - \frac{\alpha_1}{2} \|v\|^2 \geq \frac{\varepsilon}{2} \|Bu\|^2 + \lambda \|u\|^2 + \left( \frac{\alpha_1}{2} - \frac{3\varepsilon}{2} \right) \|v\|^2 - \frac{\varepsilon \alpha_1}{\sqrt{\lambda}} \|Bu\|^2 + \lambda \|u\|^2)^{1/2} \|v\|
\]
which implies the conclusion by noting
\[
\epsilon (\alpha_1 - 3\varepsilon) = \frac{\epsilon^2 \alpha_2}{\lambda}. \quad \Box
\]

Lemma 2. There exists an open ball $O = O(0, r_0)$ of $E$, centered at $0$ with radius $r_0$, such that for any bounded set $B$ of $E$, there exists $T_0(B) > 0$ (independent of $t$) such that
\[
S(t, \tau) B \subset O, \quad \forall \tau \leq t - T_0(B), \text{ fixed } t
\]
which implies that the process $S(t, \tau)$ possesses a bounded absorbing set in $E$. 

Proof. Let \( \varphi(t) = (u(t), v(t))^T \) be the solution of (10) with the initial data \( \varphi_0 = (u_0, v_0)^T \), where \( v(t) = \bar{u} + \varepsilon u \). Taking the inner product \((\cdot, \cdot)\) on both sides of the system (10) with \( \varphi(t) \), it follows

\[
\frac{1}{2} \frac{d}{dt} \| \varphi(t) \|_E^2 + (H(\varphi(t)), \varphi(t)) + (\bar{g}(u(t), \bar{u}) + \varepsilon \tilde{g}(u(t), u), \varphi(t)) = (\tilde{\varphi}, v).
\]  

(12)

By (H1) and (H2), we have

\[
e(\bar{g}(u,t), u) \geq \varepsilon c_1 \sum_{i \in \mathbb{Z}^k} G(u_i,t) = \varepsilon c_1 G(u,t),
\]  

(13)

\[
(\bar{g}(u,t), \dot{u}) = \sum_{i \in \mathbb{Z}^k} \left( g(u_i,t), \dot{u}_i \right) = \frac{d}{dt} \sum_{i \in \mathbb{Z}^k} G(u_i,t) - \sum_{i \in \mathbb{Z}^k} G_t(u_i,t) \geq \frac{d}{dt} \tilde{G}(u,t) - \frac{\varepsilon c_1}{2} \tilde{G}(u,t)
\]  

(14)

where \( \tilde{G}(u,t) = \sum_{i \in \mathbb{Z}^k} G(u_i,t) \).

Let \( \kappa = \min\{1, \frac{1}{2}\} \) and

\[
y(t) = \| \varphi(t) \|_E^2 + 2\tilde{G}(u,t) \geq \| \varphi(t) \|_E^2.
\]  

(15)

By Lemma 1, (13)–(15) and the Cauchy inequality, we easily obtain

\[
\frac{d}{dt} y + \varepsilon \kappa y \leq \frac{1}{\varepsilon c_1} \sup_{t \in \mathbb{R}} \tilde{q}(t) \| \varphi(t) \|_E^2.
\]  

(16)

By the Gronwall inequality and (15), we have

\[
\| \varphi(t) \|_E^2 \leq y(t) e^{-\varepsilon \kappa (t - \tau)} + \frac{1}{\varepsilon \kappa c_1} \sup_{t \in \mathbb{R}} \tilde{q}(t) \| \varphi(t) \|_E^2 (1 - e^{-\varepsilon \kappa (t - \tau)}).
\]  

(17)

By (H1) and (H3), it follows

\[
\sum_{i \in \mathbb{Z}^k} G(u_{i0},t) \leq \frac{1}{c_1} \sum_{i \in \mathbb{Z}^k} \left( u_{i0}, g(u_{i0},t) \right) \leq \frac{1}{c_1} \sup_{t \in \mathbb{R}} \| \varphi(t) \|_E^2,
\]  

where \( \| u_0 \| \leq r \). So, from (17) we get

\[
\| \varphi(t) \|_E^2 \leq \left( 1 + \frac{2}{c_1 \lambda} \right) r^2 e^{-\varepsilon \kappa (t - \tau)} + \frac{1}{\varepsilon \kappa c_1} \sup_{t \in \mathbb{R}} \| \tilde{q}(t) \|_E^2,
\]  

(18)

where \( \| \varphi_0 \|_E \leq r \). For any bounded set \( B \) of \( E \), \( B \subseteq O(0,r(B)) \) where \( r(B) = \sup_{y \in B} \| y \|_E \), there exists \( T_0(B) \geq 0 \) such that for any \( s \geq T_0(B) \)

\[
\left( 1 + \frac{2}{c_1 \lambda} \right) r^2 e^{-\varepsilon \kappa s} \leq \frac{1}{2} r^2_0,
\]  

where \( r^2_0 = \frac{2}{\varepsilon \kappa c_1} \sup_{t \in \mathbb{R}} \| \tilde{q}(t) \|_E^2 \). By (18), the assertion is ensured. \( \square \)

By (18), we easily obtain

**Corollary 1.** There exists \( T_0 = T_0(r_0) \geq 0 \) such that \( S(t, \tau)O \subseteq O \) for any \( \tau \leq t - T_0 \), fixed \( t \).

**Notation.** We shall always see \( O, r_0 \) and \( T_0 \) as given in Lemma 2 and Corollary 1 in the following discussion.

By Lemma 2, we know that the trajectory starting from any bounded set finally enters into \( O(0,r_0) \). From now on, we denote by \( \mathcal{D} \) the class of all families \( \mathcal{D} = \{ D(t), t \in \mathbb{R} \} \subseteq \mathcal{P}(E) \) such that \( D(t) \subseteq O(0,r_0) \), where \( \mathcal{P}(E) \) is the class of all nonempty subsets of \( E \) and \( O(0,r_0) \) is the closed ball of \( O(0,r_0) \) in \( E \). To obtain the existence of pullback \( \mathcal{D} \)-attractor for the process \( \{ S(t, \tau) \}_{t \geq \tau} \) associated with (10) in \( E \), we need to prove the pullback asymptotic compactness of \( \{ S(t, \tau) \}_{t \geq \tau} \). For this purpose, we first give an important lemma as follows:

**Lemma 3.** Let \( \varphi(t) = (u(t), v(t))^T \) be the solution of (10) with initial data \( \varphi_0 = (u_0, v_0)^T \in O \), where \( v(t) = \bar{u} + \varepsilon u \). Then for any \( \varepsilon > 0 \), there exist \( T_\varepsilon \geq T_0 \) and \( N_\varepsilon > 0 \) such that

\[
\sum_{i \in \mathbb{Z}^k} \sum_{j=1}^k \left( |B_j u(t_i) |^2 + \lambda |u_i(t) |^2 + |v_i(t) |^2 \right) \leq \varepsilon
\]  

(19)

for any \( \tau \leq t - T_\varepsilon \), fixed \( t \).
**Proof.** Choosing a smooth function \( \theta(s) \in C^1(\mathbb{R}^+, \mathbb{R}) \) satisfying

\[
\theta(s) = 0, \quad 0 \leq s \leq 1,
\]

\[
0 \leq \theta(s) \leq 1, \quad 1 \leq s \leq 2,
\]

\[
\theta(s) = 1, \quad s \geq 2
\]

and there exists a constant \( C_0 \) such that \( |\theta'(s)| \leq C_0 \) for \( s \in \mathbb{R}^+ \).

Let \( M \) be some fixed integer. For any \( (x_i)_{i \in \mathbb{Z}^k}, (y_i)_{i \in \mathbb{Z}^k} \in \ell^2 \). Set \( \tilde{x}_i = \theta\left(\frac{\|x\|}{M}\right)x_i, \tilde{x} = (\tilde{x}_i)_{i \in \mathbb{Z}^k} \) and \( \tilde{y}_i = \theta\left(\frac{\|y\|}{M}\right)y_i, \tilde{y} = (\tilde{y}_i)_{i \in \mathbb{Z}^k} \). It is easy to check that

\[
|\langle Ax, \tilde{y} \rangle - \langle A\tilde{x}, y \rangle| \leq \frac{kC_0}{\sqrt{\lambda M}}(\lambda \|x\|^2 + \|y\|^2),
\]

\( (A\tilde{x}, \tilde{x}) = (Ax, \tilde{x}) \geq \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|x\|}{M}\right) \|B_j x_i\|^2 - \frac{2kC_0}{M} \|x\|^2. \]  

(21)

Let \( t \geq T_0 \). Denote with

\[
\tilde{u}_i = \theta\left(\frac{\|\tilde{u}\|}{M}\right)u_i, \quad \tilde{v}_i = \theta\left(\frac{\|\tilde{v}\|}{M}\right)v_i, \quad \text{for any } i \in \mathbb{Z}^k.
\]

Let \( \tilde{w}(t) = (\tilde{u}, \tilde{v})^T = (\tilde{u}_i, \tilde{v}_i)_{i \in \mathbb{Z}^k} \). Taking the inner product \( (\cdot , \cdot)_E \) on both sides of the system (10) with \( \tilde{w}(t) \), we have

\[
(\dot{\tilde{w}}, \tilde{w})_E + (H(\varphi), \dot{\tilde{w}})_E = (F(\varphi, t), \dot{\tilde{w}})_E.
\]

(22)

To estimate \( (H(\varphi), \dot{\tilde{w}})_E \), we first have

\[
e(\tilde{u}, \tilde{v}) \geq \varepsilon \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) \|B_j u_i\|^2 - \frac{2\varepsilon kC_0}{M} \|u\|^2,
\]

\[
\lambda e(u, \tilde{u}) - \lambda v, \tilde{u} + \lambda u, \tilde{v} + \varepsilon^2 (u, \tilde{v}) - \varepsilon (v, \tilde{v}) = \lambda \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) \|u_i(t)\|^2 - \varepsilon \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) \|v_i(t)\|^2 + \varepsilon^2 \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) u_i v_i,
\]

\[
-(A\tilde{u}, \tilde{u}) - (A\tilde{v}, \tilde{v}) \geq -\frac{kC_0}{\sqrt{\lambda M}} \|\tilde{w}\|^2,
\]

\[
(h(v - \varepsilon u), \tilde{v}) \geq \alpha_1 \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{v}\|}{M}\right) \|v_i(t)\|^2 - \varepsilon (\alpha_2 - \varepsilon) \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{v}\|}{M}\right) u_i v_i.
\]

By a proof similar to Lemma 1, the above inequalities and Corollary 1, we obtain

\[
(H(\varphi), \dot{\tilde{w}})_E = e(A\tilde{u}, \tilde{v}) - (A\tilde{v}, \tilde{u}) + \lambda e(u, \tilde{u}) - \lambda v, \tilde{u} + (A\tilde{u}, \tilde{v}) + \lambda u, \tilde{v} + \varepsilon^2 (u, \tilde{v}) - \varepsilon (v, \tilde{v}) + (h(v - \varepsilon u), \tilde{v})
\]

\[
\geq \varepsilon \frac{\chi}{2} + \alpha_1 \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) \|v_i(t)\|^2 - \frac{2kC_0 r_0}{\lambda M}(\sqrt{\chi} + \varepsilon)
\]

(23)

where

\[
\chi = \sum_{i \in \mathbb{Z}^k} \left(\theta\left(\frac{\|\tilde{u}\|}{M}\right) \|B_j u_i(t)\|^2 + \lambda |u_i(t)|^2 + |v_i(t)|^2\right) + 2 \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) G(u_i, t).
\]

Similar to the discussion in (14), we have

\[
(\dot{\tilde{w}}, \tilde{w})_E + (\tilde{g}(u, t), \tilde{v}) = (\dot{\tilde{w}}, \tilde{w})_E + (\tilde{g}(u, t), \tilde{u} + \varepsilon \tilde{u}) \geq \frac{1}{2} \frac{d}{dt} \chi + \frac{\varepsilon \alpha_1}{2} \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) G(u_i, t).
\]

(24)

Substituting (23), (24) and the obvious inequality

\[
(\tilde{q}(t), \tilde{v}) \leq \frac{1}{\alpha_1} \sup_{\tilde{u} \in \mathbb{R}^k} \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) |q_i(t)|^2 + \frac{\alpha_1}{2} \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) |v_i(t)|^2
\]

into (22), we get

\[
\frac{d}{dt} \chi + \varepsilon \kappa \chi \leq \frac{1}{\alpha_1} \sup_{\tilde{u} \in \mathbb{R}^k} \sum_{i \in \mathbb{Z}^k} \theta\left(\frac{\|\tilde{u}\|}{M}\right) |q_i(t)|^2 + \frac{4kC_0^3}{\lambda M}(\sqrt{\chi} + \varepsilon).
\]

(25)
For any $\epsilon > 0$, choose a suitable positive integer $N_\epsilon > 0$ such that
\[
\frac{1}{\alpha_1} \sup_{t \in [0, T]} \sum_{i \in \mathbb{Z}^k} \theta \left( \frac{2||i||}{N_\epsilon} \right) |q_i(t)|^2 + \frac{8kC_0r_0^2}{\lambda N_\epsilon} (\sqrt{x} + \epsilon) \leq \frac{\epsilon \delta K}{2}.
\]  
(26)

By using (26) and the Gronwall inequality in (25),
\[
\chi \leq \left( 1 + \frac{2}{c_1 \lambda} K \left( \frac{r_0}{\chi} \right) \right)^2 e^{-\epsilon K (t - t')} + \frac{\epsilon}{2} (1 - e^{-\epsilon K (t - t')}).
\]  
(27)

Choosing $T_\epsilon = \max\{(\log(1 + \frac{2}{c_1 \lambda} K (\frac{r_0}{\chi})) + 2 \log r_0 - \log \epsilon + \log 2)/\epsilon K, T_0\}$, from (27), we obtain
\[
\sum_{i \in \mathbb{Z}^k} \left( \theta \left( \frac{2||i||}{N_\epsilon} \right) \sum_{j=1}^k \left( |B_j u_i(t)|^2 + \lambda |u_i(t)|^2 + |v_j(t)|^2 \right) \right) \leq \epsilon,
\]  
\forall \tau \leq t - T_\epsilon

which implies
\[
\sum_{i \not\in N_{N_\epsilon}} \sum_{j=1}^k \left( |B_j u_i(t)|^2 + \lambda |u_i(t)|^2 + |v_j(t)|^2 \right) \leq \sum_{i \in \mathbb{Z}^k} \left( \theta \left( \frac{2||i||}{N_\epsilon} \right) \sum_{j=1}^k \left( |B_j u_i(t)|^2 + \lambda |u_i(t)|^2 + |v_j(t)|^2 \right) \right) \leq \epsilon
\]
for all $\tau \leq t - T_\epsilon$. \qed

**Lemma 4.** The process $\{S(t, \tau)\}_{\tau \geq t}$ is pullback $\mathcal{D}$-asymptotically compact in $E$.

**Proof.** Since $O$ is an absorbing set for the process $S(t, \tau)$, without loss of generality, let $\{\varphi_n\} \subset O$. By Corollary 1, $\{S(t, -t_n)\varphi_n\} \subset O$ for a fixed $t$, where $t_n \to +\infty$ as $n \to +\infty$. Since $E$ is a Hilbert space, there exist $\varphi_0 \in E$ and a subsequence of $\{S(t, -t_n)\varphi_n\}$ (still denoted by $\{S(t, -t_n)\varphi_n\}$) such that
\[
S(t, -t_n)\varphi_n \to \varphi_0, \quad \text{weakly in } E, \quad n \to \infty.
\]  
(28)

Since $\varphi_0 \in E$, for any $\epsilon > 0$, there exists a positive integer $N_1(\epsilon) > 0$ such that
\[
\sum_{||i|| \geq N_1(\epsilon)} ||\varphi_0||^2 \leq \frac{\epsilon^2}{4}.
\]

By Lemma 3, there exists a positive integer $N_2(\epsilon) > 0$ and $T_\epsilon \geq T_0$ such that
\[
\sum_{||i|| \geq N_2(\epsilon)} ||S(t, -t_n)\varphi_{ni}||^2 \leq \frac{\epsilon^2}{4}, \quad \forall t_n \geq T_\epsilon - t.
\]

So, choosing $N_\epsilon = \max\{N_1(\epsilon), N_2(\epsilon)\}$, we have
\[
\sum_{||i|| \geq N_\epsilon} ||S(t, -t_n)\varphi_{ni} - \varphi_0||^2 \leq \sum_{||i|| \geq N_\epsilon} ||S(t, -t_n)\varphi_{ni}||^2 + \sum_{||i|| \geq N_\epsilon} ||\varphi_0||^2 \leq \frac{\epsilon^2}{2}, \quad \forall t_n \geq T_\epsilon - t.
\]  
(29)

By (28), as $n \to \infty$,
\[
S(t, -t_n)\varphi_n \to \varphi_0, \quad \text{strongly in } \mathbb{R}_{\infty, \lambda}^{(2N_\epsilon + 1)^d} \times \mathbb{R}_{\infty, \lambda}^{(2N_\epsilon + 1)^d}
\]
where $||.|| \leq N_\epsilon$ and $\mathbb{R}_{\infty, \lambda}^{(2N_\epsilon + 1)^d} \times \mathbb{R}_{\infty, \lambda}^{(2N_\epsilon + 1)^d}$ will be defined as the beginning of the next section, which is a finite-dimensional subspace of $E$. So, there exists a positive integer $n_\epsilon > 0$ such that for any $n \geq n_\epsilon$,
\[
t_n \geq T_\epsilon - t, \quad \sum_{||i|| \leq N_\epsilon} ||S(t, -t_n)\varphi_{ni} - \varphi_0||^2 \leq \frac{\epsilon^2}{2}.
\]  
(30)

Together with (29) and (30), for any $\epsilon > 0$, there exists a positive integer $n_\epsilon > 0$ such that for all $n \geq n_\epsilon$,
\[
||S(t, -t_n)\varphi_n - \varphi_0||_E \leq \epsilon
\]
which implies that $S(t, \tau)$ is pullback $\mathcal{D}$-asymptotically compact. \qed

By Theorem 1, Lemmas 2 and 4, we obtain the existence of the attractors for the process $S(t, \tau)$. 


Lemma 5. Denote by $H(\phi_n)$ with $\phi_n$ and $\bar{\phi}_n$. A process $S(t, \tau)$ is said to be (uniformly in the past) pullback asymptotically compact if there exists $T^*$ such that, given $\{(t_n, \tau_n)\}_{n \geq 1}$ a sequence satisfying
\[
\lim_{n \to +\infty} (t_n - \tau_n) = +\infty, \quad \tau_n \leq t_n \leq T^* \quad (n \geq 1)
\]
and $\{u_{0n}\}_{n \geq 1}$ a bounded sequence, the sequence $\{S(t_n, \tau_n)u_{0n}\}_{n \geq 1}$ has a convergent subsequence in $E$.

Lemma 5. The process $S(t, \tau)$ associated with (10) is (uniformly in the past) pullback asymptotically compact.

Proof. Let us define for each $n \geq 1$
\[
g_n(u, t) = \begin{cases} g(u, t + \tau_n), & \text{if } t < T^* - \tau_n, \\ 0, & \text{if } t > T^* - \tau_n, \end{cases}
\]
and $\bar{g}_n(u, t) = (g_n(u, t))_{i \in \mathbb{Z}^k}, \bar{q}_n(t) = (q_i(t))_{i \in \mathbb{Z}^k}$. Let $v_n = \bar{u}_n + \varepsilon u_n$ and $\phi_n = (u_n, v_n)^T$. Denote by $S_n(s, 0)\varphi_{0n} = \phi_n(s), s \geq 0$ with $\phi_n$ being the unique solution of
\[
\begin{cases}
\varphi_n + H(\phi_n) = F_n(\phi_n, s), & s > 0, \\
\phi_n(0) = \varphi_{0n},
\end{cases}
\]
where $H(\cdot)$ is as in (10), and
\[
F_n(\varphi, t) = \left(\begin{array}{c}
0 \\
-\bar{g}(u, t) + \bar{q}_n(t)
\end{array}\right).
\]
Then, it is not difficult to see that
\[
S(s + \tau_n, \tau_n)\varphi_{0n} = S_n(s, 0)\varphi_{0n}, \quad \text{for any } s \in [0, T^* - \tau_n].
\]
In fact, $S(s + \tau_n, \tau_n)\varphi_{0n}$ satisfies (32) for $s \in [0, T^* - \tau_n]$. Taking $s = t_n - \tau_n$, we obtain
\[
S(t_n, \tau_n)\varphi_{0n} = S_n(t_n - \tau_n, 0)\varphi_{0n}, \quad n \geq 1.
\]
Corollary 1 and Lemmas 2–4 are all true for $S_n(t_n - \tau_n, 0)$ after slight modifications. Taking into account (32) and (33), the assertion in the lemma follows directly. \hfill \Box

Theorem 4. The pullback $D$-attractor $A = \{A(t)\}_{t \in \mathbb{R}}$ defined by Theorem 3 satisfies for any fixed $T^*$
\[
\bigcup_{\tau \leq T^*} A(\tau) \quad \text{is relatively compact in } E.
\]

Proof. Denote by $\mathcal{M}$ the set of all $y \in E$ for which there exists a sequence $\{(t_n, \tau_n)\}_{n \geq 1} \subset \mathbb{R}^2$ satisfying (31), and a sequence $\{\varphi_{0n}\}_{n \geq 1} \subset O$, such that $\lim_{n \to +\infty} \|S(t_n, \tau_n)\varphi_{0n} - y\|_E = 0$.

Observe first that
\[
A(t) \subset \mathcal{M}, \quad \text{for all } t \leq T^*.
\]
In fact, by the definition of $A$, if $t \leq T^*$, and $y \in A(t)$, there exist sequences $\tau_n \leq t$ and $\{\varphi_{0n}\}_{n \geq 1} \subset O$ such that $\lim_{n \to +\infty} \|S(t_n, \tau_n)\varphi_{0n} - y\|_E = 0$ where $\tau_n \to -\infty$ as $n \to +\infty$. Consequently, taking $t_n = t$ for all $n \geq 1$, we obtain that $y \in \mathcal{M}$.

On the other hand, $\mathcal{M}$ is a relatively compact subset of $E$. In fact, if $\{y_k\}_{k \geq 1} \subset \mathcal{M}$ is a given sequence, for each $k \geq 1$, we can take a pair $(t_k, \tau_k) \in \mathbb{R}^2$ and an element $\varphi_{0k} \in O$ such that $t_k \leq T^*$, $t_k - \tau_k \geq k$, and $\|S(t_k, \tau_k)\varphi_{0k} - y_k\|_E \leq \frac{1}{k}$. Then, by Lemma 5, it is immediate that we can extract from $\{y_k\}_{k \geq 1}$ a subsequence that converges in $E$. \hfill \Box
3. Approximations of the attractors

In this section, we present the approximations to the attractors $\mathcal{A}(t), t \in \mathbb{R}$ by the uniform attractors of finite-dimensional ordinary differential systems.

Let $n \in \mathbb{N}$ be a positive integer. Set

$$
\omega = (\omega(-n, -n, \ldots, -n), \omega(-n, -n, \ldots, -n, -n+1), \ldots, \omega(-n, -n, \ldots, -n), \omega(-n, -n, \ldots, -n, -n+1), \ldots, \omega(-n, -n, \ldots, -n, -n+1), \ldots, \omega(n, n, \ldots, n, -n+1), \ldots, \omega(n, n, \ldots, n, n)).
$$

(34)

For convenience, we always denote by

$$
\mathbb{R}_{\infty}^{(2n+1)k} = \{ \omega = (\omega_i)_{i \in \mathbb{Z}^k} \in \ell^2 \mid \omega_i \text{ with subscripts of components of } \omega \text{ are ordered as in (34) and } \omega_i = 0, \|i\| > n \} \tag{35}
$$

with the same inner product and norm as those of $\ell^2$ and denote $\mathbb{R}_{\infty}^{(2n+1)k}$ the space $\mathbb{R}_{\infty}^{(2n+1)k}$ with the same inner product and norm as those of $\ell^2$. Let $E_n = \mathbb{R}_{\infty}^{(2n+1)k} \times \mathbb{R}_{\infty}^{(2n+1)k}$ with the same inner product and norm as those of $\mathbb{E}$.

Let $\tilde{u} = (\tilde{u}_i)_{i \in \mathbb{Z}^k} \in \mathbb{R}_{\infty}^{(2n+1)k}$. In this section, note that $\tilde{u}_{(i_1, \ldots, i_{j+1}, \ldots, i_k)}$ is replaced by $\tilde{u}_{(i_1, \ldots, -i_{j+1}, \ldots, i_k)}$, and $\tilde{u}_{(i_1, \ldots, -n-1, i_{j+1}, \ldots, i_k)}$ by $\tilde{u}_{(i_1, \ldots, -i_{j+1}, \ldots, i_k)}$, $j = 1, \ldots, k$ in the definition of the operator $A$ given in (4) and (5). We consider the $(2n+1)^k$-dimensional ODEs with initial data in $\mathbb{R}_{\infty}^{(2n+1)k}$:

$$
\begin{aligned}
\tilde{u} + \tilde{h}(\tilde{u}) + A\tilde{u} + \lambda \tilde{u} + \tilde{g}(\tilde{u}, t) &= \tilde{q}(t), & t \geq \tau, & t \in \mathbb{R}, \\
\tilde{u}(\tau) &= \tilde{u}_0 \in \mathbb{R}_{\infty}^{(2n+1)k}, & \tilde{u}(\tau) &= \tilde{u}_{10} \in \mathbb{R}_{\infty}^{(2n+1)k},
\end{aligned}
$$

(36)

where

$$
\tilde{u} = (\tilde{u}_i)_{i \in \mathbb{Z}^k}, \quad \tilde{h}(\tilde{u}) = (h(\tilde{u}_i))_{i \in \mathbb{Z}^k}, \quad \tilde{g}(\tilde{u}, t) = (g(\tilde{u}_i, t))_{i \in \mathbb{Z}^k} \in \mathbb{R}_{\infty}^{(2n+1)k}
$$

and $\tilde{q}(t)$ satisfying $\tilde{q}_i(t) = q_i(t)$, for $\|i\| \leq n$, while $\tilde{q}_i(t) = 0$, for $\|i\| > n$.

Let $\tilde{v} = \tilde{u} - \varepsilon \tilde{u}$ and $\tilde{\varphi} = (\tilde{u}, \tilde{v})^T$. Then (36) can be written as

$$
\tilde{\varphi} + H(\tilde{\varphi}) = \tilde{F}(\tilde{\varphi}, t), \quad \tilde{\varphi}(\tau) = \tilde{\varphi}_0 = (\tilde{u}_0, \tilde{v}_0 + \varepsilon \tilde{u}_0)^T, \quad t > \tau,
$$

(37)

where $H$ is as in (10) and

$$
\tilde{F}(\tilde{\varphi}, t) = \begin{pmatrix}
0 \\
-\tilde{g}(\tilde{u}, t) + \tilde{q}(t)
\end{pmatrix}.
$$

Obviously, the problem (37) is well-posed in $E_n$, that is, for any $\tilde{\varphi}_0 \in E_n$, there exists an unique solution $\tilde{\varphi}(t) \in C^1([\tau, +\infty), E_n)$. Furthermore, $\tilde{\varphi}(t) = \tilde{\varphi}(t, \tilde{\varphi}_0)$ is continuous on $(t, \tilde{\varphi}_0)$ in $[\tau, +\infty) \times E_n$. It implies that the mapping

$$
\begin{aligned}
S_n(t, \tau) : \tilde{\varphi}(\tau) = \tilde{\varphi}_0 = (\tilde{u}_0, \tilde{v}_0)^T \mapsto \tilde{\varphi}(t) = (\tilde{u}(t), \tilde{v}(t))^T, & E_n \mapsto E_n
\end{aligned}
$$

(38)

generates a continuous process from $E_n$ to itself for any $n \in \mathbb{N}$.

By using [9] and some computation similar to Lemmas 2–4, we obtain

**Lemma 6.** For any $\varepsilon > 0$, there exist $T(\varepsilon) \geq T_0$ and $K_n(\varepsilon) > 0$ such that the solution of $\tilde{\varphi}(t) = (\tilde{\varphi}_i(t))_{i \in \mathbb{Z}^k} = (\tilde{u}(t), \tilde{v}(t)) \in O_{E_n}(0, r_0)$ of (37) with initial data in $O_{E_n}(0, r_0)$, $\tilde{v} = \tilde{u}(t) + \varepsilon \tilde{u}(t)$ satisfying

$$
\sum_{K_n(\varepsilon) \leq \|i\| \leq n} \sum_{j=1}^k \left( |B_j \tilde{u}_i(t)|^2 + \lambda |\tilde{u}_i(t)|^2 + |\tilde{v}(t)|^2 \right) \leq \varepsilon
$$

(39)

for all $\tau \geq t - T(\varepsilon)$, fixed $t$.

**Lemma 7.** For the process $S_n(t, \tau)$ associated with (37), $S_n(t, \tau) O_{E_n}(0, r_0) \subset O_{E_n}(0, r_0)$, $S_n(t, \tau) \subset O_{E_n}(0, r_0)$, $\tau \leq t - T_0$, fixed $t$. Furthermore, the process $S_n(t, \tau)$ possesses nonempty compact uniform attractors $\mathcal{A}_n(t) \subset O_{E_n}(0, r_0)$, $t \in \mathbb{R}$ in $E_n$ for any $n \in \mathbb{N}$.

**Lemma 8.** If $\varphi_0(0) \in \mathcal{A}_n(0)$, then there exists a subsequence $\{\varphi_{n_k}(0)\}$ of $\{\varphi_0(0)\}$ and $\varphi_0 \in \mathcal{A}(0)$ such that $\varphi_{n_k}(0)$ converges to $\varphi_0$ in $E$. 
Proof. The lemma is a direct result from some computation similar to the proof of Lemma 7 in [23]. We will omit it here. □

As a direct consequence of Lemma 8, we obtain the upper semicontinuity of $A(t)$.

**Theorem 5.** $\text{dist}(A_n(t), A(t)) = 0$, as $n \to +\infty$, for any $t \in \mathbb{R}$, where the Hausdorff semidistance $\text{dist}$ is defined as in Section 2.

### 4. Dimension of the attractors

The purpose of this section is to study the dimension of the attractors for the nonautonomous lattice dynamical system by using Theorem 3.2 in [7].

For any $n \in \mathbb{N}$, let $E_n^\perp$ is orthogonal to $E_n$ and $E_n \otimes E_n^\perp = E$ where $E_n$ is defined at the beginning of Section 3. $Q_n$ is the projector mapping $E$ onto the subspace $E_n^\perp$ of codimension $2(n+1)$. A

**Lemma 9.** For any $\varphi_0, \psi_0 \in A(\tau)$, there exists a positive integer $N^* = N^*(\epsilon, r_0, q)$ and $T^* = T^*(\epsilon, r_0) > \max\{T_0, \frac{2}{\epsilon}\log 2\}$ such that for any $T \geq T^*$,

\[
\begin{align*}
\|S(t + T, \tau)\varphi_0 - S(t + T, \tau)\psi_0\|_E &\leq e^{\frac{1}{2}(K(\epsilon/\sqrt{\lambda})/\sqrt{\lambda})T} \|\varphi_0 - \psi_0\|_E, \\
\|Q_n S(t + T, \tau)\varphi_0 - Q_n S(t + T, \tau)\psi_0\|_E &\leq e^{-\frac{1}{2}T} \|\varphi_0 - \psi_0\|_E.
\end{align*}
\] (40)

**Proof.** Let $S(t, \tau)\varphi_0 = \varphi(t) = (u, v)^T$ where $v = \tilde{u} + \epsilon u$, and $S(t, \tau)\psi_0 = \psi(t) = (\tilde{v}(t), \tilde{v}(t))^T$ where $\tilde{v}(t) = \tilde{u}(t) + \epsilon \tilde{u}(t)$. Since $\varphi_0, \psi_0 \in A(\tau)$, then $\varphi(t), \psi(t) \in O$ for all $t \geq \tau$. Let $\phi(t) = S(t, \tau)\varphi_0 - S(t, \tau)\psi_0 = \varphi(t) - \psi(t) = (w(t), z(t))^T$ where $z(t) = \tilde{w}(t) + \epsilon w(t)$, then $\phi(t)$ satisfies

\[
\dot{\phi} + H(\varphi) - H(\psi) + (0, \tilde{g}(u, t) - \tilde{g}(\tilde{u}, t))^T = 0, \quad \phi(\tau) = \varphi_0 - \psi_0.
\] (42)

Similar to Lemma 1, we have

\[
\langle H(\varphi) - H(\psi), \phi \rangle_E \geq \frac{\epsilon}{2} \|\phi\|_E^2 + \frac{\alpha_1}{2} \|z\|^2.
\] (43)

By (H3),

\[
\left\| (\tilde{g}(u, t) - \tilde{g}(\tilde{u}, t), z) \right\| \leq \frac{K(\epsilon/\sqrt{\lambda})}{2\sqrt{\lambda}} \|\phi\|_E^2.
\] (44)

Taking the inner product $(\cdot, \cdot)_E$ in (42) with $\phi$, by (43), (44) and the Gronwall inequality, we have

\[
\|\psi(t) - \psi(t)\|^2 \leq e^{K(\epsilon/\sqrt{\lambda})(t - \tau)} \|\varphi_0 - \psi_0\|^2.
\] (45)

So, (40) is obtained.

Denote $\varphi_n(t) = Q_n \varphi(t), \psi_n(t) = Q_n \psi(t)$ and $\phi_n(t) = Q_n(\varphi(t) - \psi(t)) = Q_n \phi(t) = (w_n, z_n)^T$. Taking the inner product $(\cdot, \cdot)_E$ in (42) with $\phi_n$, we have

\[
\frac{d}{dt} \|\phi_n\|_E^2 + \epsilon \|\phi_n\|_E^2 + 2(\tilde{g}(u, t) - \tilde{g}(\tilde{u}, t), \phi_n) \leq 0,
\] (46)

where $\|\phi_n\|_E^2 = \sum_{\|k\| > n} \|\phi_k\|^2$.

By the mean value theorem,

\[
\left\| (\tilde{g}(u, t) - \tilde{g}(\tilde{u}, t), z_n) \right\| \leq \sum_{\|k\| > n} |g_n'(u_t + \theta k(\tilde{u}_t - u_t), t)| |w_t| \cdot |z_n|.
\]

where $\theta_k \in (0, 1), i \in \mathbb{Z}^k$. By (H3) and Lemma 3, there exists a positive integer $N^* = N^*(\epsilon, r_0, q)$ and $T^* = T^*(\epsilon, r_0) > T_0$ such that for any $\|k\| > N^*$, and $t \geq \tau + T^*$,

\[
|g_n'(u_t + \theta k(\tilde{u}_t - u_t), t)| \leq \frac{\epsilon \sqrt{\lambda}}{2}.
\]

So,

\[
\left\| (\tilde{g}(u, t) - \tilde{g}(\tilde{u}, t), z_{N^*}) \right\| \leq \frac{\epsilon}{4} \sum_{\|k\| > N^*} \|\phi_k\|^2.
\] (47)

By (46), (47) and the Gronwall inequality, we have

\[
\|\phi_{N^*}\|^2 \leq e^{-\frac{\epsilon}{2}(t-\tau)} \|\varphi_0 - \psi_0\|_E^2.
\] □
Theorem 6. (1) If \( K(r_0/\sqrt{\lambda}) < \varepsilon \sqrt{\lambda} \), then the fractal dimension \( d_f(A(t)) \) of \( A(t) \) satisfies
\[
d_f(A(t)) = 0
\]
for all \( t \in \mathbb{R} \).

(2) If \( K(r_0/\sqrt{\lambda}) \geq \varepsilon \sqrt{\lambda} \), then for any \( \sigma \geq 0 \) such that \( (2\sqrt{2})^{2(2N^*+1)} (\sqrt{2})^\sigma < 1 \) for some \( T \), the following inequality holds
\[
d_f(A(t)) \leq 2(2N^* + 1)^k + \sigma, \quad t \in \mathbb{R}
\]
for all \( t \in \mathbb{R} \), where \( N^* \) as that of Lemma 9.

Proof. Let \( l = l(T) = e^{1/(K(r_0/\sqrt{\lambda})/\sqrt{\lambda} - \varepsilon)T} \), \( \delta = \delta(T) = e^{-4/2} \) where \( T \geq T^* \) as that of Lemma 9. Since \( E_{N^*}^\bot \) is a subspace of codimension \( 2(2N^* + 1)^k \) in \( E \), by using Theorem 3.2 in [7], we easily obtain the results. \( \square \)

By the above theorem and Lemma 9, we have the following result

Corollary 2. Let \( T \geq T^* \) as that of Lemma 9. If \( K(r_0/\sqrt{\lambda}) \geq \varepsilon \sqrt{\lambda} \), then for any \( t \in \mathbb{R} \),
\[
d_f(A(t)) \leq 2(2N^* + 1)^k \left( 1 + \frac{2(K(r_0/\sqrt{\lambda})/\sqrt{\lambda} - \varepsilon)T + 4 \log 3 + 6 \log 2}{\varepsilon T - 2 \log 2} \right).
\]
Furthermore, for any \( t \in \mathbb{R} \),
\[
d_f(A(t)) \leq 2(2N^* + 1)^k \left( 1 + \frac{2(K(r_0/\sqrt{\lambda})/\sqrt{\lambda} - \varepsilon)T^* + 4 \log 3 + 6 \log 2}{\varepsilon T^* - 2 \log 2} \right).
\]

Remark. Indeed, Theorem 6 and Corollary 2 hold for the global attractor for the autonomous lattice dynamical system of [23]. That is, the attractor is finite-dimensional.

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