Remarks on Propagation of Singularities in Thermoelasticity

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The propagation of high order weak singularities for the system of homogeneous thermoelasticity in one space variable is studied by using paralinearization and a new decoupling technique introduced by the author (Microlocal analysis in nonlinear thermoelasticity, to appear). For the linear system, one shows that the nonsmooth initial data for the parabolic part lead to singularities in the hyperbolic part of solutions, even when the initial data for that part are identically zero. Both the Cauchy problem and the problem inside of a domain for the semilinear system are considered. It is shown that the propagation of high order singularities is essentially dominated by the hyperbolic operator in the system of thermoelasticity.

1. INTRODUCTION

This article is devoted to the study of singularities of solutions to the semilinear system of thermoelasticity in one space variable $x \in \mathbb{R}$,

\[ \begin{align*}
    u_{tt} - \alpha^2(t, x)u_{xx} + \gamma_1(t, x)\theta_x &= f(u, \theta) \\
    \theta_t - \beta^2(t, x)\theta_{xx} + \gamma_2(t, x)u_{tx} &= g(u, \theta),
\end{align*} \tag{1.1} \]

describing the elastic and the thermal behaviour of an elastic medium, where $u$ represents the displacement, $\theta = T_a - T_0$ is the temperature difference, and $\alpha \geq \alpha_0 > 0$, $\beta \geq \beta_0 > 0$.

Obviously, the system (1.1) is a typical hyperbolic–parabolic coupled system. It has been studied by many authors with respect to the decay rate
of solutions, the existence of global smooth solutions for small data, and the regularity of solutions to prescribe whether the qualitative properties of solutions are dominated by either the hyperbolic or the parabolic operator; see for example [8–11, 17] and references therein. By using the Fourier analysis, it was proved in [12, 15, 16] that the propagation of weak singularities in semilinear thermoelasticity is dominated by the hyperbolic operator. In [19] the author has introduced a linear transformation involving pseudodifferential operators for \((u, \theta)\) to decouple the hyperbolic and the parabolic operators in (1.1). By using this decoupling transformation, it is proved that even when the nonlinear functions \(f\) and \(g\) depend also on \(u_x\), the system (1.1) has a finite determinacy domain for the propagation of regularity. Furthermore, microlocal singularities of solutions are propagated along bicharacteristic strips of the hyperbolic operator \(\partial^2_t - \alpha^2 \partial^2_x\), which is similar to the wave equation case [1, 14].

The purpose of this paper is further to use the decoupling transformation given in [19] to study both the linear and semilinear thermoelastic system. For the linear problem, we will prove that even when the initial data of \(u\) and \(u_t\) are smooth, the nonsmooth initial datum of \(\theta\) will produce singularities for \(u\) when \(t > 0\), which coincides with the phenomenon observed in [9]. However, we shall see there is a linear nonlocal combination of \(u\) and \(\theta\) sharing a smoothing effect similar to solutions of linear parabolic equations. For the semilinear system (1.1), we will study the propagation of microlocal singularities both in interior domains and for Cauchy problems by using paralinearization.

For simplicity, in (1.1) we assume that all coefficient functions are smooth and constants out of a compact set in \(\mathbb{R}_t \times \mathbb{R}_x\), and \(f, g\) are smooth with respect to their arguments.

First, let us consider the Cauchy problem for the homogeneous linear thermoelastic system as follows:

\[
\begin{align*}
\partial^2_t - \alpha^2 \partial^2_x &+ \gamma_1(t, x)\partial_x \theta = 0 \\
\partial_t - \beta^2 \partial_x \theta + \gamma_2(t, x)u_x & = 0
\end{align*}
\tag{1.2}
\]

\(u(0, x) = u_0(x), \quad u_x(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x).\)

The following first result shows that the nonsmooth initial data for \(\theta\) lead to singularities in \(u\) even when the initial data for \(u\) are identically zero, and the second one gives a clear picture for the propagation of singularities in (1.2).

**Theorem 1.1.** For the problem (1.2), if \(\gamma_1 \neq 0, \gamma_2 \neq 0, u_0 = u_1 = 0,\) and \(\theta_0 \in H^s \setminus H^{s+1}\) for a fixed \(s \in \mathbb{R}\), then for any \(T > 0\), we have

\[
\begin{align*}
u \in C([0, T], H^{s+2}(\mathbb{R})) \cap C([0, T], H^{s+1}(\mathbb{R})) \\
\not \in L^2(0, T; H^{s+1}(\mathbb{R})) \cap H^1(0, T; H^{s+2}(\mathbb{R})).
\end{align*}
\tag{1.3}
\]
THEOREM 1.2. For the Cauchy problem (1.2), suppose for a fixed $s \in \mathbb{R}$ and a closed subset $\omega \subset \mathbb{R}$,

$$
(u_0, \theta_0) \in H^{s+1}(\mathbb{R}) \cap C_0(\mathbb{R} \setminus \omega), \quad u_1 \in H'(\mathbb{R}) \cap C_\infty(\mathbb{R} \setminus \omega). \quad (1.4)
$$

Let $\mathcal{P}$ be the union of characteristic curves for the operator $\partial_t^2 - \alpha^2(t, x) \partial_x^2$ issuing from $(0, x_0)$ for any $x_0 \in \omega$. Then, we have

$$
(u, \theta) \in C^\infty((0, +\infty) \times \mathbb{R}) \setminus \mathcal{P}. \quad (1.5)
$$

To precisely state the result on the propagation of microlocal singularities in nonlinear problems, let us introduce:

DEFINITION. Let $(\tau, \xi)$ be the dual variable of $(t, x) \in \mathbb{R}^2$. For any $-\infty < s \leq r < \infty$ and $-\infty < s_1 \leq r_1 < \infty$, $u \in H^{s-s_1} \cap H^{r_1}_m(t_0, x_0; \tau_0, \xi_0)$ means that there exist a smooth function $\phi(t, x)$, supported near $(t_0, x_0)$ with $\phi(t_0, x_0) = 1$, and a cone $K$ in $\mathbb{R}^2 \setminus 0$ about the direction $(\tau_0, \xi_0)$ such that either

$$
\langle (\tau, \xi) \rangle^{s} \langle \xi \rangle^{s_1-1} |\hat{\phi} u(\tau, \xi)| \in L^2(\mathbb{R}^2)
$$
when $s_1 \geq s$, or

$$
\langle (\tau, \xi) \rangle^{s_1} \langle \tau \rangle^{r-s_1} |\hat{\phi} u(\tau, \xi)| \in L^2(\mathbb{R}^2)
$$
when $s_1 < s$, and either

$$
\langle (\tau, \xi) \rangle^{r} \langle \xi \rangle^{r_1-1} |\hat{\phi} u(\tau, \xi)| \in L^2(\mathbb{R}^2)
$$
when $r_1 \geq r$, or

$$
\langle (\tau, \xi) \rangle^{r_1} \langle \tau \rangle^{r_1-r} (\chi_K(\tau, \xi) |\hat{\phi} u(\tau, \xi)| \in L^2(\mathbb{R}^2)
$$
when $r_1 < r$, where $\chi_K$ is the characteristic function of $K$, $\langle \tau \rangle = (1 + \tau^2)^{1/2}$, $\langle \xi \rangle = (1 + \xi^2)^{1/2}$ and $\langle (\tau, \xi) \rangle = (1 + \tau^2 + \xi^2)^{1/2}$. If $\Gamma$ is a closed conic set in $\mathbb{T}^*(\mathbb{R}^2) \setminus 0$, we shall say that $u \in H^{s-s_1} \cap H^{r_1}_m(\Gamma)$ if $u \in H^{s-s_1} \cap H^{r_1}_m(t, x; \tau, \xi)$ for all $(t, x; \tau, \xi) \in \Gamma$. When $s = s_1$ and $r = r_1$, we will denote $H^{s-s_1}$ and $H^{r_1}_m$ by $H^s$ and $H^{r_1}_m$ respectively.

For the propagation of microlocal singularities, we have the following two results for the problem inside of a domain as well as for the Cauchy problem.

THEOREM 1.3. Let $\Gamma$ be a null bicharacteristic of $\partial_t^2 - \alpha^2(t, x) \partial_x^2$ passing through $(t_0, x_0; \tau_0, \xi_0)$. For any fixed $2 < s \leq r < 2(s - 1)$, if the solutions $u \in H^{s-s-1}$ and $\theta \in H^{r-1}$ of (1.1) satisfy

$$
u \in H^{n-1}_{m_1}(t_0, x_0; \tau_0, \xi_0)
$$

then we have

$$
u \in H^s \cap H^r(\Gamma), \quad \theta \in H^{s-1-s} \cap H^{r-1-r}(\Gamma). \quad (1.7)$$
Theorem 1.4. Consider the following Cauchy problem:

\[ u_{tt} - \alpha^2(t, x)u_{xx} + \gamma_1(t, x)\theta_x = f(u, \theta) \]
\[ \theta_t - \beta^2(t, x)\theta_{xx} + \gamma_2(t, x)u_{tx} = g(u, \theta) \]  
(1.8)

\[ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x). \]

If \((u_0, \theta_0) \in H^s(\mathbb{R}) \cap H^{s'}_{ml}(x_0, \xi_0), \ u_1 \in H^{s-1}(\mathbb{R}) \cap H^{-1}_{ml}(x_0, \xi_0)\) for fixed \(\frac{3}{2} < s \leq t \leq 2s - \frac{1}{2}\), let \(\gamma(t) = (t, x(t), \tau(t), \xi(t)) \in T^*(\mathbb{R}^2)\,0\ (0 \leq t < T_0)\) be a null bicharacteristic for \(\partial_t^2 - \alpha^2(t, x)\partial_x^2\) with \(x(0) = x_0\) and \(\xi(0) = \xi_0\). Then the problem (1.8) has unique solutions:

\[ u \in C([0, T], H^s \cap H'_ml(x(t), \xi(t))) \cap C^1([0, T], \ H^{s-1} \cap H'_{ml}(x(t), \xi(t))) \]  
(1.9)

\[ \theta \in C([0, T], H^s \cap H'_{ml}(x(t), \xi(t))). \]

As a simple consequence from Theorem 1.4, we shall prove a result on the propagation of local regularity for the Cauchy problem, which improves the previous results obtained in [12, 15, 16].

Theorem 1.5. Consider the Cauchy problem (1.8). Let \(x = x(t)\) and \(x = x_2(t)\) be two characteristic curves of \(\partial_t^2 - \alpha^2(t, x)\partial_x^2\) with \(x_1(0) = x_2(0) = 0\) and \(x_1(t) < x_2(t)\) for \(t > 0\). For any fixed \(s > 3/2\), if \((u_0, \theta_0) \in H^s(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus 0)\) and \(u_1 \in H^{s-1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus 0)\), then the local solutions \(u\) and \(\theta\) of (1.8) satisfy

\[ u \in C((0, T], H^{2s-1/2}_{loc}(\{x_1(t) < x < x_2(t)\})) \]
\[ \cap C^1((0, T], H^{2s-3/2}_{loc}(\{x_1(t) < x < x_2(t)\})) \]  
(1.10)

\[ \theta \in C((0, T], H^{2s-1/2}_{loc}(\{x_1(t) < x < x_2(t)\})). \]

Remark 1.6. From our discussion, it is not difficult to see that all results in this paper can be extended into problems for semilinear thermoelastic systems in several space variables by using the decoupling transformation given in [19], which is the reason why we give the Microlocal statement in Theorem 1.4 even for the problem in one space variable.

The remainder of this paper is arranged as follows: In Section 2, we shall first recall the decoupling transformation introduced in [19] for the system (1.1) and some elementary results of paradifferential operators. Then, in Section 3 we shall easily prove Theorems 1.1 and 1.2 by using this decoupling transformation. The proofs of Theorems 1.3 and 1.4 will be given in Sections 4 and 5 respectively. As a simple consequence of Theorem 1.4, we shall prove Theorem 1.5 at the end of Section 5.
2. PRELIMINARIES

2.1. Decoupling Transformation

In this section, we shall recall a decoupling transformation from [19] for the hyperbolic-parabolic coupled system (1.1) by using an idea inspired by Taylor [18].

Denote by \( \psi \) the classical positive, self-adjoint operator with the symbol \( (\xi) = (1 + \xi^2)^{1/2} \), and set:

\[
\begin{align*}
\psi \lambda &= (\psi - \partial_x^2)/2 \\
u_+ &= (\partial_t + i\alpha \psi \lambda)u \\
u_- &= (\partial_t - i\alpha \psi \lambda)u.
\end{align*}
\]

(2.1)

Then, from (1.1) we know that \( U = (\psi \lambda u, \psi \lambda u, \theta) \) satisfies the system

\[
\begin{align*}
\partial_t U + &\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \beta^2 \lambda^2 & 0
\end{pmatrix}
U + 
\begin{pmatrix}
-i\alpha \lambda & 0 & \gamma_1 \partial_x \\
0 & i\alpha \lambda & \gamma_1 \partial_x \\
\gamma_1 \partial_x & \gamma_1 \partial_x & 0
\end{pmatrix}
U \\
+ & A_0(t, x, D_x) U = F(U)
\end{align*}
\]

(2.2)

where

\[
A_0(t, x, D_x) U = \begin{pmatrix}
(i\alpha \lambda, \alpha) + i\alpha^2 \lambda^{-1} - \alpha \rangle u - u_{2\alpha} \\
\langle i\alpha \lambda, \alpha \rangle + i\alpha^2 \lambda^{-1} + \alpha \rangle u - u_{2\alpha} \\
-\beta^2 \theta
\end{pmatrix}
\]

and

\[
F(U) = (f, f, g)'.
\]

Denote by \( \Psi^{1,0}_1(\mathbb{R}) \) the set of pseudodifferential operators of order \( k \) with symbols in \( \Psi^{1,0}_1(\mathbb{R}) \) [4] and \( \theta \) being a parameter.

As in [18, 19], for any \( m \in \mathbb{N} \), we can construct \( K_m(t, x, D_x) \in \Psi^{-1,0}_1(\mathbb{R}) \), smooth in \( t \), with the principal symbol being

\[
K_1(t, x, \xi) = \frac{i\xi}{\beta^2(\xi)^2} \begin{pmatrix}
0 & 0 & -\gamma_1 \\
0 & 0 & -\gamma_1 \\
\frac{\gamma_1}{2} & \frac{\gamma_1}{2} & 0
\end{pmatrix} \in \Psi^{-1,0}(\mathbb{R})
\]

(2.3)

such that

\[
V = (I + K_m(t, x, D_x))(u_+, u_-, \theta)'
\]

(2.4)
satisfies the decoupled system modulo $\Psi_{1,0}^{-m}$,
\[
\partial_t V + A_2(t, x, D_x)V + \tilde{A}_1(t, x, D_x)V \\
+ \tilde{A}_0(t, x, D_x)V + R_m(t, x, D_x)V = \mathcal{F}(V),
\]
where
\[
A_2(t, x, D_x) = \text{diag}[0, 0, \beta^2 \Lambda^2] \in \Psi_{1,0}^1(\mathbb{R})
\]
\[
\tilde{A}_1(t, x, D_x) = \begin{pmatrix}
-ia\Lambda & 0 & 0 \\
0 & ia\Lambda & 0 \\
0 & 0 & 0
\end{pmatrix} \in \Psi_{1,0}^1(\mathbb{R})
\]
\[
\tilde{A}_0(t, x, D_x) = \begin{pmatrix}
A_{11} & 0 \\
0 & A_{22}
\end{pmatrix} \in \Psi_{1,0}^1(\mathbb{R}).
\]
\[
R_m \in \Psi_{1,0}^{-m} \text{ and } \mathcal{F}(V) = (I + K_m)F(P_{-1}^{(1)}V_1, P_{-1}^{(2)}V_2, P_{0}V_3) \text{ with } A_{11} \text{ being a } 2 \times 2 \text{ matrix, } F \text{ being a smooth vector-valued function, } P_{-1}^{(k)}(t, x, D_x) \in \Psi_{1,0}^{-1}(\mathbb{R}) \text{ and } P_{0}(t, x, D_x) \in \Psi_{1,0}^0(\mathbb{R}) \text{ depending smoothly on } t.
\]

2.2. Paradifferential Operators

Let us briefly recall the definitions of paraproduct and paradifferential operators introduced by Bony in [3] (see also [6]): Suppose that $\psi(\theta, \eta) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n, 0))$ is nonnegative, homogeneous of order zero and that there are small $0 < \epsilon_1 < \epsilon_2$ such that
\[
\psi(\theta, \eta) = \begin{cases} 
1, & \text{when } |\theta| \leq \epsilon_1|\eta| \\
0, & \text{when } |\theta| \geq \epsilon_2|\eta|.
\end{cases}
\]
(2.6)
Moreover, let $S(\eta) \in C^\infty(\mathbb{R}^n)$ satisfy
\[
S(\eta) = \begin{cases} 
0, & \text{when } |\eta| \leq R \\
1, & \text{when } |\eta| \geq 2R.
\end{cases}
\]
(2.7)
Then, the function $\chi(\theta, \eta) = \psi(\theta, \eta)S(\eta)$ is called a para-cutoff function. For any $a, u \in \mathcal{S}'(\mathbb{R}^n)$, and $\chi(\theta, \eta)$ a para-cutoff function, we call the operator $T_a$ defined by
\[
T_a u(x) = \mathcal{F}_x^{-1} \left( \int \chi(\xi - \eta, \eta)\hat{a}(\xi - \eta)\hat{u}(\eta)d\eta \right)
\]
(2.8)
a paraproduct operator.

Suppose that $l(x, \xi)$ is homogeneous of order $m$ for $\xi \in \mathbb{R}^n$, smooth for $\xi \neq 0$, and for any $\alpha \in \mathbb{N}^n$, $D_\alpha^2 l(x, \xi)$ is in $H^s(x > n/2)$ with respect to $x$. Then the operator $T_l \in Op(\sum_{s=n/2}^m)$ is called a paradifferential operator of order $m$ with the symbol $l(x, \xi)$,
\[
T_l u(x) = \mathcal{F}_x^{-1} \left( \int \chi(\xi - \eta, \eta)\hat{l}(\xi - \eta, \eta)\hat{u}(\eta)d\eta \right).
\]
(2.9)
where \( \hat{I}(\theta, \eta) \) represents the Fourier transform of \( I(x, \eta) \) with respect to \( x \in \mathbb{R}^n \).

For convenience, let us list two elementary results of paradifferential operators and paralinearization as follows (see [6]):

**Lemma 2.1.**

1. For any \( t > n/2 \), if \( a \in H^t(\mathbb{R}^n) \), then the operator

\[
T_a : H^s \longrightarrow H^s
\]

is bounded for any \( s \in \mathbb{R} \).

2. For any \( t > n/2 \) and \( m \in \mathbb{Z} \), the operator

\[
T_i \in \text{Op}(\sum_{l-n/2}^m) : H^s \longrightarrow H^{s-m}
\]

is bounded for any \( s \in \mathbb{R} \).

3. For any two paradifferential operators \( T_{i_1} \in \text{Op}(\sum_{l-n/2}^{m_1}) \) and \( T_{i_2} \in \text{Op}(\sum_{l-n/2}^{m_2}) \) with \( m_1, m_2 \in \mathbb{R} \) and \( t > \frac{n}{2} + 1 \), we have

\[
[T_{i_1}, T_{i_2}] \in \text{Op}(\sum_{l-n/2}^{m_1+m_2-1}).
\]

**Lemma 2.2.** Suppose that \( F(y_1, \ldots, y_N) \) is smooth, and each derivative of \( F \) is bounded on any compact set \( K \subset \subset \mathbb{R}^N \). Then, for any \( u^i \in H^s(\mathbb{R}^n) \) \( (s > n/2, i = 1, \ldots, N) \), we have the paralinearization

\[
F(u_1(x), \ldots, u_N(x)) = \sum_{j=1}^N T_{\# \frac{\partial F}{\partial u^j}(u(x), \ldots, u^N(x))} \cdot u^j(x) + R(x),
\]

where \( R \in H^{2r-n/2}(\mathbb{R}^n) \).

Similar to Theorem 2.4(2) in Chapter 4 of [6], we have

**Lemma 2.3.** For any fixed \( s_0 > 1 \), given \( a \in H^{s_0}(\mathbb{R}^2) \) and \( u \in H^{s_0, k_1} \cap H^{s_2, k_2}(\Gamma) \) with \( \Gamma \) a conic set in \( T^*(\mathbb{R}^2) \), we have

\[
T_a u \in H^{s_0, k_1} \cap H^{s_2, k_2}(\Gamma),
\]

where \( s_3 = \min(s_1 + s_0 - 1, s_2) \) and \( k_3 = \min(k_1 + s_0 - 1, k_2) \).

Finally, by using Lemma 2.3 and a way similar to the proof of Lemma 3.3 in [7], we can obtain the following result on the propagation of singularities in Sobolev spaces with double indices for linear hyperbolic systems with lower order terms having paradifferential operators:

**Proposition 2.4.** Let \( N \times N \) matrices \( P_i(t, x, D_x) \in \Psi_{1,0}^1(\mathbb{R}) \), \( P_0(t, x, D_t, D_x) \), \( Q_i(t, x, D_t, D_x) \in \Psi_{0,0}^1(\mathbb{R}^2) \), let the operator \( L = D_t - P_1(t, x, D_x) \) be strictly hyperbolic with respect to \( t \), and let \( \Gamma \) be a null bicharacteristic of \( L \).
passing through $(t_0, x_0; \tau_0, \xi_0)$. For any fixed $s > 1$, given $A(t, x) = (a_{ij})_{N \times N}$ with $a_{ij} \in H^s(\mathbb{R}^2)$, denote by $T_A$ the $N \times N$ matrix $(T_{a_{ij}})$ of paraproduct operators. For the system

$$D_t u - P_1(t, x, D_x)u + P_0 T_A Q_0 u = F(t, x),$$

if $F \in H^{r,k}_m(\Gamma)$ and $u \in H^s \cap H^{r,k}_m(0, x_0; \tau_0, \xi_0)$ with $\max(r, k) \leq 2s - 1$, then we have

$$u \in H^s \cap H^{r,k}_m(\Gamma).$$

3. SYSTEMS IN LINEAR THERMOELASTICITY

In this section, we shall use the decoupled system (2.5) given in Section 2.1 to prove two results for the linear thermoelastic system.

3.1. Proof of Theorem 1.1. For any fixed $m \geq 1$, let $K_m(t, x, \xi) \in S^{-1}_{1,0}(\mathbb{R})$ be given in Section 2.1, and let

$$V(t, x) = (I + K_m(t, x, D_x))((\partial_t + i\alpha\lambda)u, (\partial_t - i\alpha\lambda)u, \theta)'.$$

(3.1)

Then from (2.5) we know that $V = (V_1, V_2, V_3)'$ satisfies the Cauchy problem

$$\begin{bmatrix}
\partial_t + \begin{pmatrix}
-\alpha \partial_x & 0 \\
0 & \alpha \partial_x
\end{pmatrix} + A_{11}(t, x, D_x)
\end{bmatrix}
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} + R_1 V = 0$$

$$(\partial_t - \beta^2 \partial_x^2)V_3 + A_{22}(t, x, D_x) V_3 + R_2 V = 0$$

$$V(0, x) = V^{(0)}(x),$$

(3.2)

where $A_{11}, A_{22} \in \Psi^0_{1,0}(\mathbb{R})$, $R_1, R_2(t, x, D_x) \in \Psi^{-m}_{1,0}(\mathbb{R})$ and

$$V^{(0)}(x) = (I + K_m(0, x, D_x))(0, 0, \theta_0(x))'.$$

(3.3)

Since the symbol of $K_m(0, x, D_x)$ is

$$K_m(0, x, \xi) = \frac{i\xi}{\beta^2(\xi)^2} \begin{pmatrix}
0 & 0 & -\gamma_1 \\
0 & 0 & -\gamma_1 \\
\frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0
\end{pmatrix} + S_{1,0}^{-2}(\mathbb{R}),$$

we have

$$V_1^{(0)}(x) = -\beta^{-2}\gamma_1(\xi)^{-2} \theta_0' + \Psi^2_{1,0} \cdot \theta_0 \in H^{s+1} \setminus H^{s+2}$$

$$V_2^{(0)}(x) = -\beta^{-2}\gamma_1(\xi)^{-2} \theta_0' + \Psi^2_{1,0} \cdot \theta_0 \in H^{s+1} \setminus H^{s+2}$$

(3.4)

$$V_3^{(0)}(x) = \theta_0 + \Psi^{-2}_{1,0} \cdot \theta_0 \in H^s \setminus H^{s+1}. $$
By applying the classical theory of linear parabolic and hyperbolic problems (cf. [4]) in (3.2) and using (3.4), we obtain

\[ V_3 \in L^2(0, T; H^{s+1}(\mathbb{R})) \cap H^1(0, T; H^{s-1}(\mathbb{R})) \subseteq C([0, T], H^s(\mathbb{R})) \]

\[ (V_1, V_2) \in C([0, T], H^{s+1}(\mathbb{R})) \]

\[ (V_1, V_2) \notin L^2(0, T; H^{s+2}(\mathbb{R})) \]

for any \( T > 0 \).

Noting that

\[ (I + K_m(t, x, \xi))^{-1} = I - K_m(t, x, \xi)(I + K_m(t, x, \xi))^{-1} \]

we know that the inverse transformation of (3.1) can be written as

\[ \left( \begin{array}{c} \partial_t + i\alpha \Delta \\ \partial_t - i\alpha \Delta \end{array} \right) u = (I - K_1(t, x, D_x) + \Psi_{1,0}^{-1}) \left( \begin{array}{c} V_1 \\ V_2 \\ V_3 \end{array} \right). \]

From (3.5) and (3.7), we can easily obtain

\[ (\partial_t \pm i\alpha \Delta) u \in C([0, T], H^{s+1}(\mathbb{R})), \]

\[ (\partial_t \pm i\alpha \Delta) u \notin L^2(0, T; H^{s+2}(\mathbb{R})) \]

which implies

\[ u \in C([0, T], H^{s+2}(\mathbb{R})) \cap C^1([0, T], H^{s+1}(\mathbb{R})) \]

\[ u \notin L^2(0, T; H^{s+3}(\mathbb{R})) \cap H^1(0, T; H^{s+2}(\mathbb{R})). \]

3.2. Proof of Theorem 1.2. As above, we know that, for any fixed \( m \in \mathbb{N} \),

\[ V(t, x) = (I + K_m(t, x, D_x))((\partial_t + i\alpha \Delta) u, (\partial_t - i\alpha \Delta) u, \theta)' \]

satisfies the problem (3.2) and the initial data

\[ (V_1, V_2)|_{t=0} = (V_1^{(0)}(x), V_2^{(0)}(x)) \in H^s(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\omega) \]

\[ V_3|_{t=0} = V_3^{(0)}(x) \in H^{s+1}(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\omega). \]

It follows that the solutions of (3.2) satisfy

\[ (V_1, V_2) \in C([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s+1}(\mathbb{R})) \]

\[ V_3 \in L^2(0, T; H^{s+2}(\mathbb{R})) \cap H^1(0, T; H^s(\mathbb{R})) \]

for any \( T > 0 \).
Furthermore, from (3.2), we have
\[
(V_1, V_2) \in \bigcap_{j=0}^{m+3} C^j([0, T], H^{s-j}(\mathbb{R})),
\]
\[
V_3 \in \bigcap_{j=0}^{m+3} H^j(0, T; H^{s+2-j}(\mathbb{R})).
\]

Therefore, by applying the regularity theory of linear parabolic and hyperbolic problems in (3.2)(3.10), we obtain
\[
(V_1, V_2) \in C^N((0, T] \times \mathbb{R}) \setminus \mathcal{D}, \quad V_3 \in C^N((0, T] \times \mathbb{R}),
\]
where \(N = N(m)\) is an integer depending on \(m \in \mathbb{N}\) satisfying \(N(m) \to +\infty\) when \(m \to +\infty\).

By using the inverse transformation of (3.1) and the arbitrariness of \(m \in \mathbb{N}\), we immediately conclude
\[
(u, \theta) \in C^{\infty}((0, +\infty) \times \mathbb{R}) \setminus \mathcal{D}.
\]

**Remark 3.1.** The result of Theorem 1.1 shows that the linear thermoelastic operators in (1.1) have not the same smoothing effect as in heat equations, which is similar to the phenomenon observed in [9]. However, from (3.2) we know that \(V_3\), a linear nonlocal combination of \(u\) and \(\theta\), has a smoothing effect as shown in (3.13).

4. SEMILINEAR SYSTEMS IN THERMOELASTICITY

4.1. Propagation of Singularities in Interior Domains

**Proof of Theorem 1.3.** From (2.4), we know that
\[
V = (I + K_1(t, x, D_x))((\partial_t + i\alpha\Lambda)u, (\partial_t - i\alpha\Lambda)u, \theta),
\]
holds the weakly coupled system
\[
\begin{bmatrix}
\partial_t + \begin{pmatrix} -\alpha \partial_x & 0 \\ 0 & \alpha \partial_x \end{pmatrix} + P_0^{(1)}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= P_0^{(2)} F\left(P_{-1}^{(1)} V_1, P_{-1}^{(2)} V_2, P_0^{(3)} V_3\right)
\]
\[
(\partial_t - \beta \partial_x^2) V_3 = P_0^{(4)} G\left(P_{-1}^{(1)} V_1, P_{-1}^{(2)} V_2, P_0^{(3)} V_3\right),
\]
where \(P_0^{(k)}(t, x, D_x) \in \Psi^0_{1,0}(\mathbb{R}), P_{-1}^{(k)}(t, x, D_x) \in \Psi^{-1}_{1,0}(\mathbb{R})\) smoothly depending on \(t\), and \(F, G\) are smooth functions with respect to their arguments.
If we define
\[ W_1 = (1 - \partial_x^2)^{-1/2}V_1, \quad W_2 = (1 - \partial_x^2)^{-1/2}V_2 \] (4.3)
then from (4.2) we deduce that \((W_1, W_2, V_3)\) satisfies
\[
\begin{align*}
\partial_t &+ \left( -\alpha \partial_x + \frac{\alpha}{\partial_x} \right) + Q_0 \left( \begin{array}{c} W_1 \\ W_2 \end{array} \right) \\
&= Q_1 \left( Q_0^{(1)} W_1, Q_0^{(2)} W_2, Q_0^{(3)} V_3 \right) \\
(\partial_t - \beta^2 \partial_x^2) V_3 &= P_0^{(4)} G \left( Q_0^{(1)} W_1, Q_0^{(2)} W_2, Q_0^{(3)} V_3 \right).
\end{align*}
\] (4.4)
where \(Q_0^{(k)}(t, x, D_x) \in \Psi^{-1}_{1,0}(\mathbb{R})\) and \(Q_1(t, x, D_x) \in \Psi^{-1}_{1,0}(\mathbb{R})\).

From the assumption (1.6), we obtain
\[
(W_1, W_2) \in H^{s-1} \cap H^{r-1}_{ml}, \quad V_3 \in H^{r-1}. \] (4.5)

Employing Lemma 2.2, we can rewrite the system for \((W_1, W_2)\) as
\[
\begin{align*}
\partial_t &+ \left( -\alpha \partial_x + \frac{\alpha}{\partial_x} \right) + Q_0 \left( \begin{array}{c} W_1 \\ W_2 \end{array} \right) \\
&= Q_1 \left( T_{F_1} Q_0^{(1)} W_1 + T_{F_2} Q_0^{(2)} W_2 \right) = Q_1 \left( T_{F_1} Q_0^{(3)} V_3 + R \right)
\end{align*}
\] (4.6)
where \(R \in H^{2r-3}\).

From (4.5), we have
\[
P_0^{(4)} G \left( Q_0^{(1)} W_1, Q_0^{(2)} W_2, Q_0^{(3)} V_3 \right) \in H^{s-1}
\]
which implies
\[
V_3 \in H^s
\] (4.7)
by using the hypoellipticity of heat operators.

Thus, we deduce
\[
Q_1 T_{F_1} Q_0^{(3)} V_3 \in H^{s,s+1}. \] (4.8)

By employing Proposition 2.4 for the system (4.6), we obtain
\[
(W_1, W_2) \in H^{\min(r-1,s),\min(r-1,s+1)}(\Gamma). \] (4.9)

(1) If \(s \leq r \leq s+1\), then we have
\[
(W_1, W_2) \in H^{s-1} \cap H^{r-1}(\Gamma) \] (4.10)
which implies
\[ P_0^{(4)} G \left( Q_0^{(1)} W_1, Q_0^{(2)} W_2, Q_0^{(3)} V_3 \right) \in H^{s-1} \cap H_{ml}^{r-1}(\Gamma) \]
and
\[ V_3 \in H^s \cap H_{ml}^r(\Gamma) \]  \hspace{1cm} (4.11)
by using the hypoellipticity of heat operators again.

From (4.10) and the system (4.2) for \((V_1, V_2)\), it follows that
\[ (V_1, V_2) \in H^{r-1} \cap H_{ml}^{r-1}(\Gamma) \]
which yields
\[ u \in H^s \cap H_{ml}^r(\Gamma), \quad \theta \in H^{s-1, r} \cap H_{ml}^{r-1, r}(\Gamma). \]  \hspace{1cm} (4.12)

(2) If \( s + 1 < r \leq s + 2 \), then from (4.9) we have
\[ (W_1, W_2) \in H^{r-1} \cap H_{ml}^{r-1}(\Gamma) \]
which yields
\[ (W_1, W_2) \in H^{s-1} \cap H_{ml}^{r-1}(\Gamma) \]  \hspace{1cm} (4.13)
by using the system (4.6).

By using (4.11) with \( r \) being replaced by \( s + 1 \), and in a way similar to that from (4.10) to (4.12), we conclude the same regularity as in (4.12).

(3) If \( r > s + 2 \), from (4.9) we obtain
\[ (W_1, W_2) \in H_{ml}^{r+1}(\Gamma) \]
which implies
\[ (W_1, W_2) \in H_{ml}^{s+1}(\Gamma) \]  \hspace{1cm} (4.14)
by using the system (4.6) again.

Similar to (4.11), we have
\[ V_3 \in H^s \cap H_{ml}^{r+1}(\Gamma). \]  \hspace{1cm} (4.15)

Combining (4.14) and (4.15), it follows that
\[ P_0^{(4)} G \left( Q_0^{(1)} W_1, Q_0^{(2)} W_2, Q_0^{(3)} V_3 \right) \in H^{r-1} \cap H_{ml}^{r+1}(\Gamma) \]
which yields
\[ V_3 \in H^s \cap H_{ml}^{r+2}(\Gamma) \]  \hspace{1cm} (4.16)
and
\[ Q^{-1} T_0 T^3_0 V_3 \in H^{s,s+1} \cap H^{s+2,s+3}_{ml}(\Gamma). \]  

Employing Proposition 2.4 for the system (4.6), we deduce
\[ (W_1, W_2) \in H^{\min(r-1,s+2), \min(r-1,s+3)}_{ml}(\Gamma). \]

Obviously, in (4.15) and (4.18) we have shown that \((W_1, W_2, V_3)\) are more regular than those given in (4.11) and (4.9) respectively.

When \(r \leq s + 4\) and \(r < 2(s - 1)\), we can obtain the same results as in (4.12) in a way similar to the above. Otherwise, by continuing this process, eventually we obtain the conclusion (1.7).

### 4.2. Propagation of Singularities for Cauchy Problems

This section is devoted to the proof of Theorem 1.4. At first, let us recall two results from [7] for Cauchy problems of linear parabolic equations and of hyperbolic equations.

**Lemma 4.1.** Consider the Cauchy problem
\[ w_t - w_{xx} = g(t, x) \]
\[ w(0, x) = w_0(x) \]
and suppose that \(\gamma(t) = \{(x(t), \xi(t)) \in T^*\mathbb{R}\setminus 0 \mid 0 \leq t < T\}\) is a smooth curve, for any fixed \(0 \leq s \leq r\), \(g \in L^2(0, T; H^{s+1}(\mathbb{R}) \cap H^{r+1}_{ml}(\gamma(t)))\) and \(w_0 \in H^s(\mathbb{R}) \cap H^r_{ml}(x(0), \xi(0))\). Then, the solution \(w(t, x)\) to (4.19) has the following regularity:
\[ w \in L^2(0, T; H^{s+1}(\mathbb{R}) \cap H^{r+1}_{ml}(\gamma(t))) \]
\[ \cap H^1(0, T; H^{r-1}(\mathbb{R}) \cap H^{s-1}_{ml}(\gamma(t))). \]

**Lemma 4.2.** Assume \(N \times N\) matrices \(P_1(t, x, D_x) \in \Psi_{1,0}^{1,1}(\mathbb{R}), P_0(t, x, D_x), Q_0(t, x, D_x, D_{xx}) \in \Psi_{1,0}^{0,0}(\mathbb{R}^2)\), the operator \(L = D_x - P_1(t, x, D_x)\) is strictly hyperbolic with respect to \(t\), and \(\gamma(t) = (t, x(t); \tau(t), \xi(t))(0 \leq t < T)\) is a null bicharacteristic of \(L\). For any fixed \(3/2 < s \leq r \leq 2s - 1, 2\), given \(A(t, x) = (a_{ij})_{N\times N}\) with \(a_{ij} \in C([0, T], H^1(\mathbb{R}))\), denote by \(T_A\) the \(N \times N\) matrix \((T_{a_{ij}})_{i,j}^{t\times j}\) of paraproduct operators. For the Cauchy problem
\[ (D_x - P_1(t, x, D_x))u + P_0 T_A Q_0 u = F(t, x) \]
\[ u(0, x) = u_0(x) \]
if \(F \in L^1(0, T; H^{r} \cap H^{s}_{ml}(x(t), \xi(t)))\) and \(u_0 \in H^s \cap H^r_{ml}(x(0), \xi(0))\), then we have
\[ u \in C([0, T], H^{r} \cap H^{s}_{ml}(x(t), \xi(t))). \]
Proof of Theorem 1.4. As in (4.4), we know that \((W_1, W_2, V_3)\) satisfies the following Cauchy problem
\[
\begin{bmatrix}
\partial_t + \begin{pmatrix}
-\alpha \partial_x & 0 \\
0 & \alpha \partial_x
\end{pmatrix} + Q^{(0)}_0 \\
W_1 \\
W_2
\end{bmatrix} = Q_{-1} F(Q^{(1)}_0 W_1, Q^{(2)}_0 W_2, Q^{(3)}_0 V_3)
\]
\[
(\partial_t - \beta^2 \partial^2_x) V_3 = P^{(4)}_0 G(Q^{(1)}_0 W_1, Q^{(2)}_0 W_2, Q^{(3)}_0 V_3)
\]
\[
t = 0 : (W_1, W_2, V_3) \in H^s \cap H^t_m(x_0, \xi_0)
\]
with the same notations as in (4.4).

First, by applying the classical theory of hyperbolic and parabolic equations in (4.23), we obtain the existence and uniqueness of local solutions
\[
W_1, W_2 \in C([0, T], H^s(\mathbb{R}))
\]
\[
V_3 \in L^2(0, T; H^{s+1}(\mathbb{R})) \cap H^1(0, T; H^{r-1}(\mathbb{R}))
\]
to the problem (4.23).

By employing the paralinearization lemma for the system of \((W_1, W_2)\) in (4.23), we obtain
\[
\begin{bmatrix}
\partial_t + \begin{pmatrix}
-\alpha \partial_x & 0 \\
0 & \alpha \partial_x
\end{pmatrix} + Q^{(0)}_0 \\
W_3 \\
W_2
\end{bmatrix} - Q_{-1}(T_{E_1} Q^{(1)}_0 W_1 + T_{E_2} Q^{(2)}_0 W_2) = Q_{-1}(T_{E_3} Q^{(3)}_0 V_3 + R).
\]
where \(R \in C([0, T], H^{2r-1/2}(\mathbb{R}))\).

By applying Lemma 4.2 in the problem of \((W_1, W_2)\), we get
\[
(W_1, W_2) \in C([0, T], H^s \cap H_{ml}^{\min(r,s+1)}(x(t), \xi(t))).
\]

(1) If \(s < r \leq s + 1\), then
\[
(W_1, W_2) \in C([0, T], H^s \cap H_{m}^{s}(x(t), \xi(t))).
\]

Obviously, from (4.24) and (4.27) we have
\[
P^{(4)}_0 G(Q^{(1)}_0 W_1, Q^{(2)}_0 W_2, Q^{(3)}_0 V_3) \in C([0, T], H^s(\mathbb{R})).
\]

Employing Lemma 4.1 for the problem of \(V_3\), it follows that
\[
V_3 \in L^2(0, T; H^{s+1} \cap H_{ml}^{r+1}(x(t), \xi(t))) \cap H^1(0, T; H^{s-1} \cap H_{ml}^{r-1}(x(t), \xi(t))).
\]
From (4.27) and (4.28), we obtain
\[ u \in C([0, T], H^s \cap H^s_{ml}(x(t), \xi(t))) \]
\[ \cap C^1([0, T], H^{s-1} \cap H^{s-1}_{ml}(x(t), \xi(t))) \] (4.29)
and
\[ \theta \in C([0, T], H^s \cap H^s_{ml}(x(t), \xi(t))). \] (4.30)

(2) If \( s + 1 < r \leq s + 2 \), from (4.28) we obtain
\[ Q^{-1} F_0 \tilde{Q}^{(3)} V_3 \in L^2(0, T; H^{s+2} \cap H^{s+2}_{ml}(x(t), \xi(t))) \]
\[ \cap H^1(0, T; H^s \cap H^s_{ml}(x(t), \xi(t))) \]
which yields
\[ (W_1, W_2) \in C([0, T], H^s \cap H^s_{ml}(x(t), \xi(t))) \] (4.31)
by using Lemma 4.2.

From (4.28) and (4.31), we have
\[ P^{(4)}_0 G(\tilde{Q}^{(1)} W_1, \tilde{Q}^{(2)} W_2, \tilde{Q}^{(3)} V_3) \in C([0, T], H^s \cap H^s_{ml}(x(t), \xi(t))) \]
which yields
\[ V_3 \in L^2(0, T; H^{s+1} \cap H^{s+1}_{ml}(x(t), \xi(t))) \]
\[ \cap H^1(0, T; H^{s-1} \cap H^{s-1}_{ml}(x(t), \xi(t))) \] (4.32)
by using Lemma 4.1.

Obviously, from (4.31) and (4.32) we obtain the same conclusions as in (4.29) and (4.30) for \( u, \theta \).

(3) If \( r > s + 2 \), then by the same argument as above we can conclude the result of the theorem.

Proof of Theorem 1.5. For any \((t_0, x_0)\) satisfying \( x_1(t_0) < x_0 < x_2(t_0) \), and \( \xi_0 \in \mathbb{R} \setminus 0 \), let \( \tau_0 \in \mathbb{R} \) be such that \( P_0 = (t_0, x_0, \tau_0, \xi_0) \) is a characteristic point for the operator \( L = \partial^2_t - \alpha^2 \partial^2_x \); i.e., \( \tau_0 = \pm \alpha(t_0, x_0) \xi_0 \).

Denote by \( \Gamma(t) = \{(t, x(t), \tau(t), \xi(t))\} \subset T^*(\mathbb{R}^2) \setminus 0 \) a null bicharacteristic of \( L \) passing through \( P_0 \). Obviously, the projection in \((t, x)\)-space of \( \Gamma(t) \) intersects with \( \{t = 0\} \) at \( x \neq 0 \), where \((u, \theta)\) is smooth by using Theorem 1.1 of [19].

Thus, by applying Theorem 1.4 we obtain
\[ u \in C([0, T], H^{2r-1/2}_{ml}(x(t), \xi(t))) \]
\[ \cap C^1([0, T], H^{2r-3/2}_{ml}(x(t), \xi(t))) \] (4.33)
\[ \theta \in C([0, T], H^{2r-1/2}_{ml}(x(t), \xi(t))). \]
If \((t_0, x_0; \tau_0, \xi_0)\) is not a characteristic point of \(L\), and \(\Gamma(t) = \{(t, x(t), \tau(t), \xi(t))\}\) is a bicharacteristic of \(L\) passing through \((t_0, x_0; \tau_0, \xi_0)\), then each point on \(\Gamma(t)\) is noncharacteristic for \(L\). By using Rauch’s lemma [13] and a classical bootstrap argument for the system (4.23), we can easily obtain (4.33) as well. Thus, from (4.33) and the arbitrariness of \((t_0, x_0)\) we immediately conclude (1.10).

It is not difficult to extend this result to the case that the initial data of \(u\) and \(\theta\) have singularities on an interval or at finite points on \(\{t = 0\}\). This result shows that for the Cauchy problem of the thermoelastic system (1.1), if the initial data have singularities on an interval, then in the lacuna issuing from this interval, the regularity of the solutions \(u\) and \(\theta\) with respect to the \(x\)-variable is similar to the case in hyperbolic problems [2].

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