Highly oscillatory waves in quasilinear hyperbolic-parabolic coupled equations

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Abstract. In this paper, we study the Cauchy problem for a quasi-linear hyperbolic-parabolic coupled system in several space variables with highly oscillatory initial data. We show that the oscillations are still propagated along the characteristics of the hyperbolic operators, and some profiles of oscillations satisfy hyperbolic problems while the other profiles of oscillations are dissipated by the parabolic effect of the system. By means of nonlinear geometric optics, we derive the formal expansions of oscillatory waves and deduce that the leading oscillation profile satisfies a nonlinear degenerate parabolic system. Furthermore, we rigorously justify the asymptotic expansion and obtain the existence of the highly oscillatory solutions in a time interval independent of the wavelength.

Key words: Hyperbolic-parabolic coupled system, Cauchy problems, nonlinear geometric optics, highly oscillatory waves, propagation and dissipation

AMS Subject Classification: 35M10, 78A05, 35B05

1 Introduction

In this paper, we consider the Cauchy problem for the following quasi-linear hyperbolic-parabolic coupled system with a highly oscillatory initial data,

\[
L(u, \partial)u - \varepsilon^2 \sum_{j,k=1}^{n} B_{jk}\partial_j\partial_k u = F(t, x, u) \tag{1.1}
\]

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\[ u(0, x) = u_0(x) + \varepsilon u_0^0(x, \frac{\varphi_0(x)}{\varepsilon}) \]  

(1.2)

where

\[ L(u, \partial) = A_0(u)\partial_t + \sum_{j=1}^{n} A_j(u)\partial_j, \]

\[ t \in \mathbb{R}^+ \text{ and } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n(n \geq 1), \]

\[ u = (u_1, u_2, \cdots, u_N)^T \] is a N-vector valued function, \( A_0(u), A_j(u)(1 \leq j \leq n), B_{jk}(u)(1 \leq j, k \leq n) \) are \( N \times N \) matrices and \( F(t, x, u) \in \mathbb{R}^N, u_0^0(x, \theta) \) is periodic in \( \theta \in T^1 = \mathbb{R}/2\pi\mathbb{Z} \), \( \varphi_0(x) \) is a given phase with \( \nabla \varphi_0(x) \neq 0 \) for all \( x \in \mathbb{R}^n \). To ensure the well-posedness of the system (1.1)(1.2), we assume that

(i) \( A_0(u), A_j(u)(1 \leq j \leq n), \) and \( F(t, x, u) \) are sufficiently smooth and \( A_0(u) \) is positively defined, \( A_j(u)(1 \leq j \leq n) \) are symmetric.

(ii) \( B_{jk}(u)(1 \leq j, k \leq n) \) are block diagonal

\[ B_{jk} = \begin{pmatrix} 0 & 0 \\ 0 & \overline{B}_{jk} \end{pmatrix} \]

and \( u = (u_1, u_{11})^T \) is the corresponding decomposition.

(ii) \( \tilde{B}_{jk} = \tilde{B}_{ij}, \sum_{j,k=1}^{n} B_{jk} w_j w_k \geq \gamma \sum_{k=1}^{n} |(w_k)_{11}|^2, \) for a positive constant \( \gamma > 0 \) with any \( w_i(1 \leq i \leq n) \in \mathbb{R}^N, w_i = ((w_i)_I, (w_i)_{II})^T. \)

The problem (1.1)(1.2) is the quasi-linear hyperbolic-parabolic system with highly oscillatory initial data. It is well known that the oscillations in the hyperbolic system are propagated along characteristic. But as to the purely parabolic system, the oscillations are damped by the dissipation when the initial data is highly oscillatory. We are interested in how the oscillations will be propagated for the hyperbolic-parabolic problem (1.1)(1.2). To this end, we shall use the approach of nonlinear geometric optics to investigate the problem. At a starting point, in this paper we only consider the propagation of oscillatory waves with two phases. We want to seek that the problem (1.1)(1.2) has a solution with the asymptotic expansion being the form

\[ u^\varepsilon(t, x) \sim U_0(t, x) + \sum_{j \geq 1} \varepsilon^j U_j(t, x, \frac{\varphi_1(t, x)}{\varepsilon}, \frac{\varphi_2(t, x)}{\varepsilon}) \]  

(1.3)

where " \( \sim \) " means that for some \( M, \)

\[ u^\varepsilon(t, x) = \left[ U_0(t, x) + \sum_{j=1}^{M} \varepsilon^j U_j(t, x, \frac{\varphi_1(t, x)}{\varepsilon}, \frac{\varphi_2(t, x)}{\varepsilon}) \right] = o(\varepsilon^M) \]  

(1.4)

in certain space and the profiles \( U_j(t, x, \theta_1, \theta_2) \) being \( 2\pi - \text{periodic} \) with respect to the variables \( \theta_1 \) and \( \theta_2, \) with \( \theta_1, \theta_2 \in T^1 = \mathbb{R}/2\pi\mathbb{Z}, \) where \( \varphi_1(t, x) \) and \( \varphi_2(t, x) \) are two phase functions which will be determined later. The results obtained in this paper show that the oscillations in the hyperbolic-parabolic problem (1.1)(1.2) are still propagated along the characteristics of the hyperbolic operators, and some profiles of oscillations satisfy hyperbolic problems while the other profiles of oscillations
are dissipated by the parabolic effect of the system. Many important fluid mechanics and physical models can be described by the hyperbolic-parabolic equations (1.1), such as the compressible viscous flows and thermoelasticity. We shall give a rigorous theory of the oscillatory waves for (1.1)(1.2).

The approach for studying the oscillatory waves (1.3) is called the nonlinear geometric optics in the literature. There have been many interesting works on the rigorous theory or formal analysis of the nonlinear geometric optics for hyperbolic problems. One can refer to the recent monograph [13], the survey lecture notes [14] and references therein. In [11] and [2], A.Majda et al. constructed a formal leading order asymptotic approximation for high frequency small amplitude resonantly interacting waves in a single variable and several space variables. J.-L. Joly, G. Metiver and J. Rauch in [4] rigorously proved the validity of the asymptotic expansions of weakly nonlinear optics in resonant interaction of oscillatory waves in one space variable. For the multidimensional semilinear hyperbolic systems, J.-L. Joly and J. Rauch justified high frequency nonlinear asymptotic expansions with single phase in [7]. By using the coherent assumption, J.-L. Joly, G. Metiver and J. Rauch [6] showed that the oscillatory Cauchy problem is well-posed and exact solutions are described asymptotically by the leading term of a formal asymptotic expansion. When one wants to construct a complete expansion, more technical assumptions are needed. In [5], J.-L. Joly, G. Metiver and J. Rauch used the coherent hypothesis of oscillation phases and the small divisor property to discuss the generic oscillatory waves described by asymptotic expansions of infinite order, they prove the approximate solutions close to exact solutions of semilinear and quasilinear hyperbolic systems. As mentioned at above, it is interesting to study the behavior of highly oscillatory waves in hyperbolic-parabolic problems. O.Gues in [1] first studied the behavior of high frequency oscillations to the characteristic boundary value problem for a semilinear parabolic system in the small viscosity limit with the single phase being associated with the the same characteristics of the boundary, in which he also considered the behavior of boundary layers. Recently, Junca [8] discussed the propagation of high frequency oscillations for one dimensional semilinear strictly hyperbolic system with a small parabolic perturbation with several phases under the transversality conditions. In this paper, we propose to study the high frequency oscillations for quasi-linear hyperbolic-parabolic systems in several space variables, for which we should pay attention on the interaction of waves, and the hyperbolic and parabolic effects on the oscillations.

The remainder of this paper is arranged as follows. In Section 2, we construct the formal asymptotic expansions of the high oscillatory solutions to the problem (1.1)(1.2). Section 3 is devoted to the study of the existence and uniqueness of the leading oscillatory profiles. The asymptotic properties of oscillations will be proved in Section 4, which rigorously justified the nonlinear geometric optics of two oscillatory waves in hyperbolic-parabolic coupled systems. In Section 5, we apply the result to study the highly oscillatory waves in two-dimensional isentropic compressible Navier-Stokes equations.
2 Formally asymptotic expansions

In this section, we study formally the asymptotic expansion (1.3) of the solution to the problem (1.1)(1.2) and derive the problems of the profiles \( U_j(t,x,\theta_1,\theta_2) \). One plugs (1.3) into (1.1) and expands \( A_0(u^r), A_j(u^r)(1 \leq j \leq n), B_{jk}(u^r)(1 \leq j, k \leq n) \) and \( F(t, x, u^r) \in \mathbb{R}^N \) by the Taylor series about \( U_0(t, x) \). Setting the coefficients of the powers \( e^l \) equal to zero for \( l = 0, 1 \) and \( l \geq 2 \) yields the equations:

\[
L(U_0, \partial) U_0 = F(t, x, U_0) + P(U_0, \partial_\theta) U_1 = 0 \quad (2.1)
\]

\[
L(U_0, \partial) U_1 - \sum_{r,s=1}^{2} D_{rs}(U_0, \nabla \varphi_1, \nabla \varphi_2) \partial_{r,s}^2 U_1 + \tilde{L}(U_0, \partial U_0) U_1
+ \tilde{P}(U_0, U_1, \partial_\theta) U_1 + P(U_0, \partial_\theta) U_2 = 0 \quad (2.2)
\]

and

\[
L(U_0, \partial) U_l - \sum_{r,s=1}^{2} D_{rs}(U_0, \nabla \varphi_1, \nabla \varphi_2) \partial_{r,s}^2 U_1 + \tilde{L}(U_0, \partial U_0) U_1
+ \tilde{P}(U_0, U_1, \partial_\theta) U_l + P(U_0, \partial_\theta) U_{l+1} + G_l = 0, \quad l \geq 2 \quad (2.3)
\]

where

\[
P(U_0, \partial_\theta) = \sum_{k=1}^{2} (A_0(U_0) \partial_\varphi_k + \sum_{j=1}^{n} A_j(U_0) \partial_j \varphi_k) \partial_{\theta_k}
\]

\[
D_{rs}(U_0, \nabla \varphi_1, \nabla \varphi_2) = \sum_{j,k=1}^{n} \partial_j \varphi_r \partial_k \varphi_s B_{jk}(U_0)
\]

\[
\tilde{L}(U_0, \partial U_0) V = \partial_u A_0(U_0) V \partial_\theta U_0 + \sum_{j=1}^{n} \partial_u A_j(U_0) V \partial_j U_0 - \partial_u F(U_0) V
\]

\[
\tilde{P}(U_0, U_1, \partial_\theta) = \sum_{k=1}^{2} [\partial_u A_0(U_0) U_1 \partial_\varphi_k + \sum_{j=1}^{n} \partial_u A_j(U_0) U_1 \partial_j \varphi_k] \partial_{\theta_k}
\]

and \( G_l(l \geq 2) \) depend smoothly on \( \{U_k\}_{k \leq l-1} \) and their derivatives up to order two.

Acting the mean value operator

\[
m_{\theta_1, \theta_2}(u) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} u(\theta_1, \theta_2) d\theta_1 d\theta_2
\]

on the equation (2.1), we deduce that \( U_0(t, x) \) satisfies the following hyperbolic system:

\[
A_0(U_0) \partial_t U_0 + \sum_{j=1}^{n} A_j(U_0) \partial_j U_0 = F(t, x, U_0). \quad (2.4)
\]

From (1.2), it is natural to impose the initial data

\[
U_0(0, x) = u_0(x) \quad (2.5)
\]

By using the classical theory for the Cauchy problem (2.4)(2.5) of quasi-linear symmetric hyperbolic system, we obtain:
Proposition 2.1 (see [9]) For any given \( u_0 \in H^s(\mathbb{R}^n) \) with \( s > \frac{n}{2} + 1 \), the problem (2.4)-(2.5) has a unique solution \( U_0 \in C([0,T];H^s(\mathbb{R}^n)) \cap C^1([0,T];H^{s-1}(\mathbb{R}^n)) \) for some \( T > 0 \).

In order to study \( U_j(j \geq 1) \), we first introduce the eigenvalues and eigenvectors of (2.4). For any fixed \( \vec{\xi} = (\xi_1, \cdots, \xi_n)^T \in \mathbb{R}^n \setminus \{0\} \), the characteristic equation

\[
\text{det}(\lambda A_0(U_0) + \sum_{j=1}^{n} \xi_j A_j(U_0)) = 0
\]

has the roots \( \{\lambda_k(U_0, \vec{\xi})\}_{k=1}^{N} \). Denote by \( \{\vec{r}_i\}_{i=1}^{n} \) and \( \{\vec{l}_i\}_{i=1}^{n} \) the associated right and left eigenvectors, respectively,

\[
(\lambda_k A_0(U_0) + \sum_{j=1}^{n} \xi_j A_j(U_0))\vec{r}_i = 0,
\]

\[
\vec{l}_i(\lambda_k A_0(U_0) + \sum_{j=1}^{n} \xi_j A_j(U_0)) = 0,
\]

with the normalization

\[
\vec{l}_i A_0 \vec{r}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}
\]

As to (2.6), we assume that

\[
\text{(H3)} \quad \forall \vec{\xi} \in \mathbb{R}^n \setminus \{0\}, \text{ (2.6) has at least two different roots. Without loss of generality, we suppose}
\]

\[
-\infty < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N < +\infty.
\]

\[
\text{(H4)} \quad \forall \vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n \setminus \{0\}, 1 \leq i \leq N, \text{ either}
\]

\[
\vec{l}_i(U_0, \vec{\xi})(\sum_{j,k=1}^{n} B_{jk}(U_0) \xi_j \xi_k)\vec{r}_i(U_0, \vec{\xi}) \equiv 0
\]

or

\[
\vec{l}_i(U_0, \vec{\xi})(\sum_{j,k=1}^{n} B_{jk}(U_0) \xi_j \xi_k)\vec{r}_i(U_0, \vec{\xi}) \geq \beta_0 |\vec{\xi}|^2 > 0
\]

for a positive constant \( \beta_0 \).

Remark 2.2 The assumption (H4) holds for many physical models, for example, it is true for the compressible Navier-Stokes equations and the equations of thermoelasticity ([12]).
The difference between (2.1) and (2.4) gives

\[ P(U_0, \partial_0)U_1 = 0. \]  

(2.10)

In order to have a solution \( U_1(t, x, \theta_1, \theta_2) \) with nontrivial \( \theta_1, \theta_2 \) dependence, which corresponds to rapid oscillation when \( \varepsilon \) tends to zero, \( \varphi_k(t, x)(k=1, 2) \) must satisfy

\[ \det(A_0(U_0))\partial_k \varphi_k + \sum_{j=1}^{n} A_j(U_0) \partial_j \varphi_k = 0, \quad k = 1, 2. \]  

(2.11)

Equation (2.11) is the eikonal equation. It asserts that \( \partial \varphi k(k=1, 2) \) belong to the characteristic variety of the symmetric hyperbolic operator \( A_0(U_0)\partial_t + \sum_{j=1}^{n} A_j(U_0)\partial_j \). Thus, we require that the phase functions \( \varphi_1(t, x) \) and \( \varphi_2(t, x) \) satisfy:

\[
\begin{cases}
\partial_t \varphi_k = \lambda_k(U_0(t, x), \nabla \varphi_k), & k = 1, 2 \\
\varphi_k(0, x) = \varphi_0(x)
\end{cases}
\]  

(2.12)

**Proposition 2.3** For \( T > 0 \) and \( s > \frac{n}{2} + 1 \) given in Proposition 2.1, assume that \( \varphi_0 \in H^s(\mathbb{R}^n) \), then there exists \( T_1, 0 < T_1 \leq T \) such that the problem (2.12) has unique solutions \( \varphi_k \in L^\infty(0, T_1; H^s(\mathbb{R}^n)) \) for \( k = 1, 2 \).

Thanks to (2.11), from (2.10) we have

\[
U_1(t, x, \theta_1, \theta_2) = \bar{U}_1(t, x) + \sigma_1(t, x, \theta_1, \theta_2) \bar{r}_1(U_0, \nabla \varphi_1) + \sigma_2(t, x, \theta_1, \theta_2) \bar{r}_2(U_0, \nabla \varphi_2)
\]  

(2.13)

for two scalar functions \( \sigma_1 \) and \( \sigma_2 \) being \( 2\pi \)-periodic in \( \theta_1, \theta_2 \), and \( \bar{U}_1(t, x) = m_{\theta_1, \theta_2}(U_1) \) denoting the mean value of \( U_1 \). Moreover, we have \( \sigma_1(t, x, \theta_1, \theta_2) = \sigma_1(t, x, \theta_1), \quad \sigma_2(t, x, \theta_1, \theta_2) = \sigma_2(t, x, \theta_2) \) in \([0, T_0]\) for some \( T_0 > 0 \), i.e. \( \sigma_1 \) (resp. \( \sigma_2 \)) is independent of \( \theta_2 \) (resp. \( \theta_1 \)).

Indeed, by plugging (2.13) into (2.10) it follows

\[ [A_0(U_0)\partial_t \varphi_1 + \sum_{j=1}^{n} A_j(U_0)\partial_j \varphi_1] \partial_\theta_1 \sigma_1 \bar{r}_1(U_0, \nabla \varphi_1) + [A_0(U_0)\partial_t \varphi_2 + \sum_{j=1}^{n} A_j(U_0)\partial_j \varphi_2] \partial_\theta_2 \sigma_2 \bar{r}_2(U_0, \nabla \varphi_1) = 0. \]  

(2.14)

Multiplying \( \bar{r}_1(U_0, \nabla \varphi_1) \) from the left of (2.14) and using (2.7), we get

\[ [\partial_t \varphi_2 + \sum_{j=1}^{n} \bar{r}_1 A_j(U_0) \bar{r}_1 \partial_j \varphi_2] \partial_\theta_2 \sigma_1 = 0. \]  

(2.15)

As \( \varphi_1(0, x) = \varphi_2(0, x) = \varphi_0(x) \), so at \( t = 0 \), we have

\[ \sum_{j=1}^{n} \bar{r}_1 A_j(U_0) \bar{r}_1 \partial_j \varphi_2 = \sum_{j=1}^{n} \bar{r}_1 A_j(U_0) \bar{r}_1 \partial_j \varphi_1 = -\partial_t \varphi_1. \]

As \( \varphi_0(0, x) = \varphi_2(0, x) = \varphi_0(x) \), so at \( t = 0 \), we have

\[ \sum_{j=1}^{n} \bar{r}_1 A_j(U_0) \bar{r}_1 \partial_j \varphi_2 = \sum_{j=1}^{n} \bar{r}_1 A_j(U_0) \bar{r}_1 \partial_j \varphi_1 = -\partial_t \varphi_1. \]  

From (2.8), we know \( \partial_t \varphi_1 \neq \partial_t \varphi_2 \), thus there is \( T_0 > 0 \), such that for \( t \in [0, T_0] \),

\[ q_1 := \partial_t \varphi_2 + \sum_{j=1}^{n} \bar{r}_1 A_j(U_0) \bar{r}_1 \partial_j \varphi_2 \neq 0. \]  

(2.16)
Thus, from (2.15) we get \( \partial_0 \sigma_1 = 0 \) on \([0, T_0]\), i.e. \( \sigma_1 = \sigma_1(t, x, \theta_1) \).

Similarly, from (2.14) one can deduce

\[
q_2 := \partial_1 \varphi_1 + \sum_{j=1}^n \tilde{l}_2 A_j(U_0) \tilde{r}_2 \partial_j \varphi_1 \neq 0
\]  

(2.17)

for \( 0 \leq t \leq T_0 \), from (2.14) it follows \( \sigma_2 = \sigma_2(t, x, \theta_1) \) on \([0, T_0]\).

Then, (2.13) becomes

\[
U_1(t, x, \theta_1, \theta_2) = \tilde{U}_1(t, x) + \sigma_1(t, x, \theta_1) \tilde{r}_1(U_0, \nabla \varphi_1) + \sigma_2(t, x, \theta_2) \tilde{r}_2(U_0, \nabla \varphi_2)
\]  

(2.18)

where \( \sigma_1, \sigma_2 \) are two scalar unknowns and \( 2\pi \) – periodic in \( \theta_1 \) and \( \theta_2 \), respectively, with mean value vanishing.

Plugging (2.18) into (2.2), and acting the mean value operator \( m_{\theta, \theta} \) on the resulting equation, we deduce that the mean value \( \bar{U}_1(t, x) = (m_{\theta, \theta} \bar{U}_1)(t, x) \) satisfies the following linear system:

\[
L(U_0, \partial) \bar{U}_1 + \bar{L}(U_0, \partial U_0) \bar{U}_1 = 0
\]  

(2.19)

with the initial data

\[
\bar{U}_1(0, x) = \bar{u}_1^0(x)
\]  

(2.20)

given in (2.9).

The difference between (2.19) and (2.2) gives

\[
\sum_{k=1}^2 \left[ (L(U_0, \partial) \sigma_k) \tilde{r}_k + \sigma_k L(U_0, \partial) \tilde{r}_k \right] - [D_{11}(U_0, \nabla \varphi_1, \nabla \varphi_2) \partial_{\theta_1}^2 \sigma_1 \tilde{r}_1

+ D_{22}(U_0, \nabla \varphi_1, \nabla \varphi_2) \partial_{\theta_2}^2 \sigma_2 \tilde{r}_2] + \sum_{j=1}^n \left[ \partial_{\theta_1}^A A_j(U_0)(\bar{U}_1 + \sigma_1 \tilde{r}_1 + \sigma_2 \tilde{r}_2) \partial_j \varphi_k \right]

\]

(2.21)

Multiplying \( \tilde{l}_1(U_0, \nabla \varphi_1) \) from the left of (2.21) and acting upon the mean value operator \( m_{\theta, \theta}(u) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta_2) d\theta_2 \), we obtain that \( \sigma_1(t, x, \theta_1) \) satisfies the following equation:

\[
\partial_t \sigma_1 + \sum_{j=1}^n (\tilde{l}_1 A_j(U_0) \tilde{r}_1) \partial_j \sigma_1 - i \sum_{j,k=1}^n \partial_j \varphi_1 \partial_k \varphi_1 \tilde{l}_1 B_{jk}(U_0) \tilde{r}_1 \partial_{\theta_1}^2 \sigma_1

+ \tilde{l}_1 [\partial_1 \varphi_1 \partial_{\theta_1} A_0(U_0) \tilde{r}_1] + \sum_{j=1}^n \partial_j \varphi_1 \partial_{\theta_1} A_j(U_0) \tilde{r}_1 \partial_1 \sigma_1 \partial_{\theta_1} \sigma_1

+ \tilde{l}_1 [\partial_1 \varphi_1 \partial_{\theta_1} A_0(U_0) \bar{U}_1] + \sum_{j=1}^n \partial_j \varphi_1 \partial_{\theta_1} A_j(U_0) \bar{U}_1 \tilde{r}_1 \partial_{\theta_1} \sigma_1

+ \tilde{l}_1 [L(U_0, \partial) \tilde{r}_1 + \bar{L}(U_0, \partial U_0) \tilde{r}_1] \sigma_1 = 0.
\]  

(2.22)
Denote by
\[
\begin{aligned}
a_l^j(t, x) &= \tilde{L}A_j(U_0)\tilde{r}_l, j = 1, \cdots, n, \\
b_l(t, x) &= \sum_{j,k=1}^n \partial_j\phi_i\partial_k\phi_i\tilde{L}B_{jk}(U_0)\tilde{r}_l, \\
c_l(t, x) &= \tilde{L}[\partial_t\phi_i, \partial_uA_0(U_0)\tilde{r}_l + \sum_{j=1}^n \partial_j\phi_i, \partial_uA_j(U_0)][\tilde{r}_l], \\
d_l(t, x) &= \tilde{L}[\partial_uA_0(U_0)\tilde{U}_1\partial_t\phi_i + \sum_{j=1}^n \partial_uA_j(U_0)\tilde{U}_j\partial_t\phi_i][\tilde{r}_l], \\
e_l(t, x) &= \tilde{L}[L(U_0, \partial)\tilde{r}_l + \tilde{L}(U_0, \overline{\partial}U_0)[\tilde{r}_l].
\end{aligned}
\]  
for \(i = 1, 2\). Obviously, the equation (2.22) can be rewritten as
\[
\partial_t\sigma_1 + \sum_{j=1}^n a_1^j(t, x)\partial_j\sigma_1 - b_1(t, x)\partial_t^2\sigma_1 + c_1(t, x)\sigma_1 \cdot \partial_t\sigma_1 + d_1(t, x)\partial_t\sigma_1 + e_1(t, x)\sigma_1 = 0
\]
Similarly, from (2.21) we can deduce that \(\sigma_2(t, x, \theta_2)\) satisfies the equation:
\[
\partial_t\sigma_2 + \sum_{j=1}^n a_2^j(t, x)\partial_j\sigma_2 - b_2(t, x)\partial_t^2\sigma_2 + c_2(t, x)\sigma_2 \cdot \partial_t\sigma_2 + d_2(t, x)\partial_t\sigma_2 + e_2(t, x)\sigma_2 = 0
\]
with the coefficients being given in (2.23). From (2.9) we know that the initial data for the equations (2.24) and (2.25) are given by
\[
\sigma_1|_{t=0} = \sigma^0_1(x, \theta_1), \quad \sigma_2|_{t=0} = \sigma^0_1(x, \theta_2).
\]

**Remark 2.4** As \(\nabla\phi_i \neq 0 (i = 1, 2)\), from the assumption (H4) we know that \(b^i(t, x) (i = 1, 2)\) is either zero or positive, so the oscillation profiles \(\sigma_1(t, x, \theta_1)\) and \(\sigma_2(t, x, \theta_1)\) satisfy the Burgers type equation as obtained in [10] for the purely hyperbolic problems, or the Burgers type with a partial viscous term showing the dissipative effect from the parabolic operators given in (1.1).

Once \(\sigma_1(t, x, \theta_1), \sigma_2(t, x, \theta_2)\) are solved from (2.24), (2.25) and (2.26), combining with the solution \(\tilde{U}_1(t, x)\) of (2.19) and (2.20), we get the leading profile \(U_1(t, x, \theta_1, \theta_2)\) uniquely. We shall discuss the problems of \(\sigma_1(t, x, \theta_1)\) and \(\sigma_2(t, x, \theta_2)\) in Section 3. In what follows, let us briefly derive problems for high order profiles in the expansion (1.3).

By induction, suppose that \(\{U_k(t, x, \theta_1, \theta_2)\}_{k \geq 0}\) are known already, we want to determine the profile \(U_{l+1}(t, x, \theta_1, \theta_2)\).

It follows from (2.3) that
\[
P(U_0, \partial\theta)U_{l+1} = \tilde{f}_l
\]
where \(\tilde{f}_l\) is a smooth function of \(\{U_k(t, x, \theta_1, \theta_2)\}_{k \leq l}\) and their derivatives up to order two.

Set
\[
U_{l+1}(t, x, \theta_1, \theta_2) = U_{l+1}^{(0)}(t, x) + \sum_{k=1}^{2} V_{l+1}^{(k)}(t, x, \theta_1, \theta_2)\tilde{r}_k(U_0, \nabla\phi_k).
\]
In order to solve $\bar{U}_{t+1}(t,x)$, acting mean value operator $m_{\theta_1, \theta_2}$ on (2.3) with $l$ being replaced by $l+1$, one gets that $\bar{U}_{t+1}(t,x)$ satisfies the following system:

$$L(U_0, \partial)\bar{U}_{t+1} + \tilde{L}(U_0, \partial U_0)\bar{U}_{t+1} + \tilde{g}_{t+1} = 0$$

(2.29)

with the initial data

$$\bar{U}_{t+1}|_{t=0} = 0,$$

(2.30)

where $\tilde{g}_{t+1}$ only depends on $\{\bar{U}_k(t,x)\}_{k \leq l}$ and their derivatives. The problem (2.29)(2.30) is similar to that one given in (2.19)(2.20).

Plugging (2.28) into (2.27) and multiplying $L(U_0, \nabla \phi_k)(k = 1, 2)$ from the left of the resulting equation to yield

$$q_1 \partial_{\theta_2} V_{t+1}^{(1)} = \tilde{l}_1(U_0, \nabla \phi_1) \cdot \tilde{f}_1$$

(2.31)

and

$$q_2 \partial_{\theta_1} V_{t+1}^{(2)} = \tilde{l}_2(U_0, \nabla \phi_2) \cdot \tilde{f}_1$$

(2.32)

where $q_1$ and $q_2$ are non-zero functions for $t \in [0, T_0]$ defined in (2.16) and (2.17) respectively. So, we get

$$V_{t+1}^{(1)}(t, x, \theta_1, \theta_2) = q_1^{-1} \int_0^{\theta_2} \tilde{l}_1(U_0, \nabla \phi_1) \cdot \tilde{f}_1 d\theta_2 + V_{t+1}^{(1)}(t, x, \theta_1, 0)$$

(2.33)

and

$$V_{t+1}^{(2)}(t, x, \theta_1, \theta_2) = q_2^{-1} \int_0^{\theta_1} \tilde{l}_2(U_0, \nabla \phi_2) \cdot \tilde{f}_1 d\theta_1 + V_{t+1}^{(2)}(t, x, 0, \theta_2).$$

(2.34)

To determine $V_{t+1}^{(1)}(t, x, \theta_1, 0)$ and $V_{t+1}^{(2)}(t, x, 0, \theta_2)$, we plug (2.33) and (2.34) into (2.28) to get

$$U_{t+1}(t, x, \theta_1, \theta_2) = \bar{U}_{t+1}(t, x) + \bar{U}_{t+1}(t, x, \theta_1, \theta_2)$$

$$+ V_{t+1}^{(1)}(t, x, \theta_1, 0)\tilde{r}_1(U_0, \nabla \phi_1) + V_{t+1}^{(2)}(t, x, 0, \theta_2)\tilde{r}_2(U_0, \nabla \phi_2)$$

where

$$\bar{U}_{t+1}(t, x, \theta_1, \theta_2) = q_1^{-1} \int_0^{\theta_2} \tilde{l}_1(\nabla \phi_1) \cdot \tilde{f}_1 d\theta_2 \cdot \tilde{r}_1(\nabla \phi_1) + q_2^{-1} \int_0^{\theta_1} \tilde{l}_2(\nabla \phi_2) \cdot \tilde{f}_1 d\theta_1 \cdot \tilde{r}_2(\nabla \phi_2)$$

is known.

Thanks to $V_{t+1}^{(1)}(t, x, \theta_1, 0)$ and $V_{t+1}^{(2)}(t, x, 0, \theta_2)$ only depend on $\theta_1$ and $\theta_2$, respectively, (2.35) is similar to (2.18). Using the same arguments as deriving the problems (2.24),(2.25),(2.26) of $\sigma_1(t, x, \theta_1)$ and $\sigma_2(t, x, \theta_2)$ from (2.21), by studying the equations (2.3) of the $(l+1)$-case we deduce that for $i = 1, 2$, $V_{t+1}^{(i)}(t, x, \theta_1, 0)$ satisfy the following linear problems:

$$\begin{cases}
\partial_t V_{t+1}^{(i)} + \sum_{j=1}^n a_j'(t,x) \partial_j V_{t+1}^{(i)} - b'(t,x) \partial_{\theta_1} V_{t+1}^{(i)} + \tilde{a}'(t,x) \partial_{\theta_1} V_{t+1}^{(i)} + c'(t,x) V_{t+1}^{(i)} = \tilde{q}_{i,t+1} \\
V_{t+1}^{(i)}|_{t=0} = 0
\end{cases}$$
where for \( i = 1, 2 \), \( a_j(t, x), b_j(t, x), c_j(t, x) \) are defined in (2.23),

\[
\ddot{d}(t, x) = \ddot{U}_1[\partial_\theta \varphi_i \partial_u A_0(U_0)u_1 + \sum_{j=1}^n \partial_\theta \varphi_i \partial_u A_j(U_0)u_1]\tilde{r}_i,
\]

(2.35)

and \( \tilde{q}_{i,t+1} \) depend only on \( \{U_k\}_{k \leq t}, \tilde{U}_{t+1}(t, x), \tilde{U}_{t+1}(t, \theta_1, \theta_2) \) and their derivatives up to order two.

Till now, we have derived the problems of the profiles \( U_{i+1}(t, x, \theta_1, \theta_2) \).

### 3 Existence of solutions to the profile equations

It is clear from (2.24), (2.25) and (2.26) that to determine the leading order profiles \( \sigma_1(t, x, \theta_1) \) and \( \sigma_2(t, x, \theta_2) \), we need to study the following problem in \( \{t > 0, x \in \mathbb{R}^n, \theta \in T^1\} \):

\[
\begin{aligned}
\partial_t \sigma + \sum_{j=1}^n a_j(t, x) \partial_j \sigma - b(t, x) \partial_\theta \sigma + c(t, x) \partial_\theta \sigma + d(t, x) \partial_\theta \sigma + e(t, x) \sigma &= 0 \\
\sigma(0, x, \theta) &= \sigma_0(x, \theta)
\end{aligned}
\]

(3.1)

where \( a_j(t, x), b(t, x), c(t, x), d(t, x), e(t, x) \in L^\infty(0, T; H^s(\mathbb{R}^n)) \) for a fixed \( s > \frac{n}{2} + 1 \) and \( T > 0 \), with \( b(t, x) \) being non-negative. The equation of (3.1) is a quasilinear degenerate parabolic equation.

Its well-posedness can be obtained by using the classical theory ([3]). For completeness, we give the main steps of the proof of the well-posedness in this section.

With respect to the nonlinear (3.1), first we consider the following linearized problem:

\[
\begin{aligned}
\partial_t \sigma + \sum_{j=1}^n a_j(t, x) \partial_j \sigma - b(t, x) \partial_\theta \sigma + c(t, x) \partial_\theta \sigma + d(t, x) \partial_\theta \sigma + e(t, x) \sigma &= f(t, x, \theta) \\
\sigma(0, x, \theta) &= \sigma_0(x, \theta)
\end{aligned}
\]

(3.2)

for given \( \omega(t, x, \theta), f(t, x, \theta) \) and \( \sigma_0(x, \theta) \) with \( m_\theta f = m_\theta \sigma_0 = m_\theta \omega = 0 \).

Thanks to (2.23) and (H4), we have

\[
b(t, x) \equiv 0, \text{ or } b(t, x) \geq \beta_0 |\nabla \varphi| \geq \beta > 0
\]

(3.3)

for a positive constant \( \beta \).

**Lemma 3.1** (Moser-type inequalities; see [9]) Suppose \( u \in H^s, \nabla u \in L^\infty, \) and \( v \in H^{s-1} \cap L^\infty, \) for any multi-index \( \alpha \) with \( |\alpha| \leq s \), we have \( \partial^\alpha (uv) - u \partial^\alpha v \in L^2 \) and

\[
\|\partial^\alpha (uv) - u \partial^\alpha v\|_{L^2} \leq C_s(\|\nabla u\|_{L^\infty} \|D^{s-1}v\|_{L^2} + \|D^s u\|_{L^2} \|u\|_{L^\infty})
\]

where \( \|D^s u\|_{L^2} = \sum_{|\alpha|=s} \|\partial^\alpha u\|_{L^2} \).

**Theorem 3.2** Suppose that \( s > \frac{n+1}{2} + 1, s \in \mathbb{N}, \) for any fixed \( T > 0, \sigma_0 \in H^s(\mathbb{R}^n \times T^1), \omega \in L^1(0, T; H^s(\mathbb{R}^n \times T^1)), f \in L^2(0, T; H^s(\mathbb{R}^n \times T^1)) \), then the problem (3.2) has a unique solution.
\( \sigma \in C([0,T]; H^s(\mathbb{R}^n \times T^1)) \) with \( \partial_0 \sigma \in C([0,T]; H^{s-1}(\mathbb{R}^n \times T^1)) \) and \( m_0 \sigma = 0 \). In addition, the following inequality holds:

\[
\max_{0 \leq t \leq T} \| \sigma(t) \|^2_{H^s(\mathbb{R}^n \times T^1)} \leq C e^{C(T^2 + \| \omega \| L_{L^2(\mathbb{R}^n \times T^1)}^1)} \left( \| \sigma_0 \|^2_{H^s(\mathbb{R}^n \times T^1)} + \int_0^T \| f(t) \|^2_{H^s(\mathbb{R}^n \times T^1)} dt \right). \tag{3.4}
\]

**Proof.** In what follows, we denote \( L^2 \) inner product by \((\cdot, \cdot)\) with respect to \((x, \theta)\) over \(\mathbb{R}^n \times T^1\) and \( C \) by a constant depending only upon the bounds of derivatives of coefficients appeared in (3.2).

**(1) L^2-estimate of the solution**

Multiplying the equation of (3.2) by \( \sigma(t, x, \theta) \), and integrating the resulting equation with respect to \((x, \theta)\) over \(\mathbb{R}^n \times T^1\), one gets

\[
\frac{1}{2} \frac{d}{dt} \| \sigma(t) \|^2_{L^2(\mathbb{R}^n \times T^1)} + (b \partial_0 \sigma, \partial_0 \sigma) = (f, \sigma) - \sum_{j=1}^n (\partial_j \sigma_0, \sigma) - (c \partial_0 \omega \sigma, \sigma) - (e \sigma, \sigma). \tag{3.5}
\]

Thanks to (3.3), it follows

\[(b \partial_0 \sigma, \partial_0 \sigma) \geq 0\]

Using the Cauchy-Schwartz inequality and the Sobolev embedding theorem, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \sigma(t) \|^2_{L^2(\mathbb{R}^n \times T^1)} \leq C(1 + \| \omega \| H^{-1}(\mathbb{R}^n \times T^1)) \left( \| \sigma \|^2_{L^2(\mathbb{R}^n \times T^1)} + \| f \|^2_{L^2(\mathbb{R}^n \times T^1)} \right) \tag{3.6}
\]

Gronwall’s inequality implies

\[
\max_{0 \leq t \leq T} \| \sigma(t) \|^2_{L^2(\mathbb{R}^n \times T^1)} \leq C e^{C(T^2 + \| \omega \| L_{L^2(\mathbb{R}^n \times T^1)}^1)} \left( \| \sigma_0 \|^2_{L^2(\mathbb{R}^n \times T^1)} + \int_0^T \| f(t) \|^2_{L^2(\mathbb{R}^n \times T^1)} dt \right). \tag{3.7}
\]

**(2) H^s-estimate of the solution**

For any multiindex \( \alpha, |\alpha| \leq s \), we set \( \sigma_\alpha = \partial^\alpha_{x, \theta} \sigma \).

Acting the operator \( \partial^\alpha_{x, \theta} \) on the equation of (3.2), and taking the inner product with \( \sigma_\alpha \) with respect to \((x, \theta) \in \mathbb{R}^n \times T^1\), we get

\[
\frac{1}{2} \frac{d}{dt} \| \sigma_\alpha \|^2_{L^2(\mathbb{R}^n \times T^1)} + \sum_{j=1}^n \left( (\partial_j \sigma_\alpha, \sigma_\alpha) - (b \partial_0^2 \sigma_\alpha, \sigma_\alpha) + (c \omega \partial_0 \sigma_\alpha, \sigma_\alpha) \right) + \left( (d \partial_0 \sigma_\alpha, \sigma_\alpha) + (e \sigma_\alpha, \sigma_\alpha) = (\partial^\alpha f, \sigma_\alpha) - (R, \sigma_\alpha) \right. \tag{3.8}
\]

where

\[ R = \sum_{j=1}^n [\partial^\alpha, \partial_j] \sigma + [\partial^\alpha, c \omega] \partial_0 \sigma + [\partial^\alpha, d] \partial_0 \sigma + [\partial^\alpha, e] \sigma - [\partial^\alpha, b] \partial_0^2 \sigma. \]

In order to estimate (3.8), we need study \( \| R \|_{L^2(\mathbb{R}^n \times T^1)} \). By Lemma 3.1, it follows

\[
\| [\partial^\alpha, \partial_j] \sigma \|_{L^2(\mathbb{R}^n \times T^1)} \lesssim \| \nabla a_j \|_{L^\infty(\mathbb{R}^n)} \| D^{s-1} \partial_j \sigma \|_{L^2(\mathbb{R}^n \times T^1)} + \| D^s a_j \|_{L^2(\mathbb{R}^n)} \| \partial_j \sigma \|_{L^\infty(\mathbb{R}^n \times T^1)} \leq \| a_j \|_{H^s(\mathbb{R}^n)} \| \sigma \|_{H^s(\mathbb{R}^n \times T^1)}.
\]
Similarly, we have

\[
\|\partial^{\alpha}c_0\partial^2\sigma\|_{L^2(\mathbb{R}^n \times T^1)} \lesssim \|c\|_{H^s(\mathbb{R}^n)}\|\omega\|_{H^s(\mathbb{R}^n \times T^1)}\|\sigma\|_{H^s(\mathbb{R}^n \times T^1)},
\]

\[
\|\partial^{\alpha}d_0\partial^2\sigma\|_{L^2(\mathbb{R}^n \times T^1)} \lesssim \|d\|_{H^s(\mathbb{R}^n)}\|\sigma\|_{H^s(\mathbb{R}^n \times T^1)},
\]

\[
\|\partial^{\alpha}e\sigma\|_{L^2(\mathbb{R}^n \times T^1)} \lesssim \|e\|_{H^s(\mathbb{R}^n)}\|\sigma\|_{H^s(\mathbb{R}^n \times T^1)}
\]

and

\[
\|\partial^{\alpha}b\partial^2_0\sigma\|_{L^2(\mathbb{R}^n \times T^1)} \lesssim \|\nabla b\|_{L^\infty(\mathbb{R}^n)}\|D^{s-1}\partial^2_0\sigma\|_{L^2(\mathbb{R}^n \times T^1)}
\]

\[-\|D^s b\|_{L^2(\mathbb{R}^n)}\|\partial^2_0\sigma\|_{L^\infty(\mathbb{R}^n \times T^1)} \leq C\|b\|_{H^s(\mathbb{R}^n)}\|\partial^2_0\sigma\|_{H^s(\mathbb{R}^n \times T^1)}.
\]

Therefore, we have

\[
(R, \sigma) \leq C(\delta)(1 + \|\omega\|_{H^s(\mathbb{R}^n \times T^1)})\|\sigma\|_{H^s(\mathbb{R}^n \times T^1)}^2 + \frac{\delta}{\omega}\|\partial^2_0\sigma\|_{H^s(\mathbb{R}^n \times T^1)}^2 (3.9)
\]

where \(\delta\) will be determined later.

When \(b(t, x) \geq \beta > 0\), the term \(-b(t, x)\partial^2_0\sigma, \sigma\) can be estimated as:

\[
-(b(t, x)\partial^2_0\sigma, \sigma) = (b(t, x)\partial_0\sigma, \partial_0\sigma) \geq \beta\|\partial^2_0\sigma\|_{L^2(\mathbb{R}^n \times T^1)}.
\]

Summing over all \(|a| \leq s\), and choosing \(\delta\|b\|_{H^s(\mathbb{R}^n)} = \frac{\delta}{\omega}\), from (3.8) we obtain

\[
\frac{d}{dt}\|\sigma\|_{H^s(\mathbb{R}^n \times T^1)} \leq C(1 + \|\omega\|_{H^s(\mathbb{R}^n \times T^1)})\|\sigma\|_{H^s(\mathbb{R}^n \times T^1)}^2 + \|f\|_{H^s(\mathbb{R}^n \times T^1)}. (3.11)
\]

When \(b(t, x) \equiv 0\), from (3.8) and (3.9) we know that (3.11) holds as well.

By using Gronwall’s inequality in (3.11) we get

\[
\max_{0 \leq t \leq T} \|\sigma(t)\|_{H^s(\mathbb{R}^n \times T^1)} \leq e^{CT} \|\omega\|_{L^1(0, T; H^s(\mathbb{R}^n \times T^1))}\|\sigma_0\|_{H^s(\mathbb{R}^n \times T^1)} + \int_0^T \|f(t)\|_{H^s(\mathbb{R}^n \times T^1)} dt. (3.12)
\]

From the a priori estimate (3.12), the existence and uniqueness of the solution to the problem (3.2) is easily obtained.

In the following, we study the nonlinear problem (3.1).

**Theorem 3.3** Suppose that \(s > \frac{n+1}{2} + 1\), \(s \in \mathbb{N}, \sigma_0 \in H^s(\mathbb{R}^n \times T^1)\), then there exists \(T_0 \in (0, T]\) such that the problem (3.1) has the unique solution \(\sigma \in C([0, T_0]; H^s(\mathbb{R}^n \times T^1))\), and \(\partial_t \sigma \in C([0, T_0]; H^{s-1}(\mathbb{R}^n \times T^1))\).

**Proof.** Step 1. The iterative scheme. We construct the approximate solution to the problem (3.1) by the following iterative scheme:

\[
\begin{cases}
\partial_t \sigma^{n+1} + \sum_{j=1}^n a_j(t, x)\partial_j \sigma^{n+1} - b(t, x)\partial^2_0 \sigma^{n+1} + c(t, x)\sigma^n \partial_0 \sigma^{n+1} \\
\quad + d(t, x)\partial_0 \sigma^{n+1} + e(t, x)\sigma^{n+1} = 0 \\
\sigma^{n+1}(0, x, \theta) = \sigma_0(x, \theta)
\end{cases}
\]

\[12\]
and \( \sigma^0(0, x, \theta) = 0 \).

Suppose that \( \sigma^n \) is constructed in \( C([0, T]; H^s(\mathbb{R}^n \times T^1)) \), then by Theorem 3.2 there is a unique solution \( \sigma^{n+1} \in C([0, T]; H^s(\mathbb{R}^n \times T^1)) \) to the problem (3.13). This shows that the construction can be carried and thus defines a sequence \( \{\sigma^n\}_{n \geq 1} \subseteq C([0, T]; H^s(\mathbb{R}^n \times T^1)) \) satisfying (3.13).

**Step2. Uniform bounds.** We shall show that there are constants \( M \) and \( T_0 \in (0, T] \) such that for all \( n \),

\[
\sup_{0 \leq t \leq T_0} \|\sigma^n(t)\|_{H^s(\mathbb{R}^n \times T^1)} \leq M. \tag{3.14}
\]

Let \( M = C(\|\sigma_0\|_{H^s(\mathbb{R}^n \times T^1)} + 1) \), here \( C \) denotes the constant given in the estimate (3.4), and assume that \( \sup_{0 \leq t \leq T_0} \|\sigma^{n-1}(t)\|_{H^s(\mathbb{R}^n \times T^1)} \leq M \) holds. By choosing \( T_0 \in (0, T] \) such that \( Ce^{\frac{1}{2}(CT_0+MT_0)} \leq 1 \), from the estimate (3.4), (3.14) holds for all \( n \).

**Step3. Convergence.** Let \( \omega^n = \sigma^{n+1} - \sigma^n \). It follows from (3.13) that \( \omega^n \) solves the following problem:

\[
\begin{align*}
\frac{\partial \omega^n}{\partial t} &+ \sum_{j=1}^{n} a_j(t, x) \frac{\partial \omega^n}{\partial x_j} - b(t, x) \frac{\partial ^2 \omega^n}{\partial t^2} + e(t, x) \sigma^n \frac{\partial \omega^n}{\partial \theta} + d(t, x) \frac{\partial \omega^n}{\partial \theta} + e(t, x) \omega^n \\
\omega^n(0, x, \theta) &\equiv 0
\end{align*}
\]

(3.15)

Knowing the uniform bounds (3.14), and using the estimate (3.4), we obtain that for all \( n \geq 2 \) and \( t \leq T_0 \):

\[
\|\omega^n(t)\|_{H^{s-1}(\mathbb{R}^n \times T^1)} \leq C(C) \int_0^t \|\omega^{n-1}(\tau)\|_{H^{s-1}(\mathbb{R}^n \times T^1)} d\tau,
\]

This implies that the sequence \( \sigma^n \) convergence in \( C([0, T_0]; H^{s-1}(\mathbb{R}^n \times T^1)) \), the limit is clearly a solution of (3.1).

**Step4. Uniqueness.** Assume that \( \sigma^1 \) and \( \sigma^2 \) are two solutions of the nonlinear problem (3.1) in \( C([0, T_0]; H^s(\mathbb{R}^n \times T^1)) \). Let \( \omega = \sigma^1 - \sigma^2 \). Using the similar argument as in Step 3 for the problem of \( \omega \), there is a positive constant \( C \) such that for all \( t \in [0, T_0] \):

\[
\|\omega(t)\|_{H^{s-1}(\mathbb{R}^n \times T^1)} \leq C \int_0^t \|\omega(\tau)\|_{H^{s-1}(\mathbb{R}^n \times T^1)} d\tau,
\]

which implies \( \omega = 0 \), namely, the system (3.1) has a unique solution. \( \blacksquare \)

By applying the results obtained in Theorems 3.2 and 3.4 for the problems of \( \sigma_1(t, x, \theta_1) \), \( \sigma_2(t, x, \theta_2) \) and \( U_{s+1}(t, x, \theta_1, \theta_2) \) given in Section 2, we conclude

**Corollary 3.4** For any fixed \( s > \frac{n+1}{2} + 1 \), assume that \( u_0, \bar{w}_0 \in H^s(\mathbb{R}^n) \), \( \varphi_0 \in H^{s+1}(\mathbb{R}^n) \), \( \sigma^1_0, \sigma^2_0 \in H^s(\mathbb{R}^n \times T^1) \), then for all integer \( k \geq 1 \), the oscillation profile \( U_k \in H^{s-2k+2}(\mathbb{R}^n \times T^1 \times T^1) \).

### 4 Convergence and approximation theorems

In this section, we shall show the existence of the exact solution to (1.1) and (1.2) having the asymptotic expansion (1.3) to certain order in a time interval independent of \( \varepsilon \). First, we introduce
two spaces as in [1]. Denote by
\[ B^m_\rho = \left\{ u^\varepsilon \in C([0,T]; H^m(\mathbb{R}^n)) \mid \max_{0 \leq t \leq T} \| \partial^k u^\varepsilon(t) \|_{L^2(\mathbb{R}^n)} \leq \rho \varepsilon^{-k}, \ 0 < \varepsilon \leq 1, 0 \leq k \leq m \right\} \]
and
\[ A^m_\rho = \left\{ v^\varepsilon \in C([0,T]; W^{m,\infty}(\mathbb{R}^n)) \mid \| v^\varepsilon \|_{L^\infty([0,T] \times \mathbb{R}^n)} \leq \rho, \right. \\
\left. \| \partial^k v^\varepsilon \|_{L^\infty([0,T] \times \mathbb{R}^n)} \leq \rho \varepsilon^{-k}, \ 0 < \varepsilon \leq 1, 1 \leq k \leq m \right\} \]
where \( \partial^k = \partial^{i_1}_1 \partial^{i_2}_2 \cdots \partial^{i_n}_n, \alpha_i \in \mathbb{N}, \) with \( \sum_{i=1}^{n} |\alpha_i| = k. \) Defining
\[ u^{\varepsilon,M}(t, x) = U_0(t, x) + \varepsilon U_1(t, x, \frac{\varphi_1(t,x)}{\varepsilon}, \frac{\varphi_2(t,x)}{\varepsilon}) + \cdots + \varepsilon^M U_M(t, x, \frac{\varphi_1(t,x)}{\varepsilon}, \frac{\varphi_2(t,x)}{\varepsilon}), \]
then from the formal discussion given in §2, \( u^{\varepsilon,M}(t, x) \) satisfies
\[ \begin{cases} 
\mathcal{L}(u^{\varepsilon,M})u^{\varepsilon,M} = F(t, x, u^{\varepsilon,M}) + \varepsilon^M g^\varepsilon(t, x, \frac{\varphi_1(t,x)}{\varepsilon}, \frac{\varphi_2(t,x)}{\varepsilon}) \\
 u^{\varepsilon,M}|_{t=0} = u_0(x) + \varepsilon (u_0^0(x) + \varepsilon \sigma_1 \varphi_1^0) t_1(\nabla \varphi_0) + \varepsilon \sigma_2 (\nabla \varphi_0) t_2(\nabla \varphi_0) 
\end{cases} \quad (4.1) \]
where \( \mathcal{L}(u) = A_0(u) \partial_t + \sum_{j=1}^{n} A_j(u) \partial_j - \varepsilon^2 \sum_{j,k=1}^{n} B_{jk}(u) \partial_j \partial_k \) and \( g^\varepsilon(t, x, \frac{\varphi_1(t,x)}{\varepsilon}, \frac{\varphi_2(t,x)}{\varepsilon}) \) is \( B^m_\rho \) with \( m = s - 2M. \) Suppose that \( u^\varepsilon(t, x) \) satisfies (1.1)(1.2), and set \( u^\varepsilon = u^{\varepsilon,M} + w^\varepsilon, \) then \( w^\varepsilon \) satisfies the following problem:
\[ \begin{cases} 
\mathcal{L}(u^{\varepsilon,M} + w^\varepsilon)u^\varepsilon = F(t, x, u^{\varepsilon,M} + w^\varepsilon) - F(t, x, u^{\varepsilon,M}) - \varepsilon^M g^\varepsilon(t, x, \frac{\varphi_1}{\varepsilon}, \frac{\varphi_2}{\varepsilon}) \\
 w^\varepsilon|_{t=0} = 0 
\end{cases} \quad (4.2) \]
The main result in this section is the following one,

**Theorem 4.1** Fixed \( m \in \mathbb{N}, m > \frac{n+1}{2} + 3, \ M \geq m + 1, \ s > \frac{n+1}{2} + 2M + m - 1, \) assume that \( u_0, u^0_1 \in H^s(\mathbb{R}^n), \varphi_0 \in H^{s+1}(\mathbb{R}^n), \sigma_1^0, \sigma_2^0 \in H^s(\mathbb{R}^n \times T^1) \) and \( \rho > 0, \) then there exist \( \varepsilon_\rho \geq 0 \) and \( \sigma \geq 0 \) such that the Cauchy problem (1.1)(1.2) admits a unique solution \( u^\varepsilon \in u^{\varepsilon,M} + \varepsilon M B^m_\rho, \) for all \( 0 < \varepsilon \leq \varepsilon_\rho. \) Here \( u^{\varepsilon,M} \in A^{m+1}_\rho \) is the approximate solution to (1.1)(1.2) satisfying (4.1).

**Remark 4.2** In the above theorem, we need \( u^{\varepsilon,M} \in A^{m+1}_\rho. \) By virtue of the discussion given in previous sections, we have already known \( u^{\varepsilon,M} \in C([0,T]; H^{s-2M+2}(\mathbb{R}^n \times T^1 \times T^1)) \) when \( s > \frac{n+1}{2} + 1. \) If we choose \( s > \frac{n+1}{2} + 2M + m - 1, \) then \( u^{\varepsilon,M} \in A^{m+1}_\rho \) thanks to the Sobolev embedding.

In order to prove Theorem 4.1, we only need to study the problem (4.2). For the need of the following discussions, we rewrite (4.2) as the following form:
\[ \begin{cases} 
\mathcal{L}(u^{\varepsilon,M} + w^\varepsilon)u^\varepsilon = A_0(u^{\varepsilon,M} + w^\varepsilon) \left\{ (A_0^{-1} F)(t, x, u^{\varepsilon,M} + w^\varepsilon) - (A_0^{-1} F)(t, x, u^{\varepsilon,M}) \right\} \\
\quad + (A_0^{-1} \mathcal{L}(u^{\varepsilon,M}) - (A_0^{-1} \mathcal{L})(u^{\varepsilon,M} + w^\varepsilon))u^{\varepsilon,M} - \varepsilon^M A_0^{-1}(u^{\varepsilon,M})g^\varepsilon(t, x, \frac{\varphi_1}{\varepsilon}, \frac{\varphi_2}{\varepsilon}) \\
 w^\varepsilon|_{t=0} = 0 
\end{cases} \quad (4.3) \]
To study (4.3), let us consider the following linear problem:

\[
\begin{cases}
    L(u^\varepsilon + v)u = A_0(u^\varepsilon + v)\{(A_0^{-1}F)(t, x, u^\varepsilon + v) - (A_0^{-1}F)(t, x, u^\varepsilon)\} \\
    + [(A_0^{-1}L)(u^\varepsilon) - (A_0^{-1}L)(u^\varepsilon + v)]u^\varepsilon - \varepsilon^M A_0^{-1}(u^\varepsilon)g(t, x, \frac{u}{\varepsilon}, \frac{\partial u}{\partial x}) \\
    u|_{t=0} = 0
\end{cases}
\]  

(4.4)

for a given function \( v \in \varepsilon^M \mu^m \).

We shall adapt the idea of [1] to study the problem (4.4). For \( m \in \mathbb{N} \), we define a norm with a parameter \( \mu \) in \( H^m(\mathbb{R}^n) \):

\[
|u|_{m, \mu} = \sum_{0 \leq k \leq m} \mu^{m-k} \| \partial^k u \|_{L^2(\mathbb{R}^n)}
\]

and we also denote by \( \| \cdot \|_{m, \mu, \lambda} \) a norm with a weight \( \lambda \) defined on \( L^2(0, T; H^m(\mathbb{R}^n)) \):

\[
\| u \|_{m, \mu, \lambda} = \left( \int_0^T \| u(t) \|_{m, \mu}^2 e^{-2\lambda t} dt \right)^{\frac{1}{2}}
\]

and for \( \Omega_T = [0, T] \times \mathbb{R}^n \), denote \( H^{1, m}(\Omega_T) \) by the space of \( u \in L^2(0, T; H^m(\mathbb{R}^n)) \) and \( \partial_t u \in L^2(0, T; H^{m-1}(\mathbb{R}^n)) \), equipped with the norm:

\[
|||u|||_{m, \mu, \lambda} = ||u||_{m, \mu, \lambda} + \frac{1}{\lambda} ||\partial_t u||_{m-1, \mu, \lambda}.
\]

To the norm \( | \cdot |_{m, \mu} \), the following Hölder inequality holds:

**Lemma 4.3** (see [1]) If \( \alpha_1 + \cdots + \alpha_r \leq k \leq m \), \( \alpha_i \in \mathbb{N} \), \( a_i \in L^\infty(\mathbb{R}^n) \) \( \cap H^m(\mathbb{R}^n) \),

\[
\mu^{m-k} (\partial^{a_1} a_1) \cdots \partial^{a_r} a_r \|_{m, \mu} \leq C \sum_{i=1}^r |a_i|_{m, \mu} (\prod_{j \neq i} |a_j|_{L^\infty})
\]

(4.5)

where and in what follows, we denote \( \| \cdot \|_{L^\infty(\mathbb{R}^n)} \) by \( | \cdot |_{L^\infty} \).

In the following discussion, we shall also use the notations:

\[
\begin{align*}
|u|_{Lip} & = |u|_{L^\infty} + \sum_{0 \leq j \leq n} |\partial_j u|_{L^\infty} & \text{for all} & u \in Lip(\mathbb{R}^n) \\
|u|_* & = \sup_{0 \leq t \leq T} |u(t)|_{L^\infty} & \text{for all} & u \in C([0, T]; L^\infty(\mathbb{R}^n)) \\
|u|^{*} & = \sup_{0 \leq t \leq T} |u(t)|_{Lip} & \text{for all} & u \in C([0, T]; Lip(\mathbb{R}^n)).
\end{align*}
\]

As in [1], for the problem (4.4), we have

**Proposition 4.4** For \( R > 0 \), integer \( m > \frac{n}{2} \) be fixed, there exist a positive constant \( C_R \) and a positive function \( h(\lambda) \) such that for all \( v \in H^{1, m}(\Omega_T) \) satisfies \( |v|^* \leq R \), the problem (4.4) admits a unique solution \( u \in H^{1, m}(\Omega_T) \). Moreover, there is \( \lambda_0 > 0 \) such that for all \( \lambda \geq \lambda_0 \), the following inequality holds,

\[
\frac{e^{-M}}{\sqrt{\lambda}} |u(t)|_{m, \frac{\mu}{\lambda}} + |||u|||_{m, \frac{\mu}{\lambda}, \lambda} + \frac{\varepsilon}{2} \sqrt{\frac{1}{\lambda}} ||Du_{11}||_{m, \frac{\mu}{\lambda}, \lambda} \leq C_R (1 + |u|^* + |Du|^*) ||v||_{m, \frac{\mu}{\lambda}, \lambda} + \varepsilon^{M-m} h(\lambda).
\]

(4.6)

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The proof of this proposition mainly follows the argument given in [1]. For completeness, we give the main steps of the proof. To establish (4.6), we first consider the $L^2$-estimate.

**Lemma 4.5** Let $R > 0$ be fixed, there exists $C > 0$, $\lambda_0 > 0$ such that if $v \in L^\infty(0, T; \text{Lip}(\mathbb{R}^n))$ satisfies $|v|^* \leq R$, then for $u \in L^2(0, T; L^2(\mathbb{R}^n))$, $L((u^{\varepsilon,M} + v)u) \in L^2(0, T; L^2(\mathbb{R}^n))$, and for all $\lambda, \mu$ satisfy $\lambda \geq \lambda_0$, $\mu > 0$, the following estimate holds:

$$e^{-\lambda t}|u(t)|_{0, \mu} + \sqrt{\lambda}||u||_{0, \mu, \lambda} + \varepsilon \sqrt{\gamma}||Du_{II}||_{0, \mu, \lambda} \leq \frac{C}{\sqrt{\lambda}}||L((u^{\varepsilon,M} + v)u)||_{0, \mu, \lambda}, \quad 0 \leq t \leq T \tag{4.7}$$

**Proof.** Let $w = e^{-\lambda t}u$, then $w$ satisfies

$$\lambda A_0 w + L((u^{\varepsilon,M} + v)w) = e^{-\lambda t}L((u^{\varepsilon,M} + v)u)$$

Taking inner product by $w$ and integrating the resulting equation over $\mathbb{R}^n$, we obtain

$$\int_0^t \frac{d}{dt}(A_0 w, w)dt + \int_0^t ((2\lambda A_0 - \partial_t A_0 - \sum_{j=1}^n \partial_j A_j)w, w)dt + 2\varepsilon^2 \int_0^t \sum_{j,k=1}^n ([B_{jk}\partial_j w, \partial_k w] + (\partial_k B_{jk}\partial_j w, w)]dt$$

$$= 2\int_0^t (e^{-\lambda t}L((u^{\varepsilon,M} + v)u, w)dt$$

By virtue of (H1)(H2) and the Cauchy-Schwartz inequality, there exists $\lambda_0 > 0$, for all $\lambda \geq \lambda_0$, we have

$$||w(t)||^2_{L^2} + \int_0^t \lambda ||w(\tau)||^2_{L^2}d\tau + \varepsilon^2 \int_0^t ||Du_{II}(\tau)||^2_{L^2}d\tau$$

$$\leq \frac{C}{\lambda} \int_0^t ||e^{-\lambda t}L((u^{\varepsilon,M} + v)u)||^2_{L^2}d\tau$$

Thus, the estimate (4.7) holds. 

To study the estimate of high order norms, we first have the following results on commutators:

**Lemma 4.6** Fixed $R > 0$, $m \in \mathbb{N}$, assume that $u, Du, v \in H^m(\mathbb{R}^n) \cap \text{Lip}(\mathbb{R}^n)$ and $|v|_{\text{Lip}} \leq R$, then for any $l \leq m$, $\varepsilon, \mu$ satisfy $0 < \varepsilon < 1 \leq \varepsilon \mu$, the following estimates hold, for all $1 \leq j, k \leq n$,

$$\mu^{-l}||\partial^j A_j u^{\varepsilon,M} + v\partial_j ||_{0, \mu, \lambda} \leq C(\|u\|_{m, \mu} + |u|_{\text{Lip}}|v|_{m, \mu}), \tag{4.8}$$

$$\mu^{-l}||\partial^j B_{jk} u^{\varepsilon,M} + v\partial_j \partial_k ||_{0, \mu, \lambda} \leq C(\|Du_{II}\|_{m, \mu} + |Du_{II}|_{\text{Lip}}|v|_{m, \mu}). \tag{4.9}$$

**Proof.** (1) Thanks to Moser’s inequality, we have

$$\mu^{-l}||\partial^j A_j (u^{\varepsilon,M} + v)\partial_j ||_{0, \mu, \lambda}$$

$$\leq C\mu^{-l}(1||\partial A_j (u^{\varepsilon,M} + v)||_{L^\infty}||\partial^j u||_{0, \mu, \lambda} + ||\partial^j A_j (u^{\varepsilon,M} + v)||_{0, \mu, \lambda}||\partial_j w||_{L^\infty})$$

$$\leq C(R)|u|_{m, \mu} + C\mu^{-l}||\partial^j A_j (u^{\varepsilon,M} + v)||_{0, \mu, \lambda}||u||_{\text{Lip}},$$
and $\mu^{m-1}q A_j(u^{\varepsilon,M} + v)$ can be expressed as a sum of terms of the form:

$$\mu^{m-1}q (u^{\varepsilon,M}, v) \partial^\alpha u^{\varepsilon,M} \partial^\beta v$$

where $|\alpha| + |\beta| = l$, $\phi(u^{\varepsilon,M}, v)$ is smooth in their arguments.

If $\alpha = 0$, we have

$$|\mu^{m-1}q (u^{\varepsilon,M}, v) \partial^\alpha u^{\varepsilon,M} \partial^\beta v|_{0,\mu} \leq C \mu^{m-1}r |u^{\varepsilon,M}|_{L^\infty} |\partial^\beta v|_{0,\mu} \leq C_1 |v|_{m,\mu}$$

and if $|\alpha| \geq 1$, there has

$$|\mu^{m-1}q (u^{\varepsilon,M}, v) \partial^\alpha u^{\varepsilon,M} \partial^\beta v|_{0,\mu} \leq C(\varepsilon\mu)^{1-|\alpha|} \mu^{|\beta|} |\partial^\beta v|_{0,\mu} \leq C_2 |v|_{m,\mu}$$

Summing the above inequalities, we get the result (4.8).

(2) Using the similar arguments and noting the structure of $B_{jk}$ given in (H2), we have

$$\mu^{m-1}r (\partial^j, B_{jk}(u^{\varepsilon,M} + v) \partial_j \partial_k u|_{0,\mu} \leq C \mu^{m-1}r (|\partial B_{jk}(u^{\varepsilon,M} + v)|_{L^\infty} |\partial^j u_{II}|_{0,\mu} + |\partial^j B_{jk}(u^{\varepsilon,M} + v)|_{0,\mu} |\partial_j \partial_k u_{II}|_{L^\infty}) \leq C(\|Du_{II}\|_{m,\mu} + \|Du_{II}\|_{L^p}|v|_{m,\mu})$$

Lemma 4.7 Fixed $R > 0$, $m \in \mathbb{N}$, suppose that for any $0 \leq l \leq m$, $u, v, f \in L^2(0,T;H^l(\mathbb{R}^n))$ and $u, v, Du \in C([0,T];L^\infty(\mathbb{R}^n))$, $|v|_\ast \leq R$. If $L(u^{\varepsilon,M} + v)u = f$, then $u \in H^{1,l}(\Omega_T)$ and for any $0 \leq \varepsilon \leq 1 \leq \varepsilon \mu$, the following estimate holds:

$$|\partial u|_{l-1,\mu} \leq |f|_{l-1,\mu} + C(\|u\|_{l,\mu} + \varepsilon^2 |Du_{II}|_{l,\mu} + (\|u\|_\ast + \varepsilon^2 |Du_{II}|_\ast) |v|_{l,\mu})$$

(4.10)

with $C$ being independent of $0 \leq t \leq T$.

Proof. From the equation, obviously we have

$$|\partial u|_{l-1,\mu} \leq |f|_{l-1,\mu} + \sum_{j=1}^n |A_j(u^{\varepsilon,M} + v) \partial_j u|_{l-1,\mu} + \varepsilon^2 \sum_{j,k=1}^n |B_{jk}(u^{\varepsilon,M} + v) \partial_j \partial_k u|_{l-1,\mu}$$

We first discuss $\sum_{j=1}^n |A_j(u^{\varepsilon,M} + v) \partial_j u|_{l-1,\mu}$. Since $\mu^{l-1-i} \partial^i (A_j(u^{\varepsilon,M} + v) \partial_j u)$ can be expressed as a sum of terms of the form:

$$\phi(u^{\varepsilon,M}, v) \mu^{l-1-i} \partial^\alpha u^{\varepsilon,M} \partial^\beta v \partial^\gamma u$$

where $|\alpha| + |\beta| + |\gamma| = i + 1 \leq l$, $|\gamma| \geq 1$.

If $\alpha = 0$, by virtue of (4.5), there has

$$|\phi(u^{\varepsilon,M}, v) \mu^{l-1-i} \partial^\alpha u^{\varepsilon,M} \partial^\beta v \partial^\gamma u|_{0,\mu} \leq C \mu^{-(|\beta|+|\gamma|)} |\partial^\beta v \partial^\gamma u|_{0,\mu} \leq C |v|_{l,\mu} |u|_{L^\infty} + R|u|_{l,\mu}$$
If $|\alpha| \geq 1$, we have

$$
|\phi(u^{\varepsilon,M}, v)\mu^{l-1-i} \partial^\alpha u^{\varepsilon,M} \partial^\beta v \partial^\gamma u|_{0, \mu} 
\leq C(\mu \varepsilon)^{1-|\alpha|} \mu^{-(|\beta|+|\gamma|)} \| \partial^\beta v \partial^\gamma u \|_{0, \mu} 
\leq C_1(|v|_{l, \mu}|u|_{L^\infty} + R|u|_{l, \mu}).
$$

Thus, we obtain

$$
\sum_{j=1}^{n} |A_j(u^{\varepsilon,M} + v) \partial_j u|_{l-1, \mu} \leq C(|v|_{l, \mu}|u|_{L^\infty} + R|u|_{l, \mu}).
$$

To estimate $\varepsilon^2 \sum_{j,k=1}^{n} |B_{jk}(u^{\varepsilon,M} + v) \partial_j \partial_k u|_{l-1, \mu}$, we use the similar arguments and note the structure of $B_{jk}$ given in (H2) to obtain

$$
\varepsilon^2 \sum_{j,k=1}^{n} |B_{jk}(u^{\varepsilon,M} + v) \partial_j \partial_k u|_{l-1, \mu} \leq C \varepsilon^2 (|v|_{l, \mu}|Du||_{L^\infty} + R|Du||_{l, \mu})
$$

Thus, we conclude the estimate (4.10).  

Now, we consider the high order estimate of the solution to (4.4) to complete the proof of Proposition 4.4.

**Proof of Proposition 4.4:** Acting the operator $\partial^l (1 \leq l \leq m)$ on (4.4), one gets

$$
\begin{align*}
L(u^{\varepsilon,M} + v) \partial^l u &= A_0(u^{\varepsilon,M} + v)[\partial^l f - \varepsilon^M \partial^l A_0^{-1}(u^{\varepsilon,M})g^r + q] \\
\partial^l u|_{l=0} &= 0
\end{align*}
$$

(4.11)

where

$$
f = (A_0^{-1}F)(t, x, u^{\varepsilon,M} + v) - (A_0^{-1}F)(t, x, u^{\varepsilon,M}) + [(A_0^{-1}L)(u^{\varepsilon,M}) - (A_0^{-1}L)(u^{\varepsilon,M} + v)]u^{\varepsilon,M},
$$

and

$$
q = \sum_{j=1}^{n} [(A_0^{-1}A_j)(u^{\varepsilon,M} + v) \partial_j \partial^l u - \varepsilon^2 \sum_{j,k=1}^{n} [(A_0^{-1}B_{jk})(u^{\varepsilon,M} + v) \partial_j \partial_k \partial^l u].
$$

Using (4.7) for the problem (4.11) and noting the positively defined property of $A_0$, we have

$$
\frac{\varepsilon^{-M}}{\sqrt{\lambda}} \mu^{m-l} ||\partial^l u||_{0, \mu, \lambda} + \mu^{m-l} ||\partial^l u||_{0, \mu, \lambda} + \varepsilon \sqrt{\frac{\gamma}{\lambda}} \mu^{m-l} ||\partial^l Du||_{1, 0, \mu, \lambda} \leq C \sqrt{\lambda} \left[ ||f||_{m, \mu, \lambda} + \left( \int_0^T e^{-2\lambda \mu^{2m-2l}} ||q||_{0, \mu, \lambda}^2 dt \right)^{\frac{1}{2}} + \varepsilon^M \|A_0^{-1}(u^{\varepsilon,M})g^r\|_{m, \mu, \lambda} \right] (4.12)
$$

$$
\varepsilon^M \|A_0^{-1}(u^{\varepsilon,M})g^r\|_{m, \mu, \lambda} \leq C \sqrt{\lambda} \left[ ||f||_{m, \mu, \lambda} + \left( \int_0^T e^{-2\lambda \mu^{2m-2l}} ||q||_{0, \mu, \lambda}^2 dt \right)^{\frac{1}{2}} + \varepsilon^M \|A_0^{-1}(u^{\varepsilon,M})g^r\|_{m, \mu, \lambda} \right]. (4.13)
$$

In what follows, we estimate each term on the right hand of the above inequality. Obviously, we have

$$
\mu^{m-l} ||\partial^l \{(A_0^{-1}F)(u^{\varepsilon,M} + v) - (A_0^{-1}F)(u^{\varepsilon,M})\}|_{0, \mu} \leq \mu^{m-l} ||\partial^l \{G(u^{\varepsilon,M}, v)\}|_{0, \mu}
$$
where $G(u^{\varepsilon,M}, B)$ is smooth in their arguments, which implies

$$
\mu^{m-l} |\partial^{l} f \{ (A_{0}^{-1} F)(u^{\varepsilon,M} + v) - (A_{0}^{-1} F)(u^{\varepsilon,M}) \}|_{0,\mu} \\
\leq C \mu^{m-l} |\partial^{\alpha} u^{\varepsilon,M} \partial^{\beta} v|_{0,\mu} \quad (|\alpha| + |\beta| = l) \\
\leq C \mu^{m-l} |\partial^{\alpha} u^{\varepsilon,M}|_{L^{\infty}} |\partial^{\beta} v|_{0,\mu}
$$

(4.14)

If $\alpha = 0$, from (4.14) we have

$$
\mu^{m-l} |\partial^{l} f \{ (A_{0}^{-1} F)(u^{\varepsilon,M} + v) - (A_{0}^{-1} F)(u^{\varepsilon,M}) \}|_{0,\mu} \leq C |v|_{m,\mu}.
$$

and if $|\alpha| \geq 1$, there has

$$
\mu^{m-l} |\partial^{l} f \{ (A_{0}^{-1} F)(u^{\varepsilon,M} + v) - (A_{0}^{-1} F)(u^{\varepsilon,M}) \}|_{0,\mu} \leq C (\varepsilon \mu)^{1-|\alpha|} \mu^{m-l-|\beta|} |\partial^{\beta} v|_{0,\mu} \leq C_{1} |v|_{m,\mu}.
$$

Thus, we get

$$
||(A_{0}^{-1} F)(t, x, u^{\varepsilon,M} + v) - (A_{0}^{-1} F)(t, x, u^{\varepsilon,M})||_{m,\mu,\lambda} \leq C ||v||_{m,\mu,\lambda}.
$$

Secondly, in the same argument as above, it follows

$$
\mu^{m-l} \sum_{j=1}^{n} |\partial^{l} \{ [(A_{0}^{-1} A_{j})(u^{\varepsilon,M}) - (A_{0}^{-1} A_{j})(u^{\varepsilon,M} + v)] \partial_{j} u^{\varepsilon,M} \}|_{0,\mu} \\
\leq C \mu^{m-l} |\partial^{\alpha} u^{\varepsilon,M} \partial^{\beta} v|_{0,\mu} \quad (|\alpha| + |\beta| = l + 1, |\alpha| \geq 1) \\
\leq C (\mu \varepsilon)^{1-|\alpha|} \mu^{m-l-|\beta|} |\partial^{\beta} v|_{0,\mu} \\
\leq C |v|_{m,\mu}
$$

and

$$
\varepsilon^{2} \mu^{m-l} \sum_{j,k=1}^{n} |\partial^{l} \{ [(A_{0}^{-1} B_{jk})(u^{\varepsilon,M}) - (A_{0}^{-1} B_{jk})(u^{\varepsilon,M} + v)] \partial_{j} \partial_{k} u^{\varepsilon,M} \}|_{0,\mu} \\
\leq C \varepsilon^{2} \mu^{m-l} |\partial^{\alpha} u^{\varepsilon,M} \partial^{\beta} v|_{0,\mu} \quad (|\alpha| + |\beta| = l + 2, |\alpha| \geq 2) \\
\leq C (\mu \varepsilon)^{3-|\alpha|} \mu^{m-l-|\beta|} |\partial^{\beta} v|_{0,\mu} \\
\leq C |v|_{m,\mu}
$$

Therefore, we get

$$
||f||_{m,\mu,\lambda} \leq C ||v||_{m,\mu,\lambda}.
$$

(4.15)

On the other hand, by virtue of (4.8) and (4.9), we have

$$
\mu^{m-l} \sum_{j=1}^{n} |A_{0}^{-1} A_{j}(u^{\varepsilon,M} + v) \partial_{j} \partial^{l} u|_{0,\mu} \leq C (|u|_{m,\mu} + |u|_{Lip}|v|_{m,\mu}) \\
\mu^{m-l} \sum_{j,k=1}^{n} |A_{0}^{-1} B_{jk}(u^{\varepsilon,M} + v) \partial_{j} \partial_{k} \partial^{l} u|_{0,\mu} \leq C (|Du_{I}|_{m,\mu} + |Du_{II}|_{Lip}|v|_{m,\mu})
$$
which implies
\[
\left( \int_0^T e^{-2M\mu^{2m-2l}}|q|_{0,\mu}^2 dt \right)^{\frac{1}{2}} \leq C(||u||_{m,\mu,\lambda} + ||D_u||_{m,\mu,\lambda} + (|u|_{Lip} + |D_u||_{Lip}||V||_{m,\mu,\lambda}) \tag{4.16}
\]
Finally, we have
\[
\mu^{m-l}||\partial_x^\alpha u^\varepsilon A_0^{-1}(u^\varepsilon)g^\varepsilon||_{0,\mu} \leq C\varepsilon^M \mu^{m-l}||\partial_x^\alpha u^\varepsilon \partial^\beta g^\varepsilon||_{0,\mu} \quad (||\alpha| + |\beta| = l)
\]
which gives rise to
\[
\varepsilon^M \mu^{m-l}||\partial_x^\alpha u^\varepsilon \partial^\beta g^\varepsilon||_{0,\mu} \leq C\varepsilon^M \mu^{m-l}e^{M-l} \leq C\varepsilon^{M-m}(\mu \varepsilon)^{m-l},
\]
as $$\alpha = 0$$, or
\[
\varepsilon^M \mu^{m-l}||\partial_x^\alpha u^\varepsilon \partial^\beta g^\varepsilon||_{0,\mu} \leq C\varepsilon^M \mu^{m-l}e^{1-|\alpha|} \leq C\varepsilon^{M-m}(\mu \varepsilon)^{m-l}
\]
when $$|\alpha| \geq 1$$. Thus, we have
\[
\mu^{m-l}||\partial_x^\alpha u^\varepsilon A_0^{-1}(u^\varepsilon)g^\varepsilon||_{0,\mu} \leq C\varepsilon^{M-m}(\mu \varepsilon)^{m-l}. \tag{4.17}
\]
On the other hand, by virtue of Lemma 4.7, we have
\[
||\partial_t u||_{m-1,\mu} \leq |f|_{m-1,\mu} + ||\varepsilon M A_0^{-1}(u^\varepsilon)g^\varepsilon||_{m-1,\mu} \tag{4.18}
\]
\[
+C||u||_{m,\mu} + \varepsilon^2|Du||_{Lip}|v|_{m,\mu} \tag{4.19}
\]
Combining the above estimates (4.15),(4.16),(4.17),(4.18) and (4.12), using the definition of $$||\cdot||_{m,\mu,\lambda}$$ and choosing $$\mu = \frac{1}{\beta}$$, there exists a positive function $$h(\lambda)$$, $$\lambda_0 > 0$$, when $$\lambda \geq \lambda_0$$, (4.5) holds. We complete the proof of Proposition 4.4. \[\Box\]

In order to apply the estimate (4.6) to study the nonlinear problem (4.3), we need an embedding result for the weighted norm $$||\cdot||_{m,\mu,\lambda}$$ given in [1]:

**Lemma 4.8** Fixed $$m \in \mathbb{N}$$, $$m > \frac{n+1}{2}$$, and $$0 < \delta < 1$$, there exists a function $$c(\lambda)$$ such that for all $$u \in H^{1,m}(\Omega_T) \cap C([0,T];L^\infty(\mathbb{R}^n))$$, $$\lambda \geq 1$$, $$\mu > 0$$, the following inequality holds:
\[
||u||_{\lambda} \leq \frac{1}{\mu^\delta} c(\lambda)||u||_{m,\mu,\lambda}.
\]

For the nonlinear problem (4.3), we construct the following iterative scheme:
\[
\begin{cases}
\mathcal{L}(u^\varepsilon + w^\nu)w^{\nu+1} = A_0(u^\varepsilon + w^\nu)(A_0^{-1}F)(t,x,u^\varepsilon + w^\nu) - (A_0^{-1}F)(t,x,u^\varepsilon) \\
+[(A_0^{-1} \mathcal{L})(u^\varepsilon) - (A_0^{-1} \mathcal{L})(u^\varepsilon + w^\nu)]u^\varepsilon - \varepsilon^M A_0^{-1}(u^\varepsilon)g^\varepsilon(t,x,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}) \tag{4.20}
\end{cases}
\]
with $$w^0(t,x) \equiv 0$$. 

20
**Lemma 4.9** Fixed $R > 0$, $m > \frac{n+1}{2} + 2$, $M \geq m + 1$ and $s > \frac{n+1}{2} + 2M + m$, then for any fixed $\lambda \geq \max \{4C_R, \lambda_0\}$, there exists $0 < \varepsilon_0 \leq 1$, for all $\nu \in \mathbb{N}$, $0 < \varepsilon \leq \varepsilon_0$, the following estimates hold:

\[
\begin{align*}
\|w^\nu\|_{m, \frac{2}{\lambda}} &\leq 4\varepsilon^{M-m}h(\lambda) \\
\|Dw^\nu_f\|_{m, \frac{2}{\lambda}} &\leq \frac{8}{R^{\frac{1}{2}}}\varepsilon^{M-m-1}h(\lambda) \\
|w^\nu|^s &\leq R.
\end{align*}
\] (4.21)

where $h(\lambda)$, $C_R$ and $\lambda_0$ are given in Proposition 4.4.

**Proof.** By virtue of Lemma 4.8, when $m > \frac{n+1}{2} + 2$, choosing $\mu = \frac{3}{4}$ and $\delta = \frac{1}{4}$, we have

\[
|Dw^\nu|_s \leq \frac{1}{\mu^s c(\lambda)}\|Dw^\nu\|_{m-1, \mu, \lambda} \leq \varepsilon^{\frac{3}{4}}c_1(\lambda)\|w^\nu\|_{m, \frac{2}{\lambda}}.
\]

and

\[
|D^2w^\nu|_s \leq \frac{1}{\mu^s c(\lambda)}\|D^2w^\nu\|_{m-2, \mu, \lambda} \leq \varepsilon^{\frac{3}{4}}c_1(\lambda)\|w^\nu\|_{m, \frac{2}{\lambda}}.
\]

Thus, for any $\nu$, $|w^\nu|^s \leq \varepsilon^{\frac{3}{4}}c_1(\lambda)\|w^\nu\|_{m, \frac{2}{\lambda}}$.

For convenience, in what follows, we set $\alpha_\nu = \|w^\nu\|_{m, \frac{2}{\lambda}}$, $\beta_\nu = \|Dw^\nu_f\|_{m, \frac{2}{\lambda}}$.

When $\nu = 1$, by using Proposition 4.4 for the problem (4.20) of $w^1$ it follows $\alpha_1 \leq 4\varepsilon^{M-m}h(\lambda)$ immediately. Assume that (4.21) is satisfied up to order $\nu$, then thanks to Proposition 4.4,

\[
\alpha_{\nu+1} \leq \frac{C_R}{\lambda}(1 + |Dw^{\nu+1}|^s)\alpha_\nu + \varepsilon^{M-m}h(\lambda)
\]

\[
\leq \frac{4C_R}{\lambda}(1 + \varepsilon^{\frac{3}{4}}c(\lambda)\alpha_{\nu+1})\varepsilon^{M-m}h(\lambda) + \varepsilon^{M-m}h(\lambda)
\]

Fixed $\lambda \geq \max \{4C_R, \lambda_0\}$ firstly and next choosing $\varepsilon_0 < 1$ sufficiently small such that $\varepsilon_0^{\frac{3}{4}}c_1(\lambda)h(\lambda) < \frac{1}{2}$ and $4\varepsilon_0^{\frac{3}{4}}c_1(\lambda)h(\lambda) < R$, then for all $0 < \varepsilon \leq \varepsilon_0$, we

\[
\alpha_{\nu+1} \leq 4\varepsilon^{M-m}h(\lambda)
\]

and

\[
|w^\nu|^s \leq 4\varepsilon^{\frac{3}{4}}c_1(\lambda)\varepsilon^{M-m}h(\lambda) \leq R.
\]

Using Proposition 4.4 again, we obtain

\[
\beta_\nu \leq \frac{8}{\sqrt{7}}\varepsilon^{M-m-1}\sqrt{h(\lambda)}.
\]

\[\blacksquare\]

**Lemma 4.10** The sequence $\{w^\nu\}_{\nu=1}^\infty$ converges to $w$ in $H^{1, m^{-1}}$ when $\nu$ goes to infinity.

**Proof.** Let $\Delta_{\nu+1} = w^{\nu+1} - w^\nu$, then from (4.20) we know that $\Delta_{\nu+1}$ satisfies the following problem

\[
\begin{align*}
\mathcal{L}^\nu \Delta_{\nu+1} &= A_0^\nu [(A_0^\nu)^{-1} F^\nu - (A_0^{\nu-1})^{-1} F^{\nu-1}] - [(A_0^\nu)^{-1} \mathcal{L}^\nu - (A_0^{\nu-1})^{-1} \mathcal{L}^{\nu-1}](u^{\varepsilon, M} + w^\nu) \\
&= A_0^\nu h^\nu \\
\Delta_{\nu+1}|_{t=0} &= 0
\end{align*}
\] (4.22)
where the notations $A_j^\nu = A_j(w^{\varepsilon,M} + w^\nu)(0 \leq j \leq n)$, $F^\nu = F(w^{\varepsilon,M} + w^\nu)$ and $\mathcal{L}^\nu = A_0(w^{\varepsilon,M} + w^\nu)\partial_t + \sum_{j=1}^n A_j(w^{\varepsilon,M} + w^\nu)\partial_j - \varepsilon^2 \sum_{j,k=1}^n B_{jk}(w^{\varepsilon,M} + w^\nu)\partial_j\partial_k$.

Acting the operator $\partial^l(l \leq m - 1)$ on the above problem, we have

$$\mathcal{L}^\nu \partial^l \Delta_{\nu+1} = A_0^\nu (\partial^l h_\nu + r_\nu) \quad (4.23)$$

where $r_\nu = \sum_{j=1}^n [(A_{0,\nu})^{-1} A_j^\nu \partial_j, \partial^l] \Delta_{\nu+1} - \varepsilon^2 \sum_{j,k=1}^n [(A_{0,\nu})^{-1} B_{jk}^\nu \partial_j\partial_k, \partial^l] \Delta_{\nu+1}$. Applying Lemma 4.5 and the positively defined property of $A_0^\nu$, we obtain the following estimate:

$$||\Delta_{\nu+1}||_{m-1,\mu,\lambda} + \varepsilon \sqrt{\frac{\gamma}{\lambda}} ||D(\Delta_{\nu+1})_{II}||_{m-1,\mu,\lambda} \leq C \frac{\lambda}{\varepsilon} [||h_\nu||_{m-1,\mu,\lambda} + \left(\int_0^T \varepsilon - 2\mu^2 - 2\mu l^2 |r_\nu|_{m,\mu}^2 dt\right)^{\frac{1}{2}}] \quad (4.24)$$

On the other hand, using the estimate (4.10) with $B,l$ being replaced by $w^\nu, m - 1$, respectively, one gets

$$||\partial^l \Delta_{\nu+1}||_{m-2,\mu,\lambda} \leq ||h_\nu||_{m-2,\mu,\lambda} + C ||\Delta_{\nu+1}||_{m-1,\mu} + \varepsilon^2 ||D(\Delta_{\nu+1})_{II}||_{m-1,\mu,\lambda} + (||\Delta_{\nu+1}||_{m-1,\mu} + \varepsilon^2 ||D(\Delta_{\nu+1})_{II}||_{m-1,\mu,\lambda}) ||w^\nu||_{m-1,\mu} \quad (4.25)$$

In order to estimate $||\Delta_{\nu+1}||_{m-1,\mu,\lambda}$, we consider the estimates of the right hands of (4.24)(4.25).

First, we have

$$\mu^{(m-1)-2|\beta|} ||\partial^l [(A_0^\nu)^{-1} F^\nu - (A_0^\nu)^{-1} F^\nu_{-1}]||_{0,\mu} \leq C \mu^{(m-1)-2|\beta|} ||\partial^l w^{\varepsilon,M} \partial^\nu \partial^l w^{\nu-1} \partial^\nu \Delta_{\nu}||_{0,\mu} \quad (4)$$

If $\alpha = 0$, by virtue of (4.5), we have

$$\mu^{(m-1)-2|\beta|} ||\partial^l [(A_0^\nu)^{-1} F^\nu - (A_0^\nu)^{-1} F^\nu_{-1}]||_{0,\mu} \leq C \mu^{(m-1)-2|\beta|} ||\partial^\nu w^{\varepsilon,M} \partial^\nu \partial^\nu w^{\nu-1} \partial^\nu \Delta_{\nu}||_{0,\mu} \leq C ||w^\nu||_{m-1,\mu} + ||w^{\nu-1}||_{m-1,\mu} ||\Delta_{\nu}||_{m-1,\mu}$$

If $\alpha \geq 1$, there has

$$\mu^{(m-1)-2|\beta|} ||\partial^l [(A_0^\nu)^{-1} F^\nu - (A_0^\nu)^{-1} F^\nu_{-1}]||_{0,\mu} \leq C (\mu^{1-|\alpha|} \mu^{(m-1)-|\beta|-|\gamma|} ||\partial^\nu w^{\varepsilon,M} \partial^\nu w^{\nu-1} \partial^\nu \Delta_{\nu}||_{0,\mu} \leq C ||w^\nu||_{m-1,\mu} + ||w^{\nu-1}||_{m-1,\mu} ||\Delta_{\nu}||_{m-1,\mu}$$

Thus, we get

$$||(A_0^\nu)^{-1} F^\nu - (A_0^\nu)^{-1} F^\nu_{-1}||_{m-1,\mu} \leq C ||w^\nu||_{m-1,\mu} + ||w^{\nu-1}||_{m-1,\mu} ||\Delta_{\nu}||_{m-1,\mu}$$
Similarly, we can obtain

$$
\mu^{(m-1)-1}[\partial^j \{((A_0^\nu)^{-1}A_j^\nu - (A_0^\nu^{-1})^{-1}A_j^\nu^{-1}] \partial_j(u^\nu,M + w^\nu)\}]_{0,\mu} \\
\leq C[|w^\nu|_{m-1,\mu} + |w^\nu^{-1}|_{m-1,\mu}]|\Delta_{\nu}|_* + |\Delta_{\nu}|_{m-1,\mu}
$$

and

$$
\varepsilon^2 \mu^{(m-1)-1}[\partial^j \{((A_0^\nu)^{-1} B_j^\nu - (A_0^\nu^{-1})^{-1} B_j^\nu^{-1}] \partial_j \partial_k(u^\nu,M + w^\nu)\}]_{0,\mu} \\
\leq C \varepsilon^2[|Dw^\nu_{jk}|_{m-1,\mu} + |w^\nu|_{m-1,\mu} + |w^\nu^{-1}|_{m-1,\mu}]|\Delta_{\nu}|_* + |\Delta_{\nu}|_{m-1,\mu}
$$

Hence, it follows

$$
|h_{\nu}|_{m-1,\mu} \leq C[|w^\nu|_{m-1,\mu} + |w^\nu^{-1}|_{m-1,\mu} + \varepsilon^2 |Dw^\nu_{jk}|_{m-1,\mu}]|\Delta_{\nu}|_* + |\Delta_{\nu}|_{m-1,\mu}
$$

On the other hand, by using the estimates (4.8)(4.9) with $B, u, m$ being replaced by $w^\nu, \Delta_{\nu+1}, m-1$, respectively, we have

$$
\mu^{(m-1)-1}[((A_0^\nu)^{-1} A_j^\nu \partial_j, \partial^j]_{\Delta_{\nu+1}}|_{0,\mu} \leq C(|\Delta_{\nu+1}|_{m-1,\mu} + |\Delta_{\nu+1}|_{Lip}|w^\nu|_{m-1,\mu})
$$

and

$$
\mu^{(m-1)-1}[((A_0^\nu)^{-1} B_j^\nu \partial_j, \partial^j]_{\Delta_{\nu+1}}|_{0,\mu} \leq C(|D(\Delta_{\nu+1})_{II}|_{m-1,\mu} + |D(\Delta_{\nu+1})_{II}|_{Lip}|w^\nu|_{m-1,\mu})
$$

Combining the above inequalities with (4.24)(4.25), and choosing $\mu = \frac{1}{2}$, there exists $\lambda_1 > 0$ such that for all $\lambda \geq \lambda_1$, the following inequality holds:

$$
\eta_{\nu+1} := |||\Delta_{\nu+1}|||_{m-1,\frac{\lambda}{2},\lambda} \leq C\left(|||w^\nu|||_{m-1,\frac{\lambda}{2},\lambda} + |||w^\nu^{-1}|||_{m-1,\frac{\lambda}{2},\lambda} + \varepsilon^2 ||Dw^\nu_{jk}|||_{m-1,\frac{\lambda}{2},\lambda}|\Delta_{\nu}|_* + \eta_{\nu} + (\varepsilon^2 |D(\Delta_{\nu+1})_{II}|_* + |\Delta_{\nu+1}|_{\lambda})|||w^\nu|||_{m-1,\frac{\lambda}{2},\lambda}\right) (4.26)
$$

where $\eta_{\nu+1} = |||\Delta_{\nu+1}|||_{m-1,\frac{\lambda}{2},\lambda}$.

By virtue of Lemma 4.9, there exists $\Psi(\lambda)$ and $0 < \varepsilon_0 \leq 1$, such that for any $\nu \in \mathbb{N}, 0 < \varepsilon \leq \varepsilon_0$, the following inequalities hold:

$$
|||w^\nu|||_{m-1,\frac{\lambda}{2},\lambda} \leq \Psi(\lambda), \quad ||Dw^\nu_{II}|||_{m-1,\frac{\lambda}{2},\lambda} \leq \Psi(\lambda)
$$

Thus, there exists a constant $C_1$ independent of $\lambda, \varepsilon$ such that (4.26) becomes

$$
\eta_{\nu+1} \leq \frac{C_1}{\lambda} \left(|||\Delta_{\nu}|_* + |\Delta_{\nu+1}|_* + |D(\Delta_{\nu+1})_{II}|_*\right) \Psi(\lambda) + \eta_{\nu} \right) (4.27)
$$

By Lemma 4.6, when $m-1 > \frac{m+1}{2} + 2, \delta = \frac{1}{4}$, there exists $\Psi_1(\lambda)$, the following inequalities hold:

$$
|||\Delta_{\nu}|_* \leq \varepsilon^\frac{1}{4} \Psi_1(\lambda)|||\Delta_{\nu}|||_{m-1,\frac{\lambda}{2},\lambda},
$$

$$
|D\Delta_{\nu+1}|_* \leq \varepsilon^\frac{1}{4} \Psi_1(\lambda)|||\Delta_{\nu+1}|||_{m-1,\frac{\lambda}{2},\lambda}.
$$
Therefore, from (4.27) we get
\[ \eta_{\nu+1} \leq C_1 \left\{ \varepsilon^4 \Psi(\lambda) \Psi_1(\lambda) \eta_{\nu+1} + C_1 \left( 1 + \varepsilon^4 \Psi(\lambda) \Psi_1(\lambda) \right) \eta_{\nu} \right\} \]

Fixed \( \lambda > \max(4C_1, 4C_R, \lambda_0, \lambda_1) \) firstly and next choosing \( \varepsilon_1 \) sufficiently small, for all \( 0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1) \), we can obtain \( \eta_{\nu+1} \leq \frac{1}{C} \eta_{\nu} \). Then the series \( \sum_{\nu=0}^{\infty} \Delta_{\nu} \) converges in \( H^{1,m-1}(\Omega_T) \). which yields that \( w^\nu \) converges to \( w \) in \( H^{1,m-1}(\Omega_T) \) when \( \nu \to \infty \).

Now, a passage to the limit in the iterative scheme (4.20), we get that
\[ w = \lim_{\nu \to \infty} w^\nu \]
is the solution of the problem (4.2) and \( w \in H^{1,m}(\Omega_T) \). Finally, Proposition 4.4 applied with \( B = u = w \) shows that \( w \in C(0, T; H^m(\mathbb{R}^n)) \) and \( w \in B^m_\sigma \) for some \( \sigma \) which depends only on \( \rho \), so we completes the proof of Theorem 4.1.

5 Oscillatory waves in the compressible NS equations

In this section, we apply the results obtained so far for highly oscillatory waves in the compressible viscous flow with small viscosity limit. Consider the isentropic compressible Navier-Stokes equations in two space variables:
\[
\begin{cases}
\partial_t \rho + (v \cdot \nabla) \rho + \rho \nabla \cdot v = 0 \\
\rho (\partial_t v + (v \cdot \nabla) v) + \nabla p = \nabla \cdot (2\mu p + \lambda I_2 \nabla \cdot v)
\end{cases}
\]
where \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^2\), \( p = \frac{1}{2} \{ \partial_x v_1, \partial_x v_2 \}_{1 \times 2} \) is \( 2 \times 2 \) matrix with \( v = (v_1, v_2)^T \), \( p = p(\rho) \) is the equation of state, \( \mu \) and \( \lambda \) denote the coefficient and the second coefficient of viscosity, respectively, with \( \mu > 0 \) and \( \mu' = \mu + \lambda \geq 0 \). Here we assume that \( \mu = \varepsilon^2 \), \( \mu' = D\varepsilon^2 \) and rewrite equations (5.1) in terms of \( V = (p, v_1, v_2)^T \) as:
\[
A_0(V) \partial_t V + A_1(V) \partial_{x_1} V + A_2(V) \partial_{x_2} V - \varepsilon^2 (B_{11} \partial_{x_1}^2 V + B_{22} \partial_{x_2}^2 V + B_{12} \partial_{x_1 x_2} V) = 0
\]
where
\[
A_0(V) = \begin{pmatrix} \frac{\partial p}{\rho} & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}, \quad A_1(V) = \begin{pmatrix} \frac{\partial p}{\rho} v_1 & 1 & 0 \\ 1 & \rho v_1 & 0 \\ 0 & 0 & \rho v_1 \end{pmatrix}, \quad A_2(V) = \begin{pmatrix} \frac{\partial p}{\rho} v_2 & 0 & 1 \\ 0 & \rho v_2 & 0 \\ 1 & 0 & \rho v_2 \end{pmatrix},
\]
\[
B_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + D & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + D & 0 \\ 0 & 0 & 1 + D \end{pmatrix}, B_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & D & 0 \end{pmatrix}.
\]
For any fixed \((\xi_1, \xi_2) \neq (0, 0)\), the symbol \(L(\lambda, \xi_1, \xi_2)\) associated to the Euler operator,

\[
L(\partial_t, \partial_x) = A_0(V)\partial_t + A_1(V)\partial_{x_1} + A_2(V)\partial_{x_2}
\]

has the eigenvalues

\[
\lambda_1 = -(\xi_1v_1 + \xi_2v_2), \lambda_{2,3} = -(\xi_1v_1 + \xi_2v_2 \pm c\sqrt{\xi_1^2 + \xi_2^2}), \quad (5.3)
\]

which are roots to the following characteristic equation:

\[
\text{det}(\lambda A_0(V) + \xi_1 A_1(V) + \xi_2 A_2(V)) = 0,
\]

with \(c = \frac{1}{\sqrt{\rho_c}}\) being the sound speed. The corresponding eigenvectors with the normalization are:

\[
\begin{align*}
\vec{r}_1 &= \frac{1}{\sqrt{\rho_c}}(0, \frac{\xi_2}{|\xi|}, -\frac{\xi_1}{|\xi|}), \vec{r}_1 = (\vec{r}_1)^T, \\
\vec{r}_2 &= \frac{1}{\sqrt{\rho_c}}(-pc, \frac{\xi_1}{|\xi|}, \frac{\xi_2}{|\xi|}), \vec{r}_2 = (\vec{r}_2)^T, \\
\vec{r}_3 &= \frac{1}{\sqrt{\rho_c}}(pc, \frac{\xi_1}{|\xi|}, \frac{\xi_2}{|\xi|}), \vec{r}_3 = (\vec{r}_3)^T.
\end{align*}
\]

(5.4)

It follows from (5.2) (5.3) and (5.4) that the assumptions (H1)-(H4) hold. If we add to (5.2) the high frequency oscillatory initial data:

\[
V(0, x) = v_0(x) + \varepsilon v_1(x, \frac{\varphi_0(x)}{\varepsilon})
\]

(5.5)

with the wave-length being the same order as the square root of the viscosity.

For the initial oscillation given in (5.5), we give the following assumption,

\[
v_1 = \bar{v}_1(x) + \sigma_0^0(x, \frac{\varphi_0(x)}{\varepsilon})\vec{r}_1(v_0, \nabla \varphi_0) + \sigma_2^0(x, \frac{\varphi_0(x)}{\varepsilon})\vec{r}_2(v_0, \nabla \varphi_0),
\]

where \(v_0, \bar{v}_1 \in H^s(\mathbb{R}^n), \sigma_0^0, \sigma_2^0 \in H^s(\mathbb{R}^n \times T^1)\) with their mean values in \(\theta \in T^1\) vanishing, and \(\varphi_0 \in H^{s+1}(\mathbb{R}^n)\) are given for a fixed \(s > 2\). Then, from Theorem 4.1, we know that the problem (5.2)(5.5) has a unique solution \(V^\varepsilon(t, x)\) for \(0 \leq t \leq T\), with the following asymptotic expansion

\[
V^\varepsilon(t, x) = V_0(t, x) + \varepsilon V_1(t, x, \frac{\varphi_1(t, x)}{\varepsilon}, \frac{\varphi_2(t, x)}{\varepsilon}) + O(\varepsilon),
\]

holding in \(L^\infty([0,T] \times \mathbb{R}^2)\). Here \(V_0 \in C([0,T], H^s(\mathbb{R}^n))\) satisfies the Cauchy problem of the Euler equations,

\[
\begin{cases}
A_0(V_0)\partial_t V_0 + A_1(V_0)\partial_{x_1} V_0 + A_2(V_0)\partial_{x_2} V_0 = 0 \\
V_0(0, x) = v_0(x)
\end{cases}
\]

(5.6)

and the leading oscillation profile \(V_1(t, x, \theta_1, \theta_2)\) is given by:

\[
V_1(t, x, \theta_1, \theta_2) = V_1(t, x) + V_1^{(1)}(t, x, \theta_1)\vec{r}_1(V_0, \nabla \varphi_1) + V_1^{(2)}(t, x, \theta_2)\vec{r}_2(V_0, \nabla \varphi_2),
\]

25
where the mean value of $V_1$, $\bar{V}_1(t, x)$ solves the following linear problem:

$$
\begin{align*}
& A_0(V_0)\partial_t \bar{V}_1 + A_1(V_0)\partial_{x_1} \bar{V}_1 + A_2(V_0)\partial_{x_2} \bar{V}_1 + \partial_t A_0(V_0)\bar{V}_1 \partial_t V_0 \\
& + \partial_{x_1} A_1(V_0)\bar{V}_1 \partial_t V_0 + \partial_{x_2} A_2(V_0)\bar{V}_1 \partial_{x_2} V_0 = 0 \\
& \bar{V}_1(0, x) = \bar{v}_1(x)
\end{align*}
$$

the oscillation phases $\varphi_1(t, x), \varphi_2(t, x)$ are given by:

$$
\begin{align*}
& \partial_t \varphi_1 + v_1 \partial_{x_1} \varphi_1 + v_2 \partial_{x_2} \varphi_1 = 0 \\
& \varphi_1(0, x) = \varphi_0(x)
\end{align*}
$$

and

$$
\begin{align*}
& \partial_t \varphi_2 + v_1 \partial_{x_1} \varphi_2 + v_2 \partial_{x_2} \varphi_2 - c(\partial_{x_1} \varphi_2)^2 + (\partial_{x_2} \varphi_2)^2 = 0 \\
& \varphi_2(0, x) = \varphi_0(x)
\end{align*}
$$

and $V_1^{(i)}(t, x, \theta_i)(i = 1, 2)$ satisfy the following problem for quasilinear degenerate parabolic equations,

$$
\begin{align*}
& \partial_t V_1^{(i)} + \sum_{j=1}^2 a_j(t, x)\partial_j V_1^{(i)} - b^1(t, x)\partial_{\theta_i}^2 V_1^{(i)} + c^1(t, x) V_1^{(i)} \partial_{\theta_i} V_1^{(i)} \\
& + d^1(t, x)\partial_{\theta_i} V_1^{(i)} + e^1(t, x) V_1^{(i)} = 0, \quad i = 1, 2 \\
& V_1^{(i)}(0, x, \theta_i) = \sigma_0(x, \theta_i)
\end{align*}
$$

where

$$
\begin{align*}
& b^1(t, x) = \frac{(\partial_{x_1} \varphi_1)^2 + (\partial_{x_2} \varphi_1)^2}{\rho_0} > 0, \\
& b^2(t, x) = \frac{(\partial_{x_1} \varphi_2)^2 + (\partial_{x_2} \varphi_2)^2}{2\rho_0} > 0,
\end{align*}
$$

with $\rho_0(t, x)$ being the density function of the solution to the problem (5.6), and $a_j(t, x), c^1(t, x), d^1(t, x), e^1(t, x)$ are given in (2.23) with $n = 2.$

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**References**


