FROM TWO-FLUID EULER-POISSON EQUATIONS TO
ONE-FLUID EULER EQUATIONS

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Abstract. We consider quasi-neutral limits in two-fluid isentropic Euler-Poisson equations arising in the modeling of unmagnetized plasmas and semiconductors. For periodic smooth solutions, we justify an asymptotic expansion in a time interval independent of the Debye length. This implies the convergence of the equations to compressible Euler equations. The proof is based on energy estimates for symmetrizable hyperbolic equations and on the exploration of the coupling between the Euler equations and the Poisson equation.

Keywords: Two-fluid Euler-Poisson system, the quasi-neutral limit, compressible Euler equations, local smooth solutions

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1. INTRODUCTION

This work is concerned with quasi-neutral limits of smooth solutions in two-fluid compressible isentropic Euler-Poisson system. This system plays an important role in the mathematical modeling and numerical simulation for plasmas and semiconductors [3, 15]. We consider an unmagnetized plasma or a semiconductor consisting of electrons of charge \( q_e = -1 \) and a single species of ions of charge \( q_i = 1 \). We denote by \( n_e, u_e \) (respectively, \( n_i, u_i \)) the scaled density and velocity vector of the electrons (respectively, ions) and by \( \phi \) the electric potential. These are all functions of the time \( t > 0 \) and the position \( x \in \mathbb{R}^d \), \( d \geq 1 \). Throughout this paper, we restrict to the case of functions being periodic in \( x \), i.e. \( x \in \mathbb{T}^d \), with \( \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \) being a \( d \)-dimensional torus.
The two-fluid isentropic Euler-Poisson system consists of a set of nonlinear conservation laws for densities and momentums coupled to a Poisson equation. It reads:

\begin{align}
\begin{cases}
\partial_t n_\nu + \text{div}(n_\nu u_\nu) = 0, \\
\partial_t (n_\nu u_\nu) + \text{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = -q_\nu n_\nu \nabla \phi - \frac{n_\nu u_\nu}{\tau_\nu}, \\
-\lambda^2 \Delta \phi = n_i - n_e,
\end{cases}
\end{align}

for \( \nu = e, i \) and \((t, x) \in (0, +\infty) \times \mathbb{T}^d \). Here \( \otimes \) stands for the tensor product and \( p_\nu \) are the pressure functions. Throughout this paper, we assume that \( p_\nu \) are smooth and strictly increasing for \( n_\nu > 0 \). This includes the usual state equation of \( \gamma \)-law:

\[ p_\nu(n) = a_\nu n^{\gamma_\nu}, \quad \text{with } \gamma_\nu \geq 1, \ a_\nu > 0, \ \nu = e, i. \]

In (1.1), the physical parameters are the scaled Debye length \( \lambda > 0 \) and the relaxation times \( \tau_\nu > 0 \). Usually, \( \lambda \) is small compared to the characteristic length of physical interest. Therefore, \( n_i \approx n_e \), which is the quasi-neutrality of the system.

Given initial conditions depending on \( \lambda \):

\[ t = 0 : \ (n_\nu, u_\nu) = (n_{\nu,0}, u_{\nu,0}), \quad \nu = e, i, \]

the aim of this paper is to establish a rigorous mathematical theory on the quasi-neutral limit of (1.1)-(1.2) as \( \lambda \to 0 \). It is easy to see that the existence of the terms with relaxation times implies the time dissipation property for \( u_\nu \), which is important to study the global existence of smooth solutions. See, for instance, [10, 9, 17] and references therein. In our limit problem as \( \lambda \to 0 \), this property is not useful in general. For simplicity, we delete these terms by letting \( \tau_\nu = \infty \). Therefore, for smooth solutions with \( n_\nu > 0 \), the second equation of (1.1) is equivalent to

\[ \partial_t u_\nu + (u_\nu \cdot \nabla)u_\nu + \nabla h_\nu(n_\nu) = -q_\nu \nabla \phi, \]

where \( u, u' \) denotes the inner product of \( \mathbb{R}^d \) and the enthalpy function \( h_\nu \) is defined by

\[ h_\nu(n_\nu) = \int_1^{n_\nu} \frac{p_\nu'(s)}{s} \, ds. \]

Since \( p_\nu \) is strictly increasing on \((0, +\infty)\), so is \( h_\nu \).

In problem (1.1)-(1.2), \( \phi \) is not determined in a unique way. To avoid this, we add a restriction condition

\[ m(\phi) = \int_{\mathbb{T}^d} \phi(\cdot, x) \, dx = 0. \]

By the Poincaré inequality (see [5]), for any fixed integer \( s' \geq 0 \), the Poisson equation in (1.1) with (1.5) gives estimate

\[ \|\nabla \phi\|_{H^{s'}(\mathbb{T}^d)} \leq C \lambda^{-2} \|n_i - n_e\|_{H^{s'}(\mathbb{T}^d)}. \]

Then, regarding \( \nabla \phi \) as a function of \( n_e \) and \( n_i \), from (1.1)-(1.3) we know that \( (n_\nu, u_\nu) \) for \( \nu = e, i \) formally satisfy a symmetrizable hyperbolic system in which \( \nabla \phi \) appearing on the right-hand side of (1.1) is a zero-th order term of \( (n_e, n_i) \). Following Kato [12], this implies that the periodic problem (1.1)-(1.2) admits a unique local smooth solution, provided that the initial data \( (n_{\nu,0}^\lambda, u_{\nu,0}^\lambda) \) for \( \nu = e, i \) are smooth. Moreover, estimate (1.6) implies that \( \phi \in C([0, T]; H^{s'+1}(\mathbb{T}^d)) \) as long as \( n_e, n_i \in C([0, T]; H^{s'}(\mathbb{T}^d)) \) for some constant \( T > 0 \) and a fixed integer \( s' \geq 0 \).

The global existence of smooth solutions to (1.1)-(1.2) was investigated in [8] for a potential flow. When the relaxation terms are present, the global smooth solutions with
their long time stability were obtained in [1, 2, 10] for solutions near constant equilibrium states and in [9] for solutions near a steady equilibrium state. We mention also results on this topic in [16, 17] for (1.1) as a particular case of the two-fluid Euler-Maxwell equations with different treatments.

Now we derive the formal quasi-neutral limit of (1.1). Let \((n^\lambda, u^\lambda, \phi^\lambda)\) be a smooth solution of (1.1) with \(\tau_\nu = \infty\). Formally, as \(\lambda \to 0\), the limit \((n_\nu, u_\nu, \phi_\nu)\) of \((n^\lambda, u^\lambda, \phi^\lambda)\) satisfies the problem

\[
\begin{align*}
\partial_t n_\nu + \text{div}(n_\nu u_\nu) &= 0, \\
\partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) &= -q_\nu \nabla \phi, \\
n_e &= n_i \overset{\text{def}}{=} n,
\end{align*}
\]

for \(\nu = e, i\). When \((n_e, u_e) = (n_i, u_i)\) at \(t = 0\), we further obtain \(u_e = u_i \overset{\text{def}}{=} u\) for all \(t \in [0, T]\) for some \(T > 0\) (see [18]). Adding the momentum equations in (1.7) for \(\nu = e, i\), we see that \((n, u)\) satisfies one-fluid compressible Euler equations

\[
\begin{align*}
\partial_t n + \text{div}(nu) &= 0, \\
\partial_t u + (u \cdot \nabla) u + \nabla h(n) &= 0,
\end{align*}
\]

with

\[
\phi = \frac{1}{2}(h_e(n) - h_i(n)) + \bar{\phi}(t), \quad h = \frac{1}{2}(h_e + h_i).
\]

The function \(\bar{\phi}\) should be chosen such that \(m(\phi) = 0\). It is clear that for smooth initial data with \(n(0, x) \geq \text{const.} > 0\), system (1.8) admits a unique local in time smooth solution satisfying \(n(t, x) \geq \text{const.} > 0\), see [12, 14]. A general approximate solution of (1.1) constructed as an asymptotic expression in power of \(\lambda^2\) is given in the next section.

For later purpose, we consider a special solution of (1.8) for which the density \(n\) depends only on \(t\). This leads to \(n = n(t)\) and

\[
\begin{align*}
n'(t) + n \text{ div } u &= 0, \\
\partial_t u + (u \cdot \nabla) u &= 0.
\end{align*}
\]

This system admits non-constant solutions. For example, in one space dimension, the solution of (1.10) is given by

\[
n(t) = \frac{n_0}{1 + a_0 t}, \quad u(t, x) = \frac{u_0 + a_0 x}{1 + a_0 t},
\]

where \(n_0 > 0\), \(u_0\) and \(a_0\) are constants. This solution is global in time and not a constant if \(a_0 > 0\).

When the ion density \(n_i\) is a given function, (1.1) can be reduced to a one-fluid Euler-Poisson system for which the formal quasi-neutral limit is the incompressible Euler system. In the framework of local smooth solutions, this limit was studied in [4] in one dimensional case and in [22] in several dimensional case, both when \(n_i\) is a constant. When \(n_i\) is not a constant, this limit was justified in [19]. Extension results for non-isentropic Euler-Poisson system were given in [20].

For the two-fluid Euler-Poisson system, a formal asymptotic expansion was first introduced in [18] and is presented in subsection 2.2 of this paper. The justification of this expansion was studied in [11], following completely the same techniques as those in [19]. In particular, for an asymptotic expansion of order \(O(\lambda^{2m})\), the authors of [11] obtained
a convergence rate of order $O(\lambda^{2(m-s)})$ in Sobolev space $H^s$ for all integer $m > s$, with $s > d/2 + 1$. In their proof as that of [19], the Poisson equation was used in a rough way (see (1.6)) to show that $\nabla \phi^\lambda$ has the same order as $\lambda^{-2}(n^\lambda_1 - n^\lambda_2)$. This leads to a loss of convergence rate of order $O(\lambda^{-s})$ in $H^s$.

In this paper, we want to improve these results. By exploring the coupling of the Euler equations and the Poisson equation, we show that $\nabla \phi^\lambda$ has the same order as $\lambda^{-1}(n^\lambda_1 - n^\lambda_2)$. This enables us to obtain a convergence rate of order $O(\lambda^{2m-s+1})$ in $H^s$ for all integer $2m > s$. In particular, when $d \leq 3$ the result holds for all integer $2m \geq s$. Moreover, when the density of the leading profiles of the asymptotic expansion depends only on $t$, we obtain a convergence rate of order $O(\lambda^{2m+1})$ for all $m \geq 1$. In view of the $L^2$ energy estimates established in subsection 3.1, these convergence rates seem optimal. The proof is based on energy estimates for symmetrizable hyperbolic equations. To simplify the proof, we make a priori estimates (see (3.6)-(3.7)) and employ an induction argument with respect to the derivative orders of solutions.

This paper is organized as follows. In the next section, we first list several useful inequalities in Sobolev spaces and recall the result on the local existence of smooth solutions to (1.1)-(1.2). Then we consider the approximate solution constructed as an asymptotic expansion in power of $\lambda^2$. The main result of the paper is Theorem 2.1, which is stated in the end of section 2. Section 3 is devoted to detailed energy estimates by taking into account the coupling between the Euler equations and the Poisson equations. Finally, the proof of Theorem 2.1 is given in section 4.

2. Preliminaries and main results

2.1. Notations and inequalities. We first introduce the following notations. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ with $d \geq 1$, we denote

$$\partial^\alpha_x = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \quad \text{with} \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$ 

For $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ and $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}^d$, $\gamma < \alpha$ stands for $\gamma \neq \alpha$ and $\gamma_j \leq \alpha_j$ for all $j = 1, 2, \cdots, d$. We denote by $\| \cdot \|_s$ the norm of the usual Sobolev space $H^s(\mathbb{T}^d)$, and by $\| \cdot \|$ and $\| \cdot \|_\infty$ the norms of $L^2(\mathbb{T}^d)$ and $L^\infty(\mathbb{T}^d)$, respectively. For convenience, we also make a convention that $\| \cdot \| = \| \cdot \|_0$. The inner product in $L^2(\mathbb{T}^d)$ is denoted by $(\cdot, \cdot)$.

The following results are needed in the proof of the main results.

**Lemma 2.1.** (Moser-type calculus inequalities, see [13, 14]) Let $s \geq 1$ be an integer. Suppose $u \in H^s(\mathbb{T}^d)$, $\nabla u \in L^\infty(\mathbb{T}^d)$ and $v \in H^{s-1}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$. Then for all multi-index $\alpha \in \mathbb{N}^d$ with $1 \leq |\alpha| \leq s$ and all smooth function $f$, we have $\partial^\alpha_x (uv) - u\partial^\alpha_x v \in L^2(\mathbb{T}^d)$, $\partial^\alpha_x f(u) \in L^2(\mathbb{T}^d)$ and

$$\| \partial^\alpha_x (uv) - u\partial^\alpha_x v \| \leq C_s \left( \| \nabla u \|_\infty \| D^{[\alpha]-1} v \| + \| D^{[\alpha]} u \| \| v \|_\infty \right),$$

$$\| \partial^\alpha_x f(u) \| \leq C_s \left( 1 + \| \nabla u \|_\infty \right)^{|\alpha|-1} \| D^{[\alpha]} u \|,$$

where the constants $C_s > 0$ depends on $\| u \|_\infty$ and $s$, and $C_s > 0$ depends only on $s$ and $\sum_{|\alpha| = s'} \| \partial^\alpha_x u \|$, $\forall s' \in \mathbb{N}$. 

Moreover, if \( s > \frac{d}{2} + 1 \), then the embedding \( H^s(\mathbb{T}^d) \hookrightarrow W^{1,\infty}(\mathbb{T}^d) \) is continuous and we have
\[
\|\partial_x^a (uv) - u \partial_x^a v\| \leq C_s \|u\|_s \|v\|_s \|u\|_{|\alpha|}.
\]

**Lemma 2.2.** For any smooth functions \( u : \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( \Phi : \mathbb{R}^d \rightarrow \mathbb{R} \), we have
\[
(2.1) \quad \left| (u \Delta \Phi, \nabla \Phi) \right| \leq C \| \nabla u \|_s \| \nabla \Phi \|_s^2,
\]
where the constant \( C > 0 \) is independent of \( u \) and \( \Phi \).

**Proof.** Let us denote \( u = (u_1, \ldots, u_d) \). Integrating by parts yields
\[
2(u \Delta \Phi, \nabla \Phi) = 2 \sum_{i,j=1}^d \left( \partial_i^2 \Phi, u_j \frac{\partial \Phi}{\partial x_j} \right)
\]
\[
= -2 \sum_{i,j=1}^d \left( \partial_i \Phi, u_j \frac{\partial \Phi}{\partial x_j} \right) - 2 \sum_{i,j=1}^d \left( \frac{\partial \Phi}{\partial x_i}, u_j \frac{\partial \Phi}{\partial x_j} \right)
\]
\[
= -2 \sum_{i,j=1}^d \left( \frac{\partial \Phi}{\partial x_i}, u_j \frac{\partial \Phi}{\partial x_j} \right)^2 - 2 \sum_{i,j=1}^d \left( \frac{\partial \Phi}{\partial x_i}, u_j \frac{\partial \Phi}{\partial x_j} \right),
\]
which implies (2.1). \( \square \)

The next result is a refined version of Lemma 2.1 in the case \( d \leq 3 \). Its proof can be found in [17]. For the sake of completeness, we give it in Appendix.

**Lemma 2.3.** Let \( s > \frac{d}{2} + 1 \) be an integer with \( d \leq 3 \). For all \( \alpha \in \mathbb{N}^d \) with \( 1 \leq |\alpha| \leq s \), if \( u \in H^s(\mathbb{T}^d) \) and \( v \in H^{s_1}(\mathbb{T}^d) \), then
\[
(2.2) \quad \| \partial_x^a (uv) - u \partial_x^a v\| \leq C_s \|\nabla u\|_{s-1} \|v\|_{|\alpha|-1}.
\]

The next result concerns the local existence of smooth solutions, which can be easily obtained by employing the theory of Kato [12] (see also [14]) for the symmetrizable hyperbolic system (1.1)-(1.3) and using the estimate (1.6).

**Proposition 2.1.** Let \( s > \frac{d}{2} + 1 \) be an integer and \((n_0^\lambda, u_0^\lambda) \in H^s(\mathbb{T}^d) \) with \( n_0^\lambda \geq 2\kappa \), for some given constant \( \kappa > 0 \) independent of \( \lambda \). Then there exist \( T_1^\lambda > 0 \) and a unique smooth solution \((n_\nu^\lambda, u_\nu^\lambda, \phi^\lambda) \) with \( n_\nu^\lambda \geq \kappa \) for \( \nu = e, i \) to periodic problem (1.1)-(1.2) with (1.5) defined in time interval \([0, T_1^\lambda]\), satisfying
\[
(n_\nu^\lambda, u_\nu^\lambda) \in C([0, T_1^\lambda]; H^s(\mathbb{T}^d)) \cap C^1([0, T_1^\lambda]; H^{s-1}(\mathbb{T}^d)), \quad \nu = e, i,
\]
\[
\phi^\lambda \in C([0, T_1^\lambda]; H^{s+1}(\mathbb{T}^d)) \cap C^1([0, T_1^\lambda]; H^s(\mathbb{T}^d)).
\]

### 2.2. Asymptotic expansions

We seek \((n_\nu^\lambda, u_\nu^\lambda, \phi^\lambda)\), an approximate solution of (1.1) under the form of an asymptotic expansion of a power series in \( \lambda \):
\[
(2.3) \quad (n_\nu^\lambda, u_\nu^\lambda, \phi^\lambda)(t,x) = \sum_{k=0}^\infty \lambda^{2k} (n_\nu^k, u_\nu^k, \phi^k)(t,x), \quad \nu = e, i,
\]
provided that the initial conditions given in (1.2) have the following expansions:

\[(n_{\nu,\lambda}, u_{\nu,\lambda})(0, x) = \sum_{k=0}^{\infty} \lambda^{2k}(\bar{n}_{\nu}^{k}, \bar{u}_{\nu}^{k})(x), \quad \nu = e, i,\]

where \((\bar{n}_{\nu}^{k}, \bar{u}_{\nu}^{k})\) are given smooth periodic functions on \(\mathbb{T}^d\). The construction of profiles \((n_{\nu}^{k}, u_{\nu}^{k}, \phi^{k})\) are similar to that of the two-fluid Euler-Maxwell equations discussed in [18], see also [11]. Precisely, substituting expression (2.3) into (1.1) and identifying the coefficients in power of \(\lambda^2\), we see that, for each \(k \geq 0\), the profiles \((n_{\nu}^{k}, u_{\nu}^{k}, \phi^{k})\) satisfy a system of partial differential equations related to (1.1). In order that the quasi-neutral limit of (1.1) leads to system (1.8)-(1.9), it is natural to suppose the quasi-neutral condition at initial time for the leading profiles, namely,

\[(\bar{n}_{\nu}^{0}, \bar{u}_{\nu}^{0}) = (\bar{n}_{\nu}^{0}, \bar{n}_{\nu}^{0}), \quad \text{on } \mathbb{T}^d.\]

Then it was proved in [18] that the leading profiles \((n_{\nu}^{0}, u_{\nu}^{0}, \phi^{0})\) satisfy (1.7)-(1.9). For simplicity, in what follows we denote

\[(n, u, \phi) = (n_{\nu}^{0}, u_{\nu}^{0}, \phi^{0}), \quad \nu = e, i.\]

For \(k \geq 1\), the profiles \((n_{\nu}^{k}, u_{\nu}^{k}, \phi^{k})\) are obtained by induction on \(k\). Assume that \((n_{\nu}^{l}, u_{\nu}^{l}, \phi^{l})\) are smooth and already determined for all \(0 \leq l \leq k-1\). Then \((n_{\nu}^{k}, u_{\nu}^{k}, \phi^{k})\) satisfies a linearized system :

\[
\begin{cases}
\partial_t n_{\nu}^{k} + \text{div}(nu_{\nu}^{k} + n_{\nu}^{k}u) = -\sum_{l=1}^{k-1} \text{div}(n_{\nu}^{l}u_{\nu}^{k-l}), & \nu = e, i, \\
\partial_t u_{\nu}^{k} + (u \cdot \nabla)u_{\nu}^{k} + (u_{\nu}^{k} \cdot \nabla)u + \nabla(h_{\nu}'(n)n_{\nu}^{k}) + q_{\nu}\nabla\phi^{k} = -\sum_{l=1}^{k-1} (u_{\nu}^{l} \cdot \nabla)u_{\nu}^{k-l} - \nabla((h_{\nu}'(n)_{l \leq k-1}), & \nu = e, i, \\
-\Delta\phi^{k-1} = n_{\nu}^{k} - n_{\nu}^{k_{e}}, & m(\phi^{k}) = 0,
\end{cases}
\]

where \(h_{\nu}^{0} = 0\) and \(h_{\nu}^{k-1}\) for \(k \geq 2\) is a function depending only on \((n_{\nu}^{l})_{0 \leq l \leq k-1}\) and is defined by

\[h_{\nu}(\sum_{k \geq 0}^{\infty} \lambda^{2k}n_{\nu}^{k}) = \sum_{k \geq 0}^{\infty} c_{\nu}^{k}\lambda^{2k},\]

with

\[c_{\nu}^{0} = h_{\nu}(n), \quad c_{\nu}^{1} = h_{\nu}'(n)n_{\nu}^{1}, \quad c_{\nu}^{k} = h_{\nu}'(n)n_{\nu}^{k} + h_{\nu}'(n)n_{\nu}^{k-1}((n_{\nu}^{l})_{l \leq k-1}), \quad \forall k \geq 2.\]

In (2.6) the convention \(\sum_{l=1}^{0} = 0\) is also used. The initial conditions of \((n_{\nu}^{k}, u_{\nu}^{k})\) are

\[(n_{\nu}^{k}, u_{\nu}^{k})(0, x) = (\bar{n}_{\nu}^{k}, \bar{u}_{\nu}^{k})(x), \quad \nu = e, i, \quad k \in \mathbb{N}.
\]

In order to determine \((n_{\nu}^{k}, u_{\nu}^{k}, \phi^{k})\) for \(k \geq 1\), the last equation in (2.6) suggests a compatibility condition on the given data :

\[(\bar{n}_{\nu}^{k} - \bar{n}_{\nu}^{k_{e}}) = -\Delta\phi^{k-1}(0, \cdot),\]

where \(\phi^{k-1}\) is solved with \((n_{\nu}^{l-1}, u_{\nu}^{l-1})\) in the \(k-1\) step. The unusual problem (2.6)-(2.7) can be solved through solving other two classical problems of variables

\[n_{\nu}^{k} = n_{\nu}^{k_{e}} + n_{\nu}^{k_{i}}, \quad u_{\nu}^{k} = u_{\nu}^{k_{e}} \pm u_{\nu}^{k_{i}}.\]

Indeed, adding the linearized Euler equations in (2.6) for \(\nu = e, i\) to eliminate \(\phi^{k}\), we get that \((n_{\nu}^{k}, u_{\nu}^{k})\) satisfies linearized compressible Euler equations and \(u_{\nu}^{k}\) satisfies linearized
incompressible Euler equations with pressure $-2\phi^k$, both with given source terms. Once $(n^k_\nu, u^k_\nu)$ and $(u^k, \phi^k)$ are known, we obtain $(n^k_\nu, u^k_\nu)$ for $\nu = e, i$ from (2.9) together with the last equation in (2.6). We refer to [18] for more details.

We conclude the above discussion in the following result.

**Proposition 2.2.** Assume that the initial profiles $(\bar{n}^k_\nu, \bar{u}^k_\nu)_{k \geq 0}$ given in (2.4) are sufficiently smooth with $\bar{n}^0_\nu > 0$ in $\mathbb{T}^d$ and satisfy the compatibility conditions (2.5) and (2.8). Then there exists a unique asymptotic expansion up to any order of the form (2.3) for the solution to the problem (1.1) and (2.4), i.e. there exist $T_1 > 0$ and the unique smooth profiles $(n^k_\nu, u^k_\nu, \phi^k)_{k \geq 0}$, solutions of the problems (1.7)-(1.9) and (2.6) with (2.7) in time interval $[0, T_1]$. The solution satisfies $n_\nu > 0$ in $[0, T_1] \times \mathbb{T}^d$. In particular, the formal quasi-neutral limit as $\lambda \to 0$ of the two-fluid compressible Euler-Poisson system (1.1) is the (one-fluid) compressible Euler system (1.8) with (1.9).

**Remark 2.1.** The time interval $[0, T_1]$ is determined by solving the nonlinear compressible Euler equations (1.8) for the leading order profiles.

### 2.3. Error estimates and main results

Let $m \in \mathbb{N}$ be a fixed integer and $(n^\lambda_\nu, u^\lambda_\nu, \phi^\lambda)$ be the exact solution to problem (1.1)-(1.2) defined in time interval $[0, T^\lambda]$. We denote by $(n^m_{\nu, \lambda}, u^m_{\nu, \lambda}, \phi^m_{\lambda})$ the approximate solution of order $m$ defined in $[0, T^\lambda]$ by

\[
(n^m_{\nu, \lambda}, u^m_{\nu, \lambda}, \phi^m_{\lambda})(t, x) = \sum_{k=0}^m \lambda^{2k}(n^k_\nu, u^k_\nu, \phi^k)(t, x), \quad \nu = e, i,
\]

where $(n^k_\nu, u^k_\nu, \phi^k)_{0 \leq k \leq m}$ are constructed in the previous subsection. The convergence of the asymptotic expansion (2.3) is to establish the limit $(n^\lambda_\nu, u^\lambda_\nu, \phi^\lambda) \to (n^m_{\nu, \lambda}, u^m_{\nu, \lambda}, \phi^m_{\lambda})$ and its convergence rate as $\lambda \to 0$ in a time interval independent of $\lambda$, when the convergence holds at $t = 0$.

For $\nu = e, i$, define the remainders $(R^\lambda_{n_e}, R^\lambda_{u_e}, R^\lambda_\phi)$ by

\[
\begin{aligned}
\partial_t n^m_{\nu, \lambda} + \text{div}(n^m_{\nu, \lambda} u^m_{\nu, \lambda}) &= R^\lambda_{n_e}, \\
\partial_t u^m_{\nu, \lambda} + (u^m_{\nu, \lambda} \cdot \nabla) u^m_{\nu, \lambda} + \nabla h_\nu(n^m_{\nu, \lambda}) &= -q_\nu \nabla \phi^m_{\lambda} + R^\lambda_{u_e}, \\
-\lambda^2 \Delta \phi^m_{\lambda} &= n^m_{i, \lambda} - n^m_{e, \lambda} + R^\lambda_\phi.
\end{aligned}
\]

(2.10)

It is clear that the convergence rate depends strongly on the order of the remainders with respect to $\lambda$. Since the profiles $(n^k_\nu, u^k_\nu, \phi^k)_{0 \leq k \leq m}$ are sufficiently smooth, a straightforward computation gives the following result.

**Proposition 2.3.** Let the assumptions of Proposition 2.2 hold. Then for all integers $m \geq 0$ and $s \geq 0$, the remainders $(R^\lambda_{n_e}, R^\lambda_{u_e}, R^\lambda_\phi)$ satisfy

\[
\sup_{0 \leq t \leq T^\lambda} \| (R^\lambda_{n_e}(t, \cdot), R^\lambda_{u_e}(t, \cdot), R^\lambda_\phi(t, \cdot)) \|_s \leq C_m \lambda^{2(m+1)}, \quad \nu = e, i,
\]

(2.11)

where $C_m > 0$ is a constant independent of $\lambda$.

The main result of this paper is the following convergence result whose proof will be given in sections 3 and 4.

**Theorem 2.1.** Let $s > \frac{d}{2} + 1$ and $m \in \mathbb{N}$ be integers. We assume that (2.5) holds and

\[
\| (n^\lambda_{\nu, 0} - n^m_{\nu, \lambda}(0, \cdot), u^\lambda_{\nu, 0} - u^m_{\nu, \lambda}(0, \cdot)) \|_s \leq C_1 \lambda^{2(m+1)}, \quad \nu = e, i,
\]

(2.12)
where \( C_1 > 0 \) is a constant independent of \( \lambda \). Then there exists a constant \( C_2 > 0 \), which depends on \( T_1 \) but is independent of \( \lambda \), such that as \( \lambda \to 0 \) we have \( T_1^\lambda \geq T_1 \) and for all integer \( 2m > s \), the solution \((n^\lambda, u^\lambda, \phi^\lambda), \nu = e, i\), to the periodic problem (1.1)-(1.2) satisfies
\[
\sup_{0 \leq t \leq T_1} \| (n^\lambda(t) - n_{m,\nu,\lambda}^m(t), u^\lambda(t) - u_{m,\nu,\lambda}^m(t)) \|_s \leq C_2 \lambda^{2m-s+1}, \quad \nu = e, i,
\]
and
\[
\sup_{0 \leq t \leq T_1} \| \nabla (\phi^\lambda(t) - \phi_{m,\nu,\lambda}^m(t)) \|_s \leq C_2 \lambda^{2m-s}.
\]
In particular, when \( d \leq 3 \), (2.13)-(2.14) hold for all integer \( 2m \geq s \).

Moreover, if the density of the leading profiles, \( n \), is positive and depends only on \( t \), then for all integer \( m \geq 1 \), we have
\[
\sup_{0 \leq t \leq T_1} \| (n^\lambda(t) - n_{m,\nu,\lambda}^m(t), u^\lambda(t) - u_{m,\nu,\lambda}^m(t)) \|_s \leq C_2 \lambda^{2m+1}, \quad \nu = e, i,
\]
and
\[
\sup_{0 \leq t \leq T_1} \| \nabla (\phi^\lambda(t) - \phi_{m,\nu,\lambda}^m(t)) \|_s \leq C_2 \lambda^{2m}.
\]

**Remark 2.2.** Estimates (2.15)-(2.16) seem optimal in view of the \( L^2 \) energy estimate (see Lemma 3.1). They are obtained by overcoming a technical obstacle in the proof when the density of the leading profiles depends only on \( t \) (see the proof of Proposition 4.2).

### 3. Energy estimates

#### 3.1. Preliminaries.
By Propositions 2.1-2.2, the exact solution \((n^\lambda, u^\lambda, \phi^\lambda)\) of (1.1)-(1.2) is defined in time interval \([0, T_1^\lambda]\) and the approximate solution \((n_{m,\nu,\lambda}^m, u_{m,\nu,\lambda}^m, \phi_{m,\nu,\lambda}^m)\) is defined in time interval \([0, T_1]\), with \( T_1^\lambda > 0 \) and \( T_1 > 0 \). Let
\[
T_2^\lambda = \min(T_1^\lambda, T_1) > 0.
\]
Then the exact solution and the approximate solution are both defined in time interval \([0, T_2^\lambda]\). In this time interval, we denote
\[
(N^\lambda, U^\lambda, \Phi^\lambda) = (n^\lambda - n_{m,\nu,\lambda}^m, u^\lambda - u_{m,\nu,\lambda}^m, \phi^\lambda - \phi_{m,\nu,\lambda}^m), \quad \nu = e, i.
\]
It is easy to check that the variable \((N^\lambda, U^\lambda)\) satisfies Euler equations:
\[
\begin{cases}
\partial_t N^\lambda + (U^\lambda + u_{m,\nu,\lambda}^m) \cdot \nabla N^\lambda + (N^\lambda + n_{m,\nu,\lambda}^m) \text{ div } U^\lambda = - (N^\lambda \text{ div } u_{m,\nu,\lambda}^m + U^\lambda \cdot \nabla n_{m,\nu,\lambda}^m) - R^\lambda_{nu}, \quad \nu = e, i, \\
\partial_t U^\lambda + ((U^\lambda + u_{m,\nu,\lambda}^m) \cdot \nabla) U^\lambda + h^\nu(N^\lambda + n_{m,\nu,\lambda}^m) \nabla N^\lambda = -((U^\lambda \cdot \nabla) u_{m,\nu,\lambda}^m + (h^\nu(N^\lambda + n_{m,\nu,\lambda}^m) - h^\nu(n_{m,\nu,\lambda}^m) \nabla n_{m,\nu,\lambda}^m + q^\nu \nabla \Phi^\lambda)) - R^\lambda_{nu},
\end{cases}
\]
coupled to a Poisson equation for \( \Phi^\lambda \):
\[
-\lambda^2 \Delta \Phi^\lambda = N^\lambda_i - N^\lambda_e - R^\lambda_\phi.
\]
For \( \nu = e, i \), set
\[
W^\lambda = \begin{pmatrix} N^\lambda \\ U^\lambda \end{pmatrix}, \quad W^\lambda_e = \begin{pmatrix} W^\lambda_e \\ W^\lambda_i \end{pmatrix}, \quad W^\lambda_{nu,0} = \begin{pmatrix} n_{\nu,0}^\lambda - n_{m,\nu,\lambda}^m(0, \cdot) \\ u_{\nu,0}^\lambda - u_{m,\nu,\lambda}^m(0, \cdot) \end{pmatrix}, \quad W^\lambda_0 = \begin{pmatrix} W^\lambda_{e,0} \\ W^\lambda_{i,0} \end{pmatrix}.
\]
\begin{equation}
H_{\nu,\lambda}^1 = \begin{pmatrix}
N_\nu^\lambda \text{div } u_{\nu,\lambda}^m + U_\nu^\lambda \cdot \nabla n_{\nu,\lambda}^m \\
(U_\nu^\lambda \cdot \nabla) u_{\nu,\lambda}^m + (h'_\nu(n_\nu^\lambda + n_{\nu,\lambda}^m) - h'_\nu(n_{\nu,\lambda}^m)) \nabla n_{\nu,\lambda}^m
\end{pmatrix}, \quad H_{\nu,\lambda}^2 = \begin{pmatrix} 0 \\
q_\nu \nabla \Phi_\lambda
\end{pmatrix},
\end{equation}

and

\begin{equation}
R_\nu^\lambda = \begin{pmatrix} R_{n_\nu}^\lambda \\
R_{u_\nu}^\lambda
\end{pmatrix},
\end{equation}

where \((e_1, \ldots, e_d)\) is the canonical basis of \(\mathbb{R}^d\), \(y_j\) denotes the \(j\)th-component of \(y \in \mathbb{R}^d\) and \(I_d\) is the \(d \times d\) unit matrix. Thus problem (3.2) for unknown \(W_\nu^\lambda\) can be rewritten as

\begin{equation}
\partial_t W_\nu^\lambda + \sum_{j=1}^d A_j(n_\nu^\lambda, u_\nu^\lambda) \partial_{x_j} W_\nu^\lambda = -H_{\nu,\lambda}^1 - H_{\nu,\lambda}^2 - R_\nu^\lambda, \quad \nu = e, i,
\end{equation}

in which \(\Phi_\lambda\) and \(W_\nu^\lambda\) for \(\nu = e, i\) are linked by the Poisson equation (3.3). The initial condition of \(W_\nu^\lambda\) is

\begin{equation}
t = 0 : \quad W_\nu^\lambda = W_{\nu,0}^\lambda, \quad \nu = e, i.
\end{equation}

System (3.4) for \(W^\lambda\) is symmetrizable hyperbolic when \(n_\nu^\lambda = N_\nu^\lambda + n_{\nu,\lambda}^m > 0\). Indeed, since the density \(n\) of the leading profiles satisfies \(n \geq \text{const.} > 0\), \(n_{\nu,\lambda}^m - n = O(\lambda^2)\), and we shall prove later that \(N_\nu^\lambda\) is small for small \(\lambda\), so we have \(n_\nu^\lambda > 0\) for \(\nu = e, i\). Let

\begin{equation}
A^0_\nu(n_\nu^\lambda) = \begin{pmatrix} h'_\nu(n_\nu^\lambda) & 0 \\
0 & n_\nu^\lambda I_d
\end{pmatrix},
\end{equation}

and

\begin{equation}
\tilde{A}_j^0(n_\nu^\lambda, u_\nu^\lambda) = A^0_\nu(n_\nu^\lambda) A^0_\nu(n_\nu^\lambda, u_\nu^\lambda) = \begin{pmatrix} h'_\nu(n_\nu^\lambda) u_{\nu,j}^\lambda & p'_\nu(n_\nu^\lambda) e_j^T \\
p'_\nu(n_\nu^\lambda) e_j & n_\nu^\lambda u_{\nu,j}^\lambda I_d
\end{pmatrix}.
\end{equation}

Then for \(n_\nu^\lambda > 0\), \(A^0_\nu\) is positively definite and \(\tilde{A}_j^0\) is symmetric for all \(1 \leq j \leq d\). Moreover, similarly to the discussion given after (1.6), the potential \(\Phi_\lambda\) appearing in \(H_{\nu,\lambda}^2\) coupled to the Poisson equation (3.3) can be regarded as a lower order terms of \(W_\nu^\lambda\) even though it is non-local. Hence, system (3.4) can be regarded as being symmetrizable hyperbolic for \(W_\nu^\lambda\). Thus, the theorem of Kato for the local existence of smooth solutions can also be applied to (3.4)-(3.5) coupled with (3.3).

By standard arguments, to prove Theorem 2.1, it suffices to establish uniform estimates of \(W_\nu^\lambda\) with respect to \(\lambda\). In what follows, we denote by \(C > 0\) various constants independent of \(\lambda\). Since \(W_\nu^\lambda \in C([0, T^\lambda_2]; H^s(\mathbb{T}^d))\), the function \(t \rightarrow \|W_\nu^\lambda(t)\|_s\) is continuous on \([0, T^\lambda_2]\). From assumption (2.12) and \(2(m + 1) > 1\), there exists \(T^\lambda \in (0, T^\lambda_2]\) such that

\begin{equation}
\|W_\nu^\lambda(t)\|_s \leq C, \quad \forall \ t \in [0, T^\lambda],
\end{equation}

or

\begin{equation}
\|W_\nu^\lambda(t)\|_s \leq C\lambda, \quad \forall \ t \in [0, T^\lambda],
\end{equation}

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provided that \( \lambda > 0 \) is bounded by a constant. If \( s > d/2 + 1 \), the imbedding from \( H^s(\mathbb{T}^d) \) to \( W^{1,\infty}(\mathbb{T}^d) \) is continuous. Then in both cases (3.6) and (3.7), we have
\[
\|W^\lambda(t)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C, \quad \forall t \in [0, T^\lambda].
\]
In order that \( T^\lambda_1 \geq T_1 \), we need to show that there exists a constant \( \mu > 0 \) such that, in case (3.6),
\[
\sup_{0 \leq t \leq T^\lambda} (\|W^\lambda(t)\|_s + \lambda\|\nabla \Phi^\lambda(t)\|_s) \leq C\lambda^\mu,
\]
and in case (3.7),
\[
\sup_{0 \leq t \leq T^\lambda} (\|W^\lambda(t)\|_s + \lambda\|\nabla \Phi^\lambda(t)\|_s) \leq C\lambda^{1+\mu}.
\]
See the proof of Theorem 2.1, which is to be given in section 4.

3.2. \( L^2 \)-estimates. In what follows, we always assume that the conditions of Theorem 2.1 and (3.6) hold.

Lemma 3.1. We have
\[
\|W^\lambda(t)\|^2 + \lambda^2\|\nabla \Phi^\lambda(t)\|^2 \leq C\lambda^{4m+2}, \quad \forall t \in [0, T^\lambda].
\]
Proof. Taking the inner product of (3.4) with \( 2A^0_\nu(n^\lambda_\nu)W^\lambda_\nu \) in \( L^2(\mathbb{T}^d) \) and employing the classical energy estimate for symmetrizable hyperbolic equations, we obtain the following energy equality for \( W^\lambda_\nu \):
\[
\frac{d}{dt} (A^0_\nu(n^\lambda_\nu)W^\lambda_\nu, W^\lambda_\nu) = -2(A^0_\nu(n^\lambda_\nu)H^{1,\lambda}_\nu, W^\lambda_\nu) - 2(A^0_\nu(n^\lambda_\nu)H^{2,\lambda}_\nu, W^\lambda_\nu) - 2(A^0_\nu(n^\lambda_\nu)W^\lambda_\nu, R^\lambda_\nu) + (\text{div} A_\nu(n^\lambda_\nu, u^\lambda_\nu)W^\lambda_\nu, W^\lambda_\nu),
\]
where \((\cdot, \cdot)\) stands for the inner product of \( L^2(\mathbb{T}^d) \) and
\[
\text{div} A_\nu(n^\lambda_\nu, u^\lambda_\nu) = \partial_t A^0_\nu(n^\lambda_\nu) + \sum_{j=1}^d \partial x_j \tilde{A}^j_\nu(n^\lambda_\nu, u^\lambda_\nu).
\]
Now let us estimate each term on the right-hand side of (3.9). In view of the expressions of \( A^0_\nu \), \( \text{div} A_\nu \) and \( H^{1,\lambda}_\nu \), we have
\[
| (A^0_\nu(n^\lambda_\nu)H^{1,\lambda}_\nu, W^\lambda_\nu) | \leq C\|W^\lambda_\nu\|^2
\]
and
\[
| \text{div} A_\nu(n^\lambda_\nu, u^\lambda_\nu) | \leq C\|\nabla (n^\lambda_\nu, u^\lambda_\nu)\|_\infty \leq C\|(n^\lambda_\nu, u^\lambda_\nu)\|_s \leq C.
\]
It follows that
\[
(\text{div} A_\nu(n^\lambda_\nu, u^\lambda_\nu)W^\lambda_\nu, W^\lambda_\nu) \leq C\|W^\lambda_\nu\|^2.
\]
Moreover, since \( \|R^\lambda_\nu\|_s \leq C\lambda^{2(m+1)} \), by the Young inequality, we have
\[
| (A^0_\nu(n^\lambda_\nu)W^\lambda_\nu, R^\lambda_\nu) | \leq C\|W^\lambda_\nu\|^2 + C\lambda^{4(m+1)}.
\]
For the term containing \( H^{2,\lambda}_\nu \) in (3.9), a direct calculation gives
\[
-(A^0_\nu(n^\lambda_\nu)W^\lambda_\nu, H^{2,\lambda}_\nu) = -(q_\nu n^\lambda_\nu U^\lambda_\nu, \nabla \Phi^\lambda) = -(q_\nu (n^\lambda_\nu u^\lambda_\nu - n^m_{\nu,\lambda} u^m_{\nu,\lambda}), \nabla \Phi^\lambda) + (q_\nu N^\lambda_\nu u^m_{\nu,\lambda}, \nabla \Phi^\lambda).
\]
Summing this equality for \( \nu = e, i \) and noting \( q_e = -q_i = -1 \), we get
\[
-2 \sum_{\nu = e, i} (A_\nu^0(n_\nu^e)W^\nu_H, H^2_{\nu, \nu}) = 2((n_e^1 \nu^e - n_{\nu, e, i}^m u_{\nu, e, i}^m) - (n_i^1 \nu^i - n_{\nu, i, i}^m u_{\nu, i, i}^m), \nabla \Phi^\lambda)
\]
(3.13)
\[+ 2 \left( N_e^\lambda (u_{\nu, e}^m - u_{e, e}^m), \nabla \Phi^\lambda \right) + 2 \left( (N_i^\lambda - N_e^\lambda) u_{\nu, i}^m, \nabla \Phi^\lambda \right). \]

From (2.5) and the definition of \( u_{\nu, \nu}^m \), we have \( u_0^e = u_0^i = u \) and \( u_{\nu, e, i}^m - u_{\nu, i, i}^m = O(\lambda^2) \). Therefore,
\[
\left| (N_e^\lambda (u_{\nu, e}^m - u_{e, e}^m), \nabla \Phi^\lambda) \right| \leq \| W^\lambda \|^2 + C\lambda^4 \| \nabla \Phi^\lambda \|^2.
\]
(3.14)

Next, using the first equations in (1.1) and (2.10), we may write the first equation in (3.2) as
\[
- \text{div}(n_\nu^\nu u_\nu^\nu - n_{\nu, e, i}^m u_{\nu, e, i}^m) = \partial_t N_\nu^\nu + R_{\nu, \nu}^\nu, \quad \nu = e, i.
\]

Therefore, by an integration by parts and using the Poisson equation (3.3), we get
\[
2((n_e^1 \nu^e - n_{\nu, e, i}^m u_{\nu, e, i}^m) - (n_i^1 \nu^i - n_{\nu, i, i}^m u_{\nu, i, i}^m), \nabla \Phi^\lambda)
\]
\[= 2(\partial_t N_e^\lambda - N_i^\lambda, \Phi^\lambda) + 2(R_{n_e}^\lambda - R_{n_i}^\lambda, \Phi^\lambda)
\]
\[= 2\lambda^2(\partial_t \Delta \Phi^\lambda, \Phi^\lambda) + 2(R_{n_e}^\lambda - R_{n_i}^\lambda - \partial_t R_{\Phi}^\lambda, \Phi^\lambda).
\]

Since
\[
\| R_{n_e}^\lambda - R_{n_i}^\lambda \| + \| \partial_t R_{\Phi}^\lambda \| \leq C\lambda^{2(m+1)}
\]
and
\[
2(\partial_t \Delta \Phi^\lambda, \Phi^\lambda) = -\frac{d}{dt} \| \nabla \Phi^\lambda \|^2,
\]
it follows from the Poincaré inequality that
\[
2((n_e^1 \nu^e - n_{\nu, e, i}^m u_{\nu, e, i}^m) - (n_i^1 \nu^i - n_{\nu, i, i}^m u_{\nu, i, i}^m), \nabla \Phi^\lambda)
\]
\[\leq -\lambda^2 \frac{d}{dt} \| \nabla \Phi^\lambda \|^2 + C\lambda^2 \| \nabla \Phi^\lambda \|^2 + C\lambda^{4m+2}.
\]
(3.15)

For the last term on the right-hand side of (3.13), we use again the Poisson equation (3.3) to get
\[
((N_i^\lambda - N_e^\lambda) u_{\nu, i}^m, \nabla \Phi^\lambda) = -\lambda^2(u_{\nu, e, i}^m \Delta \Phi^\lambda, \nabla \Phi^\lambda) + (u_{\nu, i}^m R_{\Phi}^\lambda, \nabla \Phi^\lambda).
\]
It is clear that
\[
\left| (u_{\nu, e, i}^m R_{\Phi}^\lambda, \nabla \Phi^\lambda) \right| \leq \lambda^2 \| \nabla \Phi^\lambda \|^2 + C\lambda^{4m+2}.
\]

Moreover, by Lemma 2.2, we have
\[
\left| (u_{\nu, i}^m \Delta \Phi^\lambda, \nabla \Phi^\lambda) \right| \leq C \| \nabla u_{\nu, i}^m \| \| \nabla \Phi^\lambda \|.
\]

Therefore,
\[
\left| ((N_i^\lambda - N_e^\lambda) u_{\nu, i}^m, \nabla \Phi^\lambda) \right| \leq \lambda^2 \| \nabla \Phi^\lambda \|^2 + C\lambda^{4m+2}.
\]
(3.16)

Combining (3.13)-(3.16), we obtain
\[
-2 \sum_{\nu = e, i} (A_\nu^0(n_\nu^e)H^2_{\nu, \nu, \nu}, W^\lambda) \leq -\lambda^2 \frac{d}{dt} \| \nabla \Phi^\lambda \|^2 + C\| W^\lambda \|^2 + C\lambda^2 \| \nabla \Phi^\lambda \|^2 + C\lambda^{4m+2}.
\]
(3.17)

It follows from (3.9)-(3.11) and (3.17) that
\[
\frac{d}{dt} \left( \sum_{\nu = e, i} (A_\nu^0(n_\nu^e)W^\lambda_{\nu, \nu, \nu} + \lambda^2 \| \nabla \Phi^\lambda \|^2) \right) \leq C\| W^\lambda \|^2 + C\lambda^2 \| \nabla \Phi^\lambda \|^2 + C\lambda^{4m+2}.
\]
(3.18)
From the assumption (2.12), we have
\[
\|W^\lambda(0)\|_s \leq C\lambda^{2(m+1)}.
\]
It follows from the Poisson equation (3.3) together with the Poincaré inequality that
\[
\lambda^2 \|\nabla \Phi^\lambda(0)\|^2 \leq C\lambda^{4m+2}.
\]
Finally, since \(A^\nu_\nu(n_\nu)\) is positively definite, we have
\[
(A^\nu_\nu(n_\nu)W^\lambda_\nu, W^\lambda_\nu) \geq C^{-1}\|W^\lambda_\nu\|^2.
\]
Hence, \(\sum_{\nu=e,i} (A^\nu_\nu(n_\nu)W^\lambda_\nu, W^\lambda_\nu) + \lambda^2 \|\nabla \Phi^\lambda(t)\|^2\) is equivalent to \(\|W^\lambda(t)\|^2 + \lambda^2 \|\nabla \Phi^\lambda(t)\|^2\). Thus, applying the Gronwall inequality to (3.18) yields (3.8).

3.3. Higher order estimates. Let \(\alpha \in \mathbb{N}^d\) with \(1 \leq |\alpha| \leq s\). Applying \(\partial^\alpha_x\) to (3.4) and (3.3), we get
\[
\partial_t (\partial^\alpha_x W^\lambda_\nu) + \sum_{j=1}^d A^j_\nu(n_\nu, u_\nu)\partial_{x_j}(\partial^\alpha_x W^\lambda_\nu) = -\partial^\alpha_x(H^1_{\nu,\lambda} + H^2_{\nu,\lambda} + R^\lambda_\nu) + J^\alpha_{\nu,\lambda}, \quad \nu = e, i,
\]
and
\[
-\lambda^2 \Delta \partial^\alpha_x \Phi^\lambda = \partial^\alpha_x N^\lambda_e - \partial^\alpha_x N^\lambda_i - \partial^\alpha_x R^\lambda,
\]
where
\[
J^\alpha_{\nu,\lambda} = \sum_{j=1}^d [A^j_\nu(n_\nu, u_\nu)\partial_{x_j}(\partial^\alpha_x W^\lambda_\nu) - \partial^\alpha_x(A^j_\nu(n_\nu, u_\nu)\partial_{x_j}W^\lambda_\nu)].
\]

Lemma 3.2. For all \(t \in [0, T^\lambda]\) and \(\nu = e, i\), we have
\[
\frac{d}{dt}(A^0_\nu(n_\nu)\partial^\alpha_x W^\lambda_\nu, \partial^\alpha_x W^\lambda_\nu) \leq -\left(A^0_\nu(n_\nu)\partial^\alpha_x H^1_{\nu,\lambda}, \partial^\alpha_x W^\lambda_\nu\right) + C\|W^\lambda\|^2_{|\alpha|} + C\lambda^{4(m+1)}.
\]

Proof. Taking the inner product of (3.21) with \(2A^0_\nu \partial^\alpha_x W^\lambda_\nu\) in \(L^2(\mathbb{T}^d)\) yields the following energy equality for \(\partial^\alpha_x W^\lambda_\nu\):
\[
\frac{d}{dt}(A^0_\nu(n_\nu)\partial^\alpha_x W^\lambda_\nu, \partial^\alpha_x W^\lambda_\nu) = -2\left(A^0_\nu(n_\nu)\partial^\alpha_x H^1_{\nu,\lambda}, \partial^\alpha_x W^\lambda_\nu\right) - 2\left(A^0_\nu(n_\nu)\partial^\alpha_x H^2_{\nu,\lambda}, \partial^\alpha_x W^\lambda_\nu\right) - 2\left(A^0_\nu(n_\nu)\partial^\alpha_x R^\lambda_\nu, \partial^\alpha_x W^\lambda_\nu\right) + 2\left(J^\alpha_{\nu,\lambda}, \partial^\alpha_x W^\lambda_\nu\right)
\]
\[
(3.24)
\]
By (2.11) and (3.6), it is clear that
\[
\left|\left(A^0_\nu(n_\nu)\partial^\alpha_x H^1_{\nu,\lambda}, \partial^\alpha_x W^\lambda_\nu\right)\right| \leq C\|W^\lambda\|^2_{|\alpha|},
\]
\[
\left|\left(\text{div } A_\nu(n_\nu, u_\nu)\partial^\alpha_x W^\lambda_\nu, \partial^\alpha_x W^\lambda_\nu\right)\right| \leq C\|W^\lambda\|^2_{|\alpha|},
\]
and
\[
\left|\left(A^0_\nu(n_\nu)\partial^\alpha_x W^\lambda_\nu, \partial^\alpha_x R^\lambda_\nu\right)\right| \leq C\|W^\lambda\|^2_{|\alpha|} + C\lambda^{4(m+1)}.
\]
Similarly, we also write

\[ J_{\nu, \lambda}^\alpha = \sum_{j=1}^{d} \left[ (A_j^\alpha(n^\nu_{\nu, \lambda}, u^\nu_{\nu, \lambda}) - A_j^\alpha(n^m_{\nu, \lambda}, u^m_{\nu, \lambda})) \partial_{x_j}(\partial_x^{\alpha} W^\nu_{\nu, \lambda}) \right. \]

\[ - \partial_x^{\alpha} \left( (A_j^\alpha(n^\nu_{\nu, \lambda}, u^\nu_{\nu, \lambda}) - A_j^\alpha(n^m_{\nu, \lambda}, u^m_{\nu, \lambda})) \partial_{x_j} W^\nu_{\nu, \lambda} \right) \]

\[ + \sum_{j=1}^{d} \left[ A_j^\alpha(n^m_{\nu, \lambda}, u^m_{\nu, \lambda}) \partial_{x_j}(\partial_x^{\alpha} W^\nu_{\nu, \lambda}) - \partial_x^{\alpha} (A_j^\alpha(n^m_{\nu, \lambda}, u^m_{\nu, \lambda}) \partial_{x_j} W^\nu_{\nu, \lambda}) \right]. \]

Then applying Lemma 2.1 to \( J_{\nu, \lambda}^\alpha \) together with (3.6), we get

\[ |(J_{\nu, \lambda}^\alpha, \partial_x^{\alpha} W^\nu_{\nu, \lambda})| \leq C \|W^\lambda\|^2_{[\alpha]}. \]

Estimates (3.24)-(3.28) imply (3.23). \( \square \)

Regarding the term containing \( \partial_x^{\alpha} H^2_{\nu, \lambda} \) in (3.23), we have

**Lemma 3.3.** For all \( t \in [0, T^\lambda] \), we have

\[ -2 \sum_{\nu=\epsilon,i} (A^0_\nu(n^\nu_{\nu, \lambda}) \partial_x^{\alpha} H^2_{\nu, \lambda}, \partial_x^{\alpha} W^\nu_{\nu, \lambda}) \leq -\lambda^2 \frac{d}{dt} \|\nabla(\partial_x^{\alpha} \Phi^\lambda)\|^2 + C \|W^\lambda\|^2_{[\alpha]} + C \lambda^2 \|\nabla \Phi^\lambda\|^2_{[\alpha]} \]

\[ + C \lambda^{4m+2} + 2 \sum_{\nu=\epsilon,i} q_\nu \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda}) - n^\nu_{\nu, \lambda} \partial_x^{\alpha} U^\nu_{\nu, \lambda}, \nabla(\partial_x^{\alpha} \Phi^\lambda) \right). \]

**Proof.** As in the \( L^2(\mathbb{T}^d) \) estimate, we may write

\[ -(A^0_\nu(n^\nu_{\nu, \lambda}) \partial_x^{\alpha} H^2_{\nu, \lambda}, \partial_x^{\alpha} W^\nu_{\nu, \lambda}) = - \left( q_\nu n^\nu_{\nu, \lambda} \partial_x^{\alpha} U^\nu_{\nu, \lambda}, \nabla(\partial_x^{\alpha} \Phi^\lambda) \right) \]

\[ = - (q_\nu \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda}), \nabla(\partial_x^{\alpha} \Phi^\lambda)) + q_\nu \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda}) - n^\nu_{\nu, \lambda} \partial_x^{\alpha} U^\nu_{\nu, \lambda}, \nabla(\partial_x^{\alpha} \Phi^\lambda) \right) \]

\[ = -q_\nu \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda} - n^m_{\nu, \lambda} u^m_{\nu, \lambda}), \nabla(\partial_x^{\alpha} \Phi^\lambda) \right) + q_\nu \left( \partial_x^{\alpha}(N^\nu_{\nu, \lambda} u^m_{\nu, \lambda}), \nabla(\partial_x^{\alpha} \Phi^\lambda) \right) \]

\[ + q_\nu \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda}) - n^\nu_{\nu, \lambda} \partial_x^{\alpha} U^\nu_{\nu, \lambda}, \nabla(\partial_x^{\alpha} \Phi^\lambda) \right). \]

The first equations in (1.1) and (2.10) yield

\[ \text{div} \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda} - n^m_{\nu, \lambda} u^m_{\nu, \lambda}) \right) = -\partial_t \left( \omega^\alpha x N^\nu_{\nu, \lambda} \right) - \partial_x^{\alpha} R^\lambda_{\nu, \lambda}. \]

Hence,

\[ \sum_{\nu=\epsilon,i} q_\nu \text{div} \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda} - n^m_{\nu, \lambda} u^m_{\nu, \lambda}) \right) = \partial_t \left( \omega^\alpha x (N^\nu_{\nu, \lambda} - N^\lambda_{\nu, \lambda}) + \partial_x^{\alpha} (R^\lambda_{\nu, \lambda} - R^\lambda_{\nu, \lambda}) \right). \]

It follows from the Poisson equation (3.22) that

\[ -2 \sum_{\nu=\epsilon,i} q_\nu \left( \text{div} \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda} - n^m_{\nu, \lambda} u^m_{\nu, \lambda}) \right), \nabla(\partial_x^{\alpha} \Phi^\lambda) \right) = 2 \sum_{\nu=\epsilon,i} q_\nu \left( \text{div} \left( \partial_x^{\alpha}(n^\nu_{\nu, \lambda} u^\nu_{\nu, \lambda} - n^m_{\nu, \lambda} u^m_{\nu, \lambda}) \right), \partial_x^{\alpha} \Phi^\lambda \right) \]

\[ = -\lambda^2 \frac{d}{dt} \|\nabla(\partial_x^{\alpha} \Phi^\lambda)\|^2 + 2 \partial_x^{\alpha} \left( R^\lambda_{\nu, \lambda} - R^\lambda_{\nu, \lambda} - \partial_t \omega^\alpha \right)(\partial_x^{\alpha} \Phi^\lambda). \]

Similarly, we also write

\[ \sum_{\nu=\epsilon,i} q_\nu \left( \partial_x^{\alpha}(N^\nu_{\nu, \lambda} u^m_{\nu, \lambda}), \nabla(\partial_x^{\alpha} \Phi^\lambda) \right) = \left( \partial_x^{\alpha}(N^\nu_{\nu, \lambda} u^m_{\nu, \lambda} - u^m_{\nu, \lambda}), \nabla(\partial_x^{\alpha} \Phi^\lambda) \right) \]

\[ + \left( \partial_x^{\alpha}((N^\nu_{\nu, \lambda} - N^\lambda_{\nu, \lambda}) u^m_{\nu, \lambda}), \nabla(\partial_x^{\alpha} \Phi^\lambda) \right). \]
Therefore,
\[-2 \sum_{\nu = e, i} (A^0_\nu (n^\lambda_\nu) \partial_\nu^2 H^2_{\nu, \lambda}, \partial_\nu^2 W^\lambda) = -\lambda^2 \frac{d}{dt} \| \nabla (\partial_\nu^2 \Phi^\lambda) \|^2 + 2 \partial_\nu^2 (R_{n^\lambda_\nu} - R^\lambda_{n^\lambda_\nu} - \partial_t R^\lambda_{\nu^\lambda \hat{\phi}}, \partial_\nu^2 \Phi^\lambda)\]
\[+ 2 \big( \partial_\nu^2 (N^\lambda_e (u^m_{i, \lambda} - u^m_{e, \lambda})), \nabla \partial_\nu^2 \Phi^\lambda \big)\]
\[+ 2 \big( \partial_\nu^2 ((N^\lambda_i - N^\lambda_e) u^m_{i, \lambda}), \nabla \partial_\nu^2 \Phi^\lambda \big)\]
\[+ 2 \sum_{\nu = e, i} q_\nu \big( \partial_\nu^2 (n^\lambda_\nu U^\lambda_\nu) - n^\lambda_\nu \partial_\nu^2 U^\lambda_\nu, \nabla (\partial_\nu^2 \Phi^\lambda) \big)\]  
(3.30)

By (2.11), \(u^m_{i, \lambda} - u^m_{e, \lambda} = O(\lambda^2)\) and the Poincaré inequality, we have
\[|\partial_\nu^2 (R_{n^\lambda_\nu} - R^\lambda_{n^\lambda_\nu} - \partial_t R^\lambda_{\nu^\lambda \hat{\phi}}, \partial_\nu^2 \Phi^\lambda)| \leq \lambda^2 \| \nabla (\partial_\nu^2 \Phi^\lambda) \|^2 + C \lambda^{4m+2},\]
(3.31)
\[|\partial_\nu^2 ((N^\lambda_i - N^\lambda_e) u^m_{i, \lambda}), \nabla \partial_\nu^2 \Phi^\lambda)| \leq C \| W^\lambda \|^2_{[\alpha]} + C \lambda^2 \| \nabla (\partial_\nu^2 \Phi^\lambda) \|^2.\]
(3.32)

Moreover, from the Poisson equation (3.22), we also have
\[\big( \partial_\nu^2 ((N^\lambda_i - N^\lambda_e) u^m_{i, \lambda}), \nabla \partial_\nu^2 \Phi^\lambda) = -\lambda^2 \big( \partial_\nu^2 (u^m_{i, \lambda} \Delta \Phi^\lambda), \nabla \partial_\nu^2 \Phi^\lambda \big) + \big( \partial_\nu^2 (u^m_{i, \lambda} R^\lambda_{\nu^\lambda \hat{\phi}}), \nabla \partial_\nu^2 \Phi^\lambda \big),\]
with
\[\big( \partial_\nu^2 (u^m_{i, \lambda} \Delta \Phi^\lambda), \nabla \partial_\nu^2 \Phi^\lambda) \leq C \lambda^{2} \| \nabla \partial_\nu^2 \Phi^\lambda \|^2 + C \lambda^{4m+2}.\]
For \(-\lambda^2 \big( \partial_\nu^2 (u^m_{i, \lambda} \Delta \Phi^\lambda), \nabla \partial_\nu^2 \Phi^\lambda)\), we employ the Leibniz formula:
\[\partial_\nu^2 (u^m_{i, \lambda} \Delta \Phi^\lambda) = u^m_{i, \lambda} \Delta (\partial_\nu^2 \Phi^\lambda) + \sum_{\gamma < \alpha} m_{\alpha \gamma} \Delta (\partial_\nu^2 \Phi^\lambda) \partial_\nu^{\alpha - \gamma} u^m_{i, \lambda},\]
where \(m_{\alpha \gamma} > 0\) are constants. Noting that for \(\gamma < \alpha\), the highest derivative order of \(\Delta (\partial_\nu^2 \Phi^\lambda)\) equals the highest order of \(\nabla (\partial_\nu^2 \Phi^\lambda)\), it follows from Lemma 2.2 that
\[|\lambda^2 \big( \partial_\nu^2 (u^m_{i, \lambda} \Delta \Phi^\lambda), \nabla \partial_\nu^2 \Phi^\lambda) \| \leq C \lambda^{2} \| \nabla \Phi^\lambda \|^2_{[\alpha]}|,\]
Hence,
\[|\big( \partial_\nu^2 ((N^\lambda_i - N^\lambda_e) u^m_{i, \lambda}), \nabla \partial_\nu^2 \Phi^\lambda) \| \leq C \lambda^{2} \| \nabla \Phi^\lambda \|^2_{[\alpha]} + C \lambda^{4m+2}.\]
Thus (3.29) follows from (3.30)-(3.33).

It remains to control the last term on the right-hand side of (3.29). To this end, we need condition (3.7).

**Lemma 3.4.** Assume (3.7) holds. Then for all \(t \in [0, T^\lambda]\), we have
\[\left| \sum_{\nu = e, i} q_\nu \big( \partial_\nu^2 (n^\lambda_\nu U^\lambda_\nu) - n^\lambda_\nu \partial_\nu^2 U^\lambda_\nu, \nabla (\partial_\nu^2 \Phi^\lambda) \big) \right| \leq C \| W^\lambda \|^2_{[\alpha]} + C \lambda^{2} \| \nabla \Phi^\lambda \|^2_{[\alpha]} + C \lambda^{-2} \| W^\lambda \|^2_{\alpha - 1}.\]
(3.34)

Moreover, when \(d \leq 3\), (3.34) holds under condition (3.6) instead of (3.7).

**Proof.** Using \(n^\lambda_\nu = N^\lambda_\nu + n^m_{\nu, \lambda}\), we have
\[\sum_{\nu = e, i} q_\nu \big( \partial_\nu^2 (n^\lambda_\nu U^\lambda_\nu) - n^\lambda_\nu \partial_\nu^2 U^\lambda_\nu, \nabla (\partial_\nu^2 \Phi^\lambda) \big) = \sum_{\nu = e, i} q_\nu \big( \partial_\nu^2 (N^\lambda_\nu U^\lambda_\nu) - N^\lambda_\nu \partial_\nu^2 U^\lambda_\nu, \nabla (\partial_\nu^2 \Phi^\lambda) \big)\]
\[+ \sum_{\nu = e, i} q_\nu \big( \partial_\nu^2 (n^m_{\nu, \lambda} U^\lambda_\nu) - n^m_{\nu, \lambda} \partial_\nu^2 U^\lambda_\nu, \nabla (\partial_\nu^2 \Phi^\lambda) \big).\]
By Lemma 2.1 and the definition of $n_{\nu, \lambda}^m$, it is clear that

$$\sum_{\nu = e, i} q_{\nu} \left( \partial_x^\alpha (n_{\nu, \lambda}^m U_{\nu}^\lambda) - n_{\nu, \lambda}^m \partial_x^\alpha U_{\nu}^\lambda, \nabla (\partial_x^\alpha \Phi^\lambda) \right) \leq C \lambda^{-2} \| W_{\lambda} \|_{s-1}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2.$$  

Moreover, (3.7) gives $\| W_{\lambda} \|_s \leq C \lambda$, then

$$\sum_{\nu = e, i} q_{\nu} \left( \partial_x^\alpha (N_{\nu}^\lambda U_{\nu}^\lambda) - N_{\nu}^\lambda \partial_x^\alpha U_{\nu}^\lambda, \nabla (\partial_x^\alpha \Phi^\lambda) \right) \leq C \lambda^{-2} \| W_{\lambda} \|_s^2 \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2 \leq C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2,$$

which implies (3.34).

When $d \leq 3$, by Lemma 2.3 and $\| W_{\lambda} \|_s \leq C$, we have

$$\sum_{\nu = e, i} q_{\nu} \left( \partial_x^\alpha (N_{\nu}^\lambda U_{\nu}^\lambda) - N_{\nu}^\lambda \partial_x^\alpha U_{\nu}^\lambda, \nabla (\partial_x^\alpha \Phi^\lambda) \right) \leq C \lambda^{-2} \| W_{\lambda} \|_s^2 \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2 \leq C \lambda^{-2} \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2.$$

Therefore, (3.34) still holds without condition (3.7).

\[ \square \]

4. Proof of Theorem 2.1

**Proposition 4.1.** Let $s > d/2 + 1$ be an integer. Assume (3.7) holds. Then for all integer $2m \geq s$, we have

$$\sup_{0 \leq t \leq T^\lambda} \left( \| W_{\lambda}^\alpha (t) \|_s + \lambda \| \nabla \Phi^\lambda (t) \|_s \right) \leq C \lambda^{2m-s+1}.$$  

Moreover, when $d \leq 3$, (4.1) holds under condition (3.6) instead of (3.7).

**Proof.** By Lemmas 3.3-3.4, we get

$$-2 \sum_{\nu = e, i} \left( A_{\nu}^\lambda (n_{\nu}^\lambda) \partial_x^\alpha H_{\nu, \lambda}^\alpha \partial_x^\beta W_{\nu}^\lambda \right) \leq -\lambda^2 \frac{d}{dt} \| \nabla (\partial_x^\alpha \Phi^\lambda) \|_{|\alpha|}^2 + C \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2 + C \lambda^{-2} \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^{4m+2}.  
$$  

Summing (3.23) for $\nu = e, i$ together with (4.2) yields

$$\frac{d}{dt} \left( \sum_{\nu = e, i} A_{\nu}^\lambda (n_{\nu}^\lambda) \partial_x^\alpha W_{\nu}^\lambda \partial_x^\beta W_{\nu}^\lambda \right) + \lambda^2 \| \nabla \partial_x^\alpha \Phi^\lambda \|_{|\alpha|}^2 \leq C \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2 + C \lambda^{-2} \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2 \leq C \lambda^{-2} \| W_{\lambda} \|_{|\alpha|}^2 + C \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2 + C \lambda^{4m+2}.$$  

Now we deal with (4.3) by induction for $1 \leq |\alpha| \leq s$. In view of the $L^2$ estimate, we assume

$$\| W_{\lambda} \|_{|\alpha|}^2 + \lambda^2 \| \nabla \Phi^\lambda \|_{|\alpha|}^2 \leq C \lambda^{4m-2(|\alpha|-2)}.$$  

Then, (4.3) implies that
\[
\frac{d}{dt} \left( \sum_{\nu \in e, i} (A_0^\nu(n_\nu^\lambda) \partial_x^\nu W^\lambda, \partial_x^\nu W^\lambda) + \lambda^2 \| \nabla \partial_x^\alpha \tilde{\Phi}^\lambda \|^2 \right) \leq C (\| \partial_x^\nu W^\lambda \|^2 + \lambda^2 \| \nabla \partial_x^\alpha \tilde{\Phi}^\lambda \|^2) + C \lambda^{4m-2(|\alpha|-1)}.
\]

It is clear that \( \sum_{\nu \in e, i} (A_0^\nu(n_\nu^\lambda) \partial_x^\nu W^\lambda, \partial_x^\nu W^\lambda) \) is equivalent to \( \| \partial_x^\alpha W^\lambda \|^2 \). From the assumption (2.12), for all \( |\alpha| \leq s \) we have
\[
\| \partial_x^\alpha W^\lambda(0) \| \leq C \lambda^{2(m+1)}.
\]

It follows from the Poisson equation (3.3) together with the Poincaré inequality that
\[
\| \partial_x^\alpha W^\lambda(0) \|^2 + \lambda^2 \| \nabla \partial_x^\alpha \tilde{\Phi}^\lambda(0) \|^2 \leq C \lambda^{4m+2} \leq C \lambda^{4m-2(|\alpha|-1)},
\]
provided that \( \lambda > 0 \) is sufficiently small. Thus applying the Gronwall inequality yields
\[
\| \partial_x^\alpha W^\lambda \|^2 + \lambda^2 \| \nabla \partial_x^\alpha \tilde{\Phi}^\lambda \|^2 \leq C \lambda^{4m-2(|\alpha|-1)}.
\]

Combining this with (4.4) implies that
\[
\| W^\lambda \|_{|\alpha|} + \lambda \| \nabla \tilde{\Phi}^\lambda \|_{|\alpha|} \leq C \lambda^{2m-|\alpha|+1},
\]
which proves (4.1) by taking \( |\alpha| = s \).

When \( d \leq 3 \), Lemmas 3.1-3.4 hold under condition (3.6) instead of (3.7). This proves Proposition 4.1. \( \square \)

**Remark 4.1.** Estimate (4.5) implies that
\[
\sup_{0 \leq t \leq T^\lambda} \lambda^{2|\alpha|} (\| \partial_x^\alpha W^\lambda(t) \|^2 + \lambda^2 \| \nabla \partial_x^\alpha \tilde{\Phi}^\lambda(t) \|^2) \leq C \lambda^{4m+2},
\]
in which the error on the right-hand side is independent of \( \alpha \). This suggests to introduce a weighted norm
\[
\| W \|_s = \left( \sum_{|\alpha| \leq s} \lambda^{2|\alpha|} \| \partial_x^\alpha W \|^2 \right)^{\frac{1}{2}},
\]
so that estimate (4.1) is written as
\[
\sup_{0 \leq t \leq T^\lambda} (\| W^\lambda(t) \|_s + \lambda \| \nabla \tilde{\Phi}^\lambda(t) \|_s) \leq C \lambda^{2m+1}.
\]

**Proposition 4.2.** Let \( s > d/2 + 1 \) be an integer. Assume (3.7) holds. If the density of the leading profiles, \( n \), is positive and depends only on \( t \), then for all integer \( m \geq 1 \), we have
\[
\sup_{0 \leq t \leq T^\lambda} (\| W^\lambda(t) \|_s + \lambda \| \nabla \tilde{\Phi}^\lambda(t) \|_s) \leq C \lambda^{2m+1}.
\]

**Proof.** When \( n > 0 \) depends only on \( t \), we have
\[
\partial_x^\alpha (n_{\nu,\lambda}^m U^\lambda) - n_{\nu,\lambda}^m \partial_x^\alpha U^\lambda = \partial_x^\alpha ((n_{\nu,\lambda}^m - n)U^\lambda) - (n_{\nu,\lambda}^m - n) \partial_x^\alpha U^\lambda.
\]

Since \( n_{\nu,\lambda}^m - n = O(\lambda^2) \), in the proof of Lemma 3.4, estimate (3.35) becomes
\[
\left| \sum_{\nu \in e, i} q_\nu \left( \partial_x^\alpha (n_{\nu,\lambda}^m U^\lambda) - n_{\nu,\lambda}^m \partial_x^\alpha U^\lambda, \nabla (\partial_x^\alpha \tilde{\Phi}^\lambda) \right) \right| \leq C \| W^\lambda \|^2_{|\alpha|-1} + C \lambda^2 \| \nabla \tilde{\Phi}^\lambda \|^2_{|\alpha|}.
\]
Therefore,
\[ \sum_{\nu=e,i} q_\nu \left( \partial_x^2 (n^\lambda_\nu U^\lambda_\nu) - n^\lambda_\nu \partial^2_x U^\lambda_\nu, \nabla (\partial_x^2 \Phi^\lambda) \right) \leq C \| W^\lambda \|^2_{|\alpha|} + C \lambda^2 \| \nabla \Phi^\lambda \|^2_{|\alpha|}. \]

Instead of (4.3), this inequality together with Lemmas 3.2-3.3 and (3.18) allows to obtain, for all \( 0 \leq |\alpha| \leq s \),
\[ \frac{d}{dt} \left( \sum_{\nu=e,i} (A^0_\nu(n^\lambda_\nu) \partial_x^2 W^\lambda_\nu, \partial_x^2 W^\lambda_\nu) + \lambda^2 \| \nabla \partial_x^2 \Phi^\lambda \|^2 \right) \leq C \| W^\lambda \|^2_{|\alpha|} + C \lambda^2 \| \nabla \Phi^\lambda \|^2_{|\alpha|} + C \lambda^{4m+2}. \]

Summing this inequality for all \( 0 \leq |\alpha| \leq s \) and noting the equivalence between
\[ \sum_{|\alpha| \leq s} \left( \sum_{\nu=e,i} (A^0_\nu(n^\lambda_\nu) \partial_x^2 W^\lambda_\nu, \partial_x^2 W^\lambda_\nu) + \lambda^2 \| \nabla \partial_x^2 \Phi^\lambda \|^2 \right) \]
and \( \| W^\lambda \|^2_{s} + C \lambda^2 \| \nabla \Phi^\lambda \|^2_{s} \), we obtain (4.6) by the Gronwall inequality.

\[ \square \]

**Proof of Theorem 2.1.** It suffices to prove \( T^\lambda_* \geq T_1 \), i.e. \( T^\lambda_* = T_1 \). By the definitions of \( T_1, T^\lambda_1, T^\lambda_2 \) and \( T^\lambda \), we have \( T^\lambda \leq T^\lambda_2 \leq T_1 \). According to (3.7), we may replace \( T^\lambda \) by \( T^\lambda_* \in (0, T_1) \) such that \( [0, T^\lambda_*] \) is the maximum time interval on which \( W^\lambda \) exists and satisfies (3.7), i.e.
\[ \| W^\lambda (t) \|_s \leq C \lambda, \quad \forall \ t \in [0, T^\lambda_*], \]
for some constant \( C > 0 \). We want to prove \( T^\lambda_* = T_1 \).

By Proposition 4.1, for all \( 2m \geq s \) we have
\[ \sup_{0 \leq t \leq T^\lambda_*} \left( \| W^\lambda (t) \|_s + \lambda \| \nabla \Phi^\lambda (t) \|_s \right) \leq C \lambda^{2m-s+1}. \]

In particular,
\[ \| W^\lambda (T^\lambda_* \|_s + \lambda \| \nabla \Phi^\lambda (T^\lambda_*)) \|_s \leq C \lambda^{2m-s+1}. \]

If \( T^\lambda_* < T_1 \), we apply the theorem of Kato for the local existence of smooth solutions with initial data \( W^\lambda(T^\lambda_*) \). Consequently, there is \( T^\lambda > T^\lambda_* \) and a smooth solution \( W^\lambda \in C \left( [0, T^\lambda]; H^s (\mathbb{T}^d) \right) \) of (3.4)-(3.5). When \( 2m > s \) and \( \lambda \) is sufficiently small, we always have \( \lambda^{2m-s+1} < C \lambda \) for all fixed constant \( C > 0 \). Since the function \( t \rightarrow \| W^\lambda (t) \|_s \) is continuous on \([T^\lambda_*, T^\lambda]\), there exists \( T^\lambda \in (T^\lambda_*; T^\lambda) \) such that
\[ \| W^\lambda (t) \|_s \leq C \lambda, \quad \forall \ t \in [0, T^\lambda]. \]

This is contradictory to the maximality of \( T^\lambda_* \). Thus, we have proved \( T^\lambda_* = T_1 \), which implies that \( T^\lambda_1 \geq T_1 \).

Similarly, for \( \lambda \) sufficiently small, when \( d \leq 3 \) and \( 2m \geq s \), by Proposition 4.1, we always have
\[ \sup_{0 \leq t \leq T^\lambda} \left( \| W^\lambda (t) \|_s + \lambda \| \nabla \Phi^\lambda (t) \|_s \right) \leq C \lambda. \]

We also have \( \lambda \leq C \) for all fixed constant \( C > 0 \). Thus, with the same argument as above together with (3.6) instead of (3.7), we obtain \( T^\lambda_1 \geq T_1 \).

Finally, when \( m \geq 1 \) and the density of the leading profiles depends only on \( t \), the proof follows from Proposition 4.2 in a same way. \[ \square \]
Appendix : Proof of Lemma 2.3

We first prove
\[(A1) \quad \|yz\| \leq C\|y\|_1\|z\|_1, \quad \forall \ y, z \in H^1(\mathbb{T}^d).\]
Indeed, the result is obvious for \(d = 1\) because of the imbedding \(H^1(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)\). For \(d = 2, 3\), we use the Hölder inequality
\[\|yz\| \leq \|y\|_{L^6(\mathbb{T}^d)}\|z\|_{L^3(\mathbb{T}^d)}\]
and the interpolation inequality
\[\|z\|_{L^3(\mathbb{T}^d)} \leq \|z\|^{\frac{2}{3}}\|z\|^{\frac{1}{3}}_{L^6(\mathbb{T}^d)} \leq \|z\|^{\frac{1}{3}}_1\|z\|^{\frac{2}{3}}_{L^6(\mathbb{T}^d)}.\]
Moreover, the Sobolev imbedding inequality yields :
\[\|w\|_{L^6(\mathbb{T}^d)} \leq C\|w\|_1, \quad \forall \ w \in H^1(\mathbb{T}^d).\]
Applying the last inequality to \(u\) and \(v\), we obtain (A1).

The result of Lemma 2.3 is clear for \(|\alpha| = 1\) due to the continuous imbedding \(H^{s-1}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)\). For \(3 \leq |\alpha| \leq s\), the Leibniz formula gives
\[
\partial_\alpha^2(uv) - u\partial_\alpha^2v = \sum_{\gamma < \alpha} m_{\alpha\gamma} \partial_\alpha^{\alpha-\gamma}u \partial_\gamma^2v
\]
\[(A2) \quad = \sum_{|\gamma| = |\alpha| - 1} m_{\alpha\gamma} \partial_\alpha^{\alpha-\gamma}u \partial_\gamma^2v + \sum_{1 \leq |\gamma| \leq |\alpha| - 2} m_{\alpha\gamma} \partial_\alpha^{\alpha-\gamma}u \partial_\gamma^2v + \partial_\alpha^2u v,
\]
where \(m_{\alpha\gamma} > 0\) are constants. When \(|\gamma| = |\alpha| - 1\), we have \(|\alpha - \gamma| = 1\), then
\[\|\partial_\alpha^{\alpha-\gamma}u \partial_\gamma^2v\| \leq \|\partial_\alpha^{\alpha-\gamma}u\|_\infty \|\partial_\gamma^2v\| \leq C\|\nabla u\|_{s-1}\|v\|_{|\alpha|-1}.
\]
When \(1 \leq |\gamma| \leq |\alpha| - 2\), by (A1), we obtain
\[\|\partial_\alpha^{\alpha-\gamma}u \partial_\gamma^2v\| \leq C\|\partial_\alpha^{\alpha-\gamma}u\|_1 \|\partial_\gamma^2v\|_1 \leq C\|\nabla u\|_{s-1}\|v\|_{|\alpha|-1}.
\]
For the last term \(\partial_\alpha^2u v\) in (A2), we have similarly
\[\|\partial_\alpha^2u v\| \leq C\|\partial_\alpha^2u\|_1\|v\|_1 \leq C\|\nabla u\|_{s-1}\|v\|_{|\alpha|-1}, \quad \text{if} \ 3 \leq |\alpha| \leq s - 1,
\]
and
\[\|\partial_\alpha^2u v\| \leq \|\partial_\alpha^2u\|_{\infty} \leq C\|\nabla u\|_{s-1}\|v\|_{|\alpha|-1}, \quad \text{if} \ |\alpha| = s.
\]
Finally, for \(|\alpha| = 2\),
\[\partial_\alpha^2(uv) - u\partial_\alpha^2v = \sum_{|\gamma| = |\alpha|-1} m_{\alpha\gamma} \partial_\alpha^{\alpha-\gamma}u \partial_\gamma^2v + \partial_\alpha^2u v.
\]
Hence,
\[
\|\partial_\alpha^2(uv) - u\partial_\alpha^2v\| \leq \sum_{|\gamma| = |\alpha| - 1} m_{\alpha\gamma} \|\partial_\alpha^{\alpha-\gamma}u\|_\infty \|\partial_\gamma^2v\| + \|\partial_\alpha^2u\|_1 \|v\|_1
\]
\[\leq C\|\nabla u\|_{s-1}\|v\|_{|\alpha|-1}.
\]
This proves Lemma 2.3.

\[\square\]

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