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The inertia of weighted unicyclic graphs ☆

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A B S T R A C T

Let G_w be a weighted graph. The *inertia* of G_w is the triple $In(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w)$, $i_-(G_w)$, $i_0(G_w)$ are the numbers of the positive, negative and zero eigenvalues of the adjacency matrix $A(G_w)$ of G_w including their multiplicities, respectively. $i_+(G_w)$, $i_-(G_w)$ are called the *positive, negative indices of inertia* of G_w, respectively. In this paper we present a lower bound for the positive, negative indices of weighted unicyclic graphs of order n with fixed girth and characterize all weighted unicyclic graphs attaining this lower bound. Moreover, we characterize the weighted unicyclic graphs of order n with two positive, two negative and at least $n - 6$ zero eigenvalues, respectively.

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1. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* $A(G) = (a_{ij})$ of graph G of order n is a symmetric $(0,1)$-matrix such that $a_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise. A weighted graph G_w is a pair (G, w) where G is a simple graph with edge set $E(G)$, called the *underlying graph* of G_w, and w is a weight function from $E(G)$ to the set of nonzero real numbers. The *adjacency matrix* of G_w on n vertices is defined as the matrix $A(G_w) = (a_{ij})$ such that $a_{ij} = w(v_iv_j)$ if v_i is adjacent to v_j and 0 otherwise. The characteristic polynomial of G_w is the characteristic polynomial of $A(G_w)$, denoted by

$$P_{G_w}(\lambda) = \det\bigl(\lambda I - A(G_w)\bigr) = \lambda^n + a_1^*\lambda^{n-1} + \cdots + a_n^*.$$

The *inertia* of G_w is defined to be the triple $In(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w)$, $i_-(G_w)$, $i_0(G_w)$ are the numbers of the positive, negative and zero eigenvalues of $A(G_w)$ including multiplicities, respectively. $i_+(G_w)$ and $i_-(G_w)$ are called the *positive, negative indices of inertia* (abbreviated *positive, negative indices*) of G_w, respectively. The number $i_0(G_w)$ is called the *nullity* of G_w. The rank of an n-vertex graph G_w, denoted by $r(G_w)$, is defined as the rank of $A(G_w)$. Obviously, $r(G_w) = i_+(G_w) + i_-(G_w) = n - i_0(G_w)$.

A graph G_w is called *acyclic* (resp. *unicyclic*, *bipartite*) if its underlying graph G is *acyclic* (resp. *unicyclic*, *bipartite*). An *induced subgraph* of G_w is an induced subgraph of G with the same weights. For a subgraph H_w of G_w, let $G_w - H_w$ be the subgraph obtained from G_w by deleting all vertices of H_w and all incident edges. For $V' \subseteq V(G_w)$, $G_w - V'$ is the subgraph obtained from G_w by deleting all vertices in V' and all their incident edges. A vertex of a graph G_w is called *pendant* if it has degree one, and is called *quasi-pendant* if it is adjacent to a pendant vertex. For a weighted graph G_w on at least two vertices, a vertex $v \in V(G_w)$ is called *unsaturated* in G_w if there exists a maximum matching M of G in which no edge is incident with v; otherwise, v is called *saturated* in G_w.

A simple graph may be regarded as a weighted graph in which the weight of each edge is $+1$. A signed graph may be regarded as a weighted graph in which the weight of each edge is $+1$ or -1. Moreover, the sign of a signed cycle, denoted by $sgn(C)$, is defined as the sign of the product of all edge weights $+1$ or -1 on C. The signed cycle C is said to be *positive* (*or negative*) if $sgn(C) = +$ (or $sgn(C) = -$). A signed graph is said to be *balanced* if all its cycles are positive, otherwise it is called *unbalanced*.

The study of eigenvalues of a weighted graph has attracted much attention. Several results about the (Laplacian) spectral radius of weighted graphs were derived in [1,10,9,24,25]. The inertia of unweighted graphs has attracted some attention. Gregory et al. [17] studied the subadditivity of the positive, negative indices of inertia and developed certain properties of Hermitian rank which were used to characterize the biclique decomposition number. Gregory et al. [16] investigated the inertia of a partial join of two graphs and established a few relations between inertia and biclique decompositions of partial joins of graphs. Daugherty [11] characterized the inertia of unicyclic graphs in terms of matching number and obtained a linear-time algorithm for computing it. Yu et al. [27] investigated the minimal positive index of inertia among all unweighted bicyclic graphs of order n with pendant vertices, and characterized the bicyclic graphs with positive index 1 or 2. Fan et al. [13] introduced the nullity of signed graphs and characterized the unicyclic signed graphs of order n with nullity $n-2$, $n-3$, $n-4$, $n-5$, respectively. Fan et al. [12] characterized the signed graphs of order n with nullity $n-2$, $n-3$, respectively, and determined the unbalanced bicyclic signed graphs of order n with nullity $n-3$ or $n-4$ and bicyclic signed graphs of order n with nullity $n-5$. Sciriha [23] (see also [3]) characterized the unweighted graphs of order n with nullity $n-2$ or $n-3$, respectively. Chang et al. [4,5] determined the unweighted graphs of order n with nullity $n-4$ or $n-5$. Guo et al. [18] studied some relations between the matching number and the nullity. The nullity of unweighted graphs has been studied well in the literature, see [2] for a survey. However, a characterization of unweighted graphs of order n with nullity at most $n-6$ is still an open problem. There is also a large body of knowledge related to the inertia of unweighted graphs due to its many applications in chemistry (see [6,14,19,21] for details). Motivated by the above description, we shall characterize the weighted unicyclic graphs of order n with nullity at least $n-6$.

This paper is organized as follows. In Section 2, some preliminaries are introduced. In Section 3, we present a lower bound for the positive, negative indices of weighted unicyclic graphs of order n with girth k ($3 \leqslant k \leqslant n-2$) and characterize all weighted unicyclic graphs attaining this lower bound. Moreover, we characterize the weighted unicyclic graphs of order n with two positive (negative) eigenvalues and the weighted unicyclic graphs of order n with rank 4, respectively. In Section 4, we determine the weighted unicyclic graphs with rank 6. In Section 5, we characterize the weighted unicyclic graphs of order n with rank 2, 3, 5, respectively.

2. Preliminaries

Definition 2.1. Let M be a Hermitian matrix. The three types of elementary congruence matrix operations (ECMOs) of M are defined as follows:

(1) interchanging i-th and j-th rows of M, while interchanging i-th and j-th columns of M;
(2) multiplying i-th row of M by non-zero number k, while multiplying i-th column of M by k;
(3) adding i-th row of M multiplied by a non-zero number k to the j-th row, while adding i-th column of M multiplied by k to the j-th column.

Lemma 2.2 *(Sylvester's law of inertia). (See [20].) Let M be an $n \times n$ real symmetric matrix and P be an $n \times n$ nonsingular matrix. Then*

$$i_+(PMP^T) = i_+(M);$$
$$i_-(PMP^T) = i_-(M).$$

By Sylvester's law of inertia, ECMOs do not change the inertia of a Hermitian matrix. Moreover, the following result is well known.

Lemma 2.3. *Let M be an $n \times n$ real symmetric matrix and N be the real matrix obtained by bordering M as follows:*

$$N = \begin{pmatrix} M & y \\ y^T & a \end{pmatrix},$$

where y is a real column vector and a is a real number. Then

$$i_+(N) - 1 \leqslant i_+(M) \leqslant i_+(N),$$
$$i_-(N) - 1 \leqslant i_-(M) \leqslant i_-(N).$$

The following result is an immediate consequence of Lemma 2.3.

Lemma 2.4. *Let H_w be an induced subgraph of G_w. Then $i_+(H_w) \leqslant i_+(G_w)$, $i_-(H_w) \leqslant i_-(G_w)$.*

It is known that the following result holds.

Lemma 2.5. *Let G_w be a weighted bipartite graph. Then $i_+(G_w) = i_-(G_w)$.*

The following lemma is a variation of known results. For example, it is a corollary of Theorem 1.1 (b) in [16], and a slight generalization of Lemma 2.3 in [22] or Lemma 2.9 in [27].

Lemma 2.6. *Let G_w be a weighted graph containing a pendant vertex v with the unique neighbor u. Then $i_+(G_w) = i_+(G_w - u - v) + 1$, $i_-(G_w) = i_-(G_w - u - v) + 1$ and $i_0(G_w) = i_0(G_w - u - v)$.*

Example 2.7. Let P_w be a weighted path of order $n = 2s + t$. Then

$$\big(i_+(P_w), i_-(P_w), i_0(P_w)\big) = \begin{cases} (\frac{n}{2}, \frac{n}{2}, 0), & \text{if } t = 0, \\ (\frac{n-1}{2}, \frac{n-1}{2}, 1), & \text{if } t = 1. \end{cases}$$

Let u, v be two pendant vertices of a weighted graph G_w. u, v are called *pendant twins* if they have the same neighbor in G_w.

Lemma 2.8. *Let u, v be pendant twins of a weighted graph G_w. Then $i_+(G_w) = i_+(G_w - u) = i_+(G_w - v)$, $i_-(G_w) = i_-(G_w - u) = i_-(G_w - v)$.*

Proof. Let u' be the common neighbor of u, v with $w_1 = w(uu')$, $w_2 = w(vu')$. Then the adjacency matrix of G_w can be expressed as

$$A(G_w) = \left(\begin{array}{cc|c|c} 0 & 0 & w_1 & \multirow{2}{*}{0} \\ 0 & 0 & w_2 & \\ \hline w_1 & w_2 & 0 & \alpha \\ \hline \multicolumn{2}{c|}{0^t} & \alpha^t & B \end{array}\right),$$

where B is the adjacency matrix of $G_w - u - v - u'$ and the first three rows and columns are labeled by u, v and u'. So we have

$$i_+(G_w) = i_+ \left(\begin{array}{cc|c|c} 0 & 0 & 0 & \multirow{2}{*}{0} \\ 0 & 0 & w_2 & \\ \hline 0 & w_2 & 0 & \alpha \\ \hline \multicolumn{2}{c|}{0^t} & \alpha^t & B \end{array}\right) \quad \text{by the third ECMO}$$

$$= i_+ \left(\begin{array}{c|c|c} 0 & w_2 & 0 \\ \hline w_2 & 0 & \alpha \\ \hline 0^t & \alpha^t & B \end{array}\right)$$

$$= i_+(G_w - u).$$

Similarly, we have $i_+(G_w) = i_+(G_w - v)$, $i_-(G_w) = i_-(G_w - u) = i_-(G_w - v)$. □

A subgraph U of an unweighted graph G is called an *elementary subgraph* if each component of U is a single edge or a cycle. Let $p(U)$, $c(U)$ be the number of components and the number of cycles contained in an elementary subgraph U, respectively.

Lemma 2.9. *(See [8].) The coefficient of the characteristic polynomial of the weighted graph G_w can be expressed as*

$$a_i^* = \sum_{U \in \mathscr{U}_i} (-1)^{p(U)} 2^{c(U)} \prod_{e \in E(U)} \big(w(e)\big)^{\zeta(e,U)},$$

where $\mathscr{U}_i$ is the set of all elementary subgraphs U contained in the underlying graph G having exactly i vertices, $\zeta(e,U) = 1$ if e is contained in some cycle of U and 2 otherwise.

The following result is well known for unweighted graphs [7].

Lemma 2.10. *(See [7].) Let T_w be a weighted tree of order n with matching number $m(T_w)$. Then*

$$i_+(T_w) = i_-(T_w) = m(T_w), \qquad i_0(T_w) = n - 2m(T_w).$$

Proof. For completeness, we present a proof. It is natural that any elementary subgraph in T consists only of copies of K_2 and has an even number of vertices. By Lemma 2.9, the coefficients of $P_{T_w}(\lambda)$ with odd subscript are zero. So we only consider the coefficients with even subscript. If $i > m(T_w)$, there exists no elementary subgraph and $a^*_{2i} = 0$. Therefore we suppose $0 \leqslant i \leqslant m(T_w)$ in the sequel. In view of Lemma 2.9, we have $a^*_{2i} = (-1)^i \sum_{U \in \mathscr{U}_{2i}} \prod_{e \in E(U)} (w(e))^2$. So $a_{2m(T_w)}$ is the last non-zero coefficient of $P_{T_w}(\lambda)$. It yields that $i_0(T_w) = n - 2m(T_w)$. So we have $i_+(T_w) = i_-(T_w) = m(T_w)$ by Lemma 2.5. □

Remark. It follows from Lemma 2.10 that the inertia of a weighted tree is independent of the weights.

Corollary 2.11. *Let G_w be a weighted forest of order n with matching number $m(G_w)$. Then*

$$i_+(G_w) = i_-(G_w) = m(G_w), \qquad i_0(G_w) = n - 2m(G_w).$$

The following result is an extension of Theorem 5.2 in [22].

Lemma 2.12. *Let G_w be a weighted graph. Then $|i_+(G_w) - i_-(G_w)| \leqslant c(G_w)$, where $c(G_w)$ is the number of all odd cycles in G_w.*

Proof. By Lemma 2.5, it suffices to consider the non-bipartite graphs. We apply induction on the number of odd cycles in G_w. Now assume that G_w contains at least one odd cycle and $|i_+(G_w - v) - i_-(G_w - v)| \leqslant c(G_w - v)$ holds for a vertex v of some odd cycle in G_w. By Lemma 2.3, we have

$$\begin{aligned} \big|i_+(G_w) - i_-(G_w)\big| &\leqslant \big|i_+(G_w - v) - i_-(G_w - v)\big| + 1 \\ &\leqslant c(G_w - v) + 1 \\ &\leqslant c(G_w). \end{aligned}$$

This completes the proof. □

3. Inertia of weighted unicyclic graphs

Let C^w_k be a weighted cycle with vertex set $\{v_1, v_2, \ldots, v_k\}$ such that $v_i v_{i+1} \in E(C^w_k)$ $(1 \leqslant i \leqslant k-1)$, $v_1 v_k \in E(C^w_k)$. Let $w_i = w(v_i v_{i+1})$ and $w_k = w(v_k v_1)$. For C^w_k, let $W = \prod_{i=1}^k w_i$. For an even integer k, let $W_e = w_2 w_4 \cdots w_k$ and $W_o = \frac{W}{W_e}$.

Definition 3.1. A weighted even cycle C^w_k is said to be of Type A (resp. Type B) if $W_o + (-1)^{\frac{k-2}{2}} W_e = 0$ (resp. $W_o + (-1)^{\frac{k-2}{2}} W_e \neq 0$).

A weighted odd cycle C^w_k is said to be of Type C (resp. Type D) if $(-1)^{\frac{k-1}{2}} W > 0$ (resp. $(-1)^{\frac{k-1}{2}} W < 0$).

Lemma 3.2. *Let C_n^w be a weighted cycle of order n. Then*

$$\bigl(i_+\bigl(C_n^w\bigr), i_-\bigl(C_n^w\bigr), i_0\bigl(C_n^w\bigr)\bigr) = \begin{cases} (\frac{n-2}{2}, \frac{n-2}{2}, 2), & \text{if } C_n^w \text{ is of Type } A, \\ (\frac{n}{2}, \frac{n}{2}, 0), & \text{if } C_n^w \text{ is of Type } B, \\ (\frac{n+1}{2}, \frac{n-1}{2}, 0), & \text{if } C_n^w \text{ is of Type } C, \\ (\frac{n-1}{2}, \frac{n+1}{2}, 0), & \text{if } C_n^w \text{ is of Type } D. \end{cases}$$

Proof. Let $V(C_n^w) = \{v_1, v_2, \ldots, v_n\}$ and $v_i v_{i+1} \in E(C_n^w)$ $(1 \leqslant i \leqslant n-1)$, $v_1 v_n \in E(C_n^w)$. Let $w_i = w(v_i v_{i+1})$ $(1 \leqslant i \leqslant n-1)$ and $w_n = w(v_n v_1)$. Then

$$A\bigl(C_n^w\bigr) = \begin{pmatrix} 0 & w_1 & 0 & 0 & \cdots & 0 & w_n \\ w_1 & 0 & w_2 & 0 & \cdots & 0 & 0 \\ 0 & w_2 & 0 & w_3 & \cdots & 0 & 0 \\ 0 & 0 & w_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & w_{n-1} \\ w_n & 0 & 0 & 0 & \cdots & w_{n-1} & 0 \end{pmatrix}$$

Case 1. n is even. Applying ECMOs on $A(C_n^w)$, we have

$$i_+\bigl(C_n^w\bigr) = i_+ \begin{pmatrix} 0 & w_1 & 0 & 0 & \cdots & 0 & w_n \\ w_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & w_3 & \cdots & 0 & -\frac{w_2 w_n}{w_1} \\ 0 & 0 & w_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & w_{n-1} \\ w_n & 0 & -\frac{w_2 w_n}{w_1} & 0 & \cdots & w_{n-1} & 0 \end{pmatrix}$$

$$= i_+ \begin{pmatrix} 0 & w_1 & 0 & 0 & \cdots & 0 & 0 \\ w_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & w_3 & \cdots & 0 & -\frac{w_2 w_n}{w_1} \\ 0 & 0 & w_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & w_{n-1} \\ 0 & 0 & -\frac{w_2 w_n}{w_1} & 0 & \cdots & w_{n-1} & 0 \end{pmatrix}$$

$$= i_+ \begin{pmatrix} 0 & w_1 \\ w_1 & 0 \end{pmatrix} + i_+ \begin{pmatrix} 0 & w_3 & \cdots & 0 & -\frac{w_2 w_n}{w_1} \\ w_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & w_{n-1} \\ -\frac{w_2 w_n}{w_1} & 0 & \cdots & w_{n-1} & 0 \end{pmatrix}$$

$$= 1 + i_+ \begin{pmatrix} 0 & w_3 & 0 & 0 & \cdots & 0 & -\frac{w_2 w_n}{w_1} \\ w_3 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & w_5 & \cdots & 0 & (-1)^2 \frac{w_2 w_4 w_n}{w_1 w_3} \\ 0 & 0 & w_5 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & w_{n-1} \\ -\frac{w_2 w_n}{w_1} & 0 & (-1)^2 \frac{w_2 w_4 w_n}{w_1 w_3} & 0 & \cdots & w_{n-1} & 0 \end{pmatrix}$$

$$= 2 + i_+ \begin{pmatrix} 0 & w_5 & \cdots & 0 & (-1)^2 \frac{w_2 w_4 w_n}{w_1 w_3} \\ w_5 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & w_{n-1} \\ (-1)^2 \frac{w_2 w_4 w_n}{w_1 w_3} & 0 & \cdots & w_{n-1} & 0 \end{pmatrix}$$

$$= \cdots$$

$$= \frac{n-4}{2} + i_+ \begin{pmatrix} 0 & w_{n-3} & 0 & c_1 \\ w_{n-3} & 0 & w_{n-2} & 0 \\ 0 & w_{n-2} & 0 & w_{n-1} \\ c_1 & 0 & w_{n-1} & 0 \end{pmatrix}$$

$$\left(\text{where } c_1 = (-1)^{\frac{n-4}{2}} \frac{w_2 w_4 \cdots w_{n-4} w_n}{w_1 w_3 \cdots w_{n-5}}\right)$$

$$= \frac{n-4}{2} + i_+ \begin{pmatrix} 0 & w_{n-3} & 0 & 0 \\ w_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & c_2 & 0 \end{pmatrix}$$

$$\left(\text{where } c_2 = w_{n-1} + (-1)^{\frac{n-2}{2}} \frac{w_2 w_4 \cdots w_{n-2} w_n}{w_1 w_3 \cdots w_{n-3}}\right)$$

$$= \frac{n-2}{2} + i_+ \begin{pmatrix} 0 & c_2 \\ c_2 & 0 \end{pmatrix}.$$

Moreover, note that

$$i_+ \begin{pmatrix} 0 & c_2 \\ c_2 & 0 \end{pmatrix} = \begin{cases} 0, & \text{if } c_2 = 0, \\ 1, & \text{if } c_2 \neq 0. \end{cases}$$

Therefore

$$i_+(C_n^w) = \begin{cases} \frac{n-2}{2}, & \text{if } w_1 w_3 \cdots w_{n-3} w_{n-1} + (-1)^{\frac{n-2}{2}} w_2 w_4 \cdots w_{n-2} w_n = 0, \\ \frac{n}{2}, & \text{if } w_1 w_3 \cdots w_{n-3} w_{n-1} + (-1)^{\frac{n-2}{2}} w_2 w_4 \cdots w_{n-2} w_n \neq 0. \end{cases}$$

Similarly, one has

$$i_-(C_n^w) = \begin{cases} \frac{n-2}{2}, & \text{if } w_1 w_3 \cdots w_{n-3} w_{n-1} + (-1)^{\frac{n-2}{2}} w_2 w_4 \cdots w_{n-2} w_n = 0, \\ \frac{n}{2}, & \text{if } w_1 w_3 \cdots w_{n-3} w_{n-1} + (-1)^{\frac{n-2}{2}} w_2 w_4 \cdots w_{n-2} w_n \neq 0. \end{cases}$$

Case 2. n is odd. By modifying the above procedure, we have

$$\begin{aligned} i_+(C_n^w) &= \frac{n-3}{2} + i_+\begin{pmatrix} 0 & w_{n-2} & c_3 \\ w_{n-2} & 0 & w_{n-1} \\ c_3 & w_{n-1} & 0 \end{pmatrix} \quad \text{where } c_3 = (-1)^{\frac{n-3}{2}} \frac{w_2 w_4 \cdots w_{n-3} w_n}{w_1 w_3 \cdots w_{n-4}} \\ &= \frac{n-3}{2} + i_+\begin{pmatrix} 0 & w_{n-2} & 0 \\ w_{n-2} & 0 & 0 \\ 0 & 0 & c_4 \end{pmatrix} \quad \text{where } c_4 = 2(-1)^{\frac{n-1}{2}} \frac{w_2 w_4 \cdots w_{n-1} w_n}{w_1 w_3 \cdots w_{n-2}} \\ &= \frac{n-1}{2} + \begin{cases} 1, & \text{if } c_4 > 0, \\ 0, & \text{if } c_4 < 0, \end{cases} \\ i_-(C_n^w) &= \frac{n-1}{2} + \begin{cases} 1, & \text{if } c_4 < 0, \\ 0, & \text{if } c_4 > 0. \end{cases} \end{aligned}$$

It is evident that the sign of c_4 is the same as that of $(-1)^{\frac{n-1}{2}}(\prod_{i=1}^n w_i)$, which leads to the desired result. □

3.1. Minimal positive (negative) index of weighted unicyclic graphs

Let $U_{n,k}$ $(n > k)$ be an unweighted unicyclic graph obtained from a cycle C_k by attaching $n-k$ pendant vertices to a vertex of C_k.

Theorem 3.3. *Let G_w be a weighted unicyclic graph of order n with girth k $(3 \leqslant k \leqslant n-2)$. Then $i_+(G_w) \geqslant \lceil \frac{k}{2} \rceil$, $i_-(G_w) \geqslant \lceil \frac{k}{2} \rceil$. These bounds are sharp if the underlying graph of G_w is $U_{n,k}$.*

Proof. It is evident that the underlying graph of G_w must contain $U_{k+1,k}$ as an induced subgraph. Moreover, by Lemma 2.6, $i_+(U_{k+1,k}) = i_+(P_{k-1}) + 1 = \lceil \frac{k}{2} \rceil$, so by Lemma 2.4 $i_+(G_w) \geqslant \lceil \frac{k}{2} \rceil$. Similarly, $i_-(G_w) \geqslant \lceil \frac{k}{2} \rceil$. By Lemmas 2.10 and 2.6, any weighted graph with $U_{n,k}$ as its underlying graph has the same positive (negative) index as $U_{k+1,k}$. Therefore $i_+(U_{n,k}) = \lceil \frac{k}{2} \rceil$ and $i_-(U_{n,k}) = \lceil \frac{k}{2} \rceil$. □

Corollary 3.4. *Let G_w be a weighted unicyclic graph of order n with girth k $(3 \leqslant k \leqslant n-2)$. Then $i_0(G_w) \leqslant n - 2\lceil \frac{k}{2} \rceil$. The bound is sharp if the underlying graph of G_w is $U_{n,k}$.*

Corollary 3.5. *Let G_w be a weighted unicyclic graph of order n with pendant vertices. Then $i_+(G_w) \geqslant 2$, $i_-(G_w) \geqslant 2$ and $i_0(G_w) \leqslant n-4$. These bounds are sharp if the underlying graph of G_w is $U_{n,3}$ or $U_{n,4}$.*

3.2. Weighted unicyclic graphs with the minimal positive (negative) index

The following result is an extension of Theorems 3.1 and 3.3 in [15].

Lemma 3.6. *Let T_w be a weighted tree with $u \in V(T_w)$ and G_w^0 be a weighted graph different from T_w. Let G_w be a graph obtained from G_w^0 and T_w by joining u with certain vertices of G_w^0. Then the following statements hold:*

(1) *If u is a saturated vertex in T_w, then*

$$i_+(G_w) = i_+(T_w) + i_+\big(G_w^0\big) = m(T_w) + i_+\big(G_w^0\big),$$
$$i_-(G_w) = i_-(T_w) + i_-\big(G_w^0\big) = m(T_w) + i_-\big(G_w^0\big).$$

(2) *If u is an unsaturated vertex in T_w, then*

$$i_+(G_w) = i_+(T_w - u) + i_+\big(G_w^0 + u\big) = m(T_w) + i_+\big(G_w^0 + u\big),$$
$$i_-(G_w) = i_-(T_w - u) + i_-\big(G_w^0 + u\big) = m(T_w) + i_-\big(G_w^0 + u\big),$$

where $G_w^0 + u$ is the subgraph of G_w induced by the vertices of G_w^0 and u.

Let G_w be a weighted unicyclic graph and C_k^w be the unique weighted cycle of G_w. Let G_w' be the graph obtained from G_w by deleting the two neighbors of v on C_k^w and let $G_w\{v\}$ be the component of G_w' containing v. Then $G_w\{v\}$ is a weighted tree and an induced subgraph of G_w. From Lemma 3.6, it follows that:

Lemma 3.7. *Let G_w be a weighted unicyclic graph and C_k^w be the unique weighted cycle of G_w. Then the following statements hold:*

(1) *If there exists a vertex $v \in V(C_k^w)$ which is saturated in $G_w\{v\}$, then*

$$i_+(G_w) = i_+\big(G_w\{v\}\big) + i_+\big(G_w - G_w\{v\}\big),$$
$$i_-(G_w) = i_-\big(G_w\{v\}\big) + i_-\big(G_w - G_w\{v\}\big).$$

(2) *If there does not exist a vertex $v \in V(C_k^w)$ which is saturated in $G_w\{v\}$, then*

$$i_+(G_w) = i_+\big(G_w - C_k^w\big) + i_+\big(C_k^w\big),$$
$$i_-(G_w) = i_-\big(G_w - C_k^w\big) + i_-\big(C_k^w\big).$$

Let G^* be an unweighted unicyclic graph of order n obtained from a cycle C_k and a star $K_{1,n-k-1}$ of order $n-k$ by inserting an edge between a vertex on C_k and the center of $K_{1,n-k-1}$. Let $\mathcal{U}_{n,k}^w$ $(3 \leqslant k \leqslant n-2)$ be the set of weighted unicyclic graphs of order n with girth k. In the following we shall characterize all weighted unicyclic graphs with minimal positive (negative) index $\lceil \frac{k}{2} \rceil$ among all graphs in $\mathcal{U}_{n,k}^w$.

Theorem 3.8. *Let $G_w \in \mathcal{U}_{n,k}^w$ be a weighted unicyclic graph associated the unique weighted cycle C_k^w with vertex set $\{v_1, v_2, \ldots, v_k\}$. Then $i_+(G_w) = \lceil \frac{k}{2} \rceil$ if and only if either* (1) *or* (2) *holds.*

(1) *If there exists a vertex* $v_i \in V(C_k^w)$ *which is saturated in* $G_w\{v_i\}$, *then* $m(G_w\{v_i\}) = 1$ (*equivalently,* $G_w\{v_i\}$ *is a star*) *and* $m(G_w - G_w\{v_i\}) = \lfloor \frac{k-1}{2} \rfloor$.
(2) *If there does not exist a vertex* $v \in V(C_k^w)$ *which is saturated in* $G_w\{v\}$, *then* $G \cong G^*$ *and* C_k^w *is of*

$$\begin{cases} \textit{Type A for even } k, \\ \textit{Type D for odd } k. \end{cases}$$

Proof. Assume that there exists a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$. Without loss of generality, we assume that $v_i = v_1$. We shall verify the following claim.

Claim. $G_w\{v_1\}$ *is a star and* $m(G_w - G_w\{v_1\}) = \lfloor \frac{k-1}{2} \rfloor$.

Since $v_1 \in V(C_k^w)$ is saturated in $G_w\{v_1\}$, from Lemmas 2.10 and 3.7, we have

$$\begin{aligned} i_+(G_w) &= i_+\big(G_w\{v_1\}\big) + i_+\big(G_w - G_w\{v_1\}\big) \\ &= m\big(G_w\{v_1\}\big) + m\big(G_w - G_w\{v_1\}\big). \end{aligned}$$

If k is even, then

$$m\big(G_w\{v_1\}\big) + m\big(G_w - G_w\{v_1\}\big) = \frac{k}{2}.$$

Note that $m(G_w\{v_1\}) \geqslant 1$ and $m(G_w - G_w\{v_1\}) \geqslant \frac{k-2}{2}$ since $G_w\{v_1\}$ contains P_{k-1}. So it follows that $m(G_w\{v_1\}) = 1$ and $m(G_w - G_w\{v_1\}) = \frac{k-2}{2}$.

If k is odd, then

$$m\big(G_w\{v_1\}\big) + m\big(G_w - G_w\{v_1\}\big) = \frac{k+1}{2}.$$

Note that $m(G_w\{v_1\}) \geqslant 1$ and $m(G_w - G_w\{v_1\}) \geqslant \frac{k-1}{2}$. So it follows that $m(G_w\{v_1\}) = 1$ and $m(G_w - G_w\{v_1\}) = \frac{k-1}{2}$. This completes the proof of claim.

Now assume that any vertex $v \in V(C_k^w)$ is unsaturated in $G_w\{v\}$. By Lemmas 2.10 and 3.7, we have

$$\begin{aligned} i_+(G_w) &= i_+\big(C_k^w\big) + i_+\big(G_w - C_k^w\big) \\ &= i_+\big(C_k^w\big) + m\big(G_w - C_k^w\big). \end{aligned}$$

Then we have

$$m\big(G_w - C_k^w\big) = \begin{cases} 1, & \text{if } C_k^w \text{ is of Type } A, \\ 0, & \text{if } C_k^w \text{ is of Type } B, \\ 0, & \text{if } C_k^w \text{ is of Type } C, \\ 1, & \text{if } C_k^w \text{ is of Type } D. \end{cases}$$

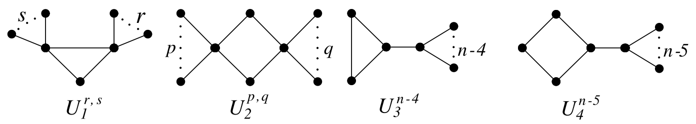

Fig. 1. Four unicyclic graphs $U_1^{r,s}$, $U_2^{p,q}$, U_3^{n-4}, U_4^{n-5}.

If $m(G_w - C_k^w) = 0$, then any vertex not on C_k^w is a pendant vertex which is adjacent to a vertex on C_k^w in G_w. This contradicts the fact that v is unsaturated in $G_w\{v\}$ for any $v \in V(C_k^w)$. So $m(G_w - C_k^w) = 1$, i.e. $G_w - C_k^w$ is a star. This implies the result. □

Similar to Theorem 3.8, we have

Theorem 3.9. *Let $G_w \in \mathcal{U}_{n,k}^w$ be a weighted unicyclic graph associated the unique weighted cycle C_k^w with vertex set $\{v_1, v_2, \dots, v_k\}$. Then $i_-(G_w) = \lceil \frac{k}{2} \rceil$ if and only if either* (1) *or* (2) *holds.*

(1) *If there exists a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$, then $m(G_w\{v_i\}) = 1$ (equivalently, $G_w\{v_i\}$ is a star) and $m(G_w - G_w\{v_i\}) = \lfloor \frac{k-1}{2} \rfloor$.*
(2) *If there does not exist a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$, then $G \cong G^*$ and C_k^w is of*

$$\begin{cases} \text{Type } A \text{ for even } k, \\ \text{Type } C \text{ for odd } k. \end{cases}$$

From Corollary 3.4 and Theorems 3.8, 3.9, it follows that:

Theorem 3.10. *Let $G_w \in \mathcal{U}_{n,k}^w$ be a weighted unicyclic graph associated the unique weighted cycle C_k^w with vertex set $\{v_1, v_2, \dots, v_k\}$. Then $i_0(G_w) = n - 2\lceil \frac{k}{2} \rceil$ if and only if either* (1) *or* (2) *holds.*

(1) *If there exists a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$, then $m(G_w\{v_i\}) = 1$ (equivalently, $G_w\{v_i\}$ is a star) and $m(G_w - G_w\{v_i\}) = \lfloor \frac{k-1}{2} \rfloor$.*
(2) *If there does not exist a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$, then $G \cong G^*$ and C_k^w is of Type A.*

3.3. Weighted unicyclic graphs with two positive (negative) eigenvalues

First we define the following four classes of unweighted unicyclic graphs (see Fig. 1). We note that these also appear in Figs. 1 and 3 of [18].

$U_1^{r,s}$ ($r, s \geqslant 0$, $r + s = n - 3$) is a unicyclic graph of order n obtained by attaching r and s pendant vertices at two different vertices of C_3, respectively; r or s is allowed to be 0.

$U_2^{p,q}$ ($p, q \geqslant 0$, $p+q=n-4$) is a unicyclic graph of order n obtained by attaching p, q pendant vertices at two nonadjacent vertices of C_4, respectively; p or q is allowed to be 0.

U_3^{n-4} is a unicyclic graph of order n obtained from C_3 and $K_{1,n-4}$ by inserting an edge between a vertex of C_3 and the center of $K_{1,n-4}$.

U_4^{n-5} is a unicyclic graph of order n obtained from C_4 and $K_{1,n-5}$ by inserting an edge between a vertex of C_4 and the center of $K_{1,n-5}$.

Theorem 3.11. *Let G_w be a weighted unicyclic graph of order n. Then $i_+(G_w)=2$ if and only if G_w is one of the following graphs: C_3^w of Type C; C_4^w of Type B; C_5^w of Type D; C_6^w of Type A; the weighted graphs with $U_1^{r,s}$ or $U_2^{p,q}$ as the underlying graph; the weighted graphs with U_3^{n-4} as the underlying graph in which the cycle C_3^w is of Type D; the weighted graphs with U_4^{n-5} as the underlying graph in which the cycle C_4^w is of Type A.*

Proof. The sufficiency can be easily verified by Lemmas 2.6 and 3.2. Next we consider the necessity. Assume that the girth of G_w is k.

If $k=n$, by virtue of Lemma 3.2, G_w is one of the following weighted cycles: C_3^w of Type C; C_4^w of Type B; C_5^w of Type D; C_6^w of Type A.

If $k=n-1$, by Lemmas 2.6 and 2.10, G_w is one of the weighted graphs with $U_1^{1,0}$ or $U_2^{1,0}$ as the underlying graph.

If $3 \leqslant k \leqslant n-2$, then by Theorem 3.3, $2=i_+(G_w) \geqslant \lceil \frac{k}{2} \rceil$, which implies that $k=3$ or 4 and $i_+(G_w)=\lceil \frac{k}{2} \rceil$. Hence it follows from Theorem 3.8 that (1) or (2) in Theorem 3.8 holds. We consider the following two cases. If (1) in Theorem 3.8 holds, i.e., there exists a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$, then $G_w\{v_i\}$ is a star and $m(G_w - G_w\{v_i\}) = \lfloor \frac{k-1}{2} \rfloor = 1$, i.e., $G_w - G_w\{v_i\}$ is a star. Hence G_w is one of the following graphs: the weighted graphs with $U_1^{r,s}$ or $U_2^{p,q}$ as the underlying graph; the weighted graphs with U_3^{n-4} as the underlying graph in which the cycle C_3^w is of Type D; the weighted graphs with U_4^{n-5} as the underlying graph in which the cycle C_4^w is of Type A.

If (2) in Theorem 3.8 holds, i.e., there does not exist a vertex $v \in V(C_k^w)$ which is saturated in $G_w\{v\}$, then G_w is U_3^{n-4} and C_3^w is Type D, or G_w is U_4^{n-5} and C_4^w is the Type A. Hence the assertion holds. □

Similar to the above result, we have

Theorem 3.12. *Let G_w be a weighted unicyclic graph of order n. Then $i_-(G_w)=2$ if and only if G_w is one of the following graphs: C_3^w of Type D; C_4^w of Type B; C_5^w of Type C; C_6^w of Type A; the weighted graphs with $U_1^{r,s}$ or $U_2^{p,q}$ as the underlying graph; the weighted graphs with U_3^{n-4} as the underlying graph in which the cycle C_3^w is of Type C; the weighted graphs with U_4^{n-5} as the underlying graph in which the cycle C_4^w is of Type A.*

Note that there does not exist weighted unicyclic graph G_w with $i_+(G_w) = 1$ and $i_-(G_w) = 3$, or $i_+(G_w) = 3$ and $i_-(G_w) = 1$ from Lemma 2.12. Combining this fact with Theorems 3.11 and 3.12, we have

Theorem 3.13. *Let G_w be a weighted unicyclic graph of order n. Then $i_0(G_w) = n - 4$ if and only if G_w is one of the following graphs: C_4^w of Type B; C_6^w of Type A; the weighted graphs with $U_1^{r,s}$ or $U_2^{p,q}$ as the underlying graph; the weighted graphs with U_4^{n-5} as the underlying graph in which the cycle C_4^w is of Type A.*

From Theorem 3.13, we get that:

Corollary 3.14. *(See [13].) Let Γ be a unicyclic signed graph of order n. Then $i_0(\Gamma) = n - 4$ if and only if Γ is one of the following signed graphs of order n: unbalance C_4; unbalance C_6; the signed graphs with $U_1^{r,s}$ or $U_2^{p,q}$ as the underlying graph; the balance signed graph with U_4^{n-5} as the underlying graph.*

Corollary 3.15. *(See [26].) Let U be an unweighted unicyclic graph of order n. Then $i_0(U) = n - 4$ if and only if U is one of the following graphs of order n: $U_1^{r,s}$, $U_2^{p,q}$, or U_4^{n-5}.*

4. Weighted unicyclic graphs with rank 6

Lemma 4.1. *Let G_w be a weighted unicyclic graph with three positive (negative) eigenvalues and girth k. Then $k \leqslant 8$.*

Proof. If $k \geqslant 9$, then G_w must contain P_8^w as an induced subgraph. By Lemma 2.4, $i_+(G_w) \geqslant 4$ ($i_-(G_w) \geqslant 4$) which is a contradiction. □

In what follows we shall characterize the weighted unicyclic graphs with three positive eigenvalues. Let $\mathcal{U}^*$ be the set of weighted unicyclic graphs without pendant twins. By Lemma 2.8, it suffices to characterize the weighted unicyclic graphs among all graphs in $\mathcal{U}^*$. Let G_1 (resp. G_2) be the unweighted graph obtained from C_8 (resp. C_7) by attaching a pendant edge on a vertex of C_8 (resp. C_7).

Theorem 4.2. *Let $G_w \in \mathcal{U}^*$ be a weighted unicyclic graph with girth k. Then:*

(1) *If $k = 7$, $i_+(G_w) = 3$ if and only if G_w is C_7^w of Type D.*
(2) *If $k = 8$, $i_+(G_w) = 3$ if and only if G_w is C_8^w of Type A.*

Proof. The sufficiency can be easily verified by Lemma 3.2. Next we consider the necessity.

If G_w is a cycle, by Lemma 3.2 G_w is C_8^w of Type A, or C_7^w of Type D. Assume that the underlying graph of G_w contains G_1 or G_2 as an induced subgraph. By Lemma 2.6,

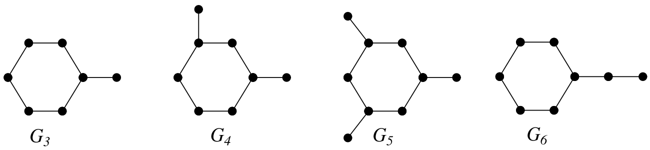

Fig. 2. Four unweighted graphs in Theorem 4.3.

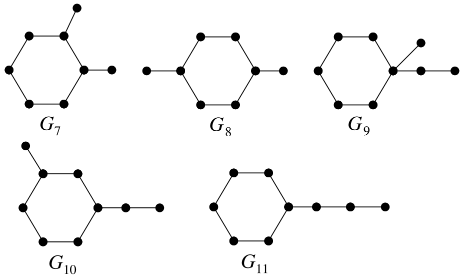

Fig. 3. Five unweighted graphs excluded by $i_+(G_w) = 3$.

we have $i_+(G_w) \geqslant i_+(G_1) = 1 + i_+(P_7) = 4$ and $i_+(G_w) \geqslant i_+(G_2) = 1 + i_+(P_6) = 4$ which contradicts the fact that $i_+(G_w) = 3$. This completes the proof. □

Theorem 4.3. *Let $G_w \in \mathcal{U}^*$ be a weighted unicyclic graph with girth 6. Then $i_+(G_w) = 3$ if and only if G_w is one of the following graphs: the weighted graphs with G_3, G_4 or G_5 (as depicted in Fig. 2) as the underlying graph; the weighted graphs with G_6 (as depicted in Fig. 2) as the underlying graph in which the cycle C_6^w is of Type A.*

Proof. The sufficiency can be easily verified by Lemmas 2.6 and 3.2. Next we consider the necessity.

By Lemmas 2.6 and 3.2, the graphs with one of G_i's ($i = 7, 8, \ldots, 11$) (as depicted in Fig. 3) as the underlying graph have four positive eigenvalues. Let $G_w \in \mathcal{U}^*$ be a weighted graph with girth 6 and positive index 3.

Case 1. $G_w - C_6^w$ is a set of isolated vertices.

If the order of the graph $G_w - C_6^w$ is 1, G_w is the graph with G_3 as the underlying graph.

If the order of the graph $G_w - C_6^w$ is 2, G_w is the graph with G_4 as the underlying graph. The graph with G_7 or G_8 as the underlying graph has four positive eigenvalues.

If the order of the graph $G_w - C_6^w$ is at least 3, G_w is the graph with G_5 as the underlying graph. Moreover, other graphs must contain G_7 as an induced subgraph and have more than three positive eigenvalues.

Case 2. $G_w - C_6^w$ contains P_2 as an induced subgraph.

Then G_w is the graph with G_6 as the underlying graph in which the cycle C_6^w is of Type A. Any other graph has more than three positive eigenvalues since its underlying graph contains one of G_i's ($i = 7, 8, \ldots, 11$) as an induced subgraph. □

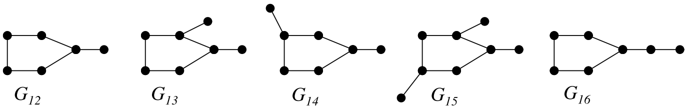

Fig. 4. Five unweighted graphs in Theorem 4.4.

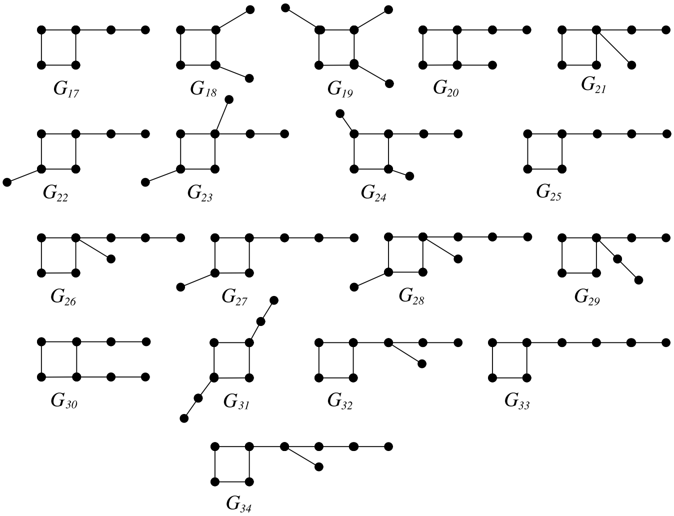

Fig. 5. Eighteen unweighted graphs in Theorem 4.5.

Similar to Theorem 4.3, we have

Theorem 4.4. *Let $G_w \in \mathcal{U}^*$ be a weighted unicyclic graph with girth* 5*. Then $i_+(G_w) = 3$ if and only if G_w is one of the following graphs*: *the weighted graph with one of G_i's* ($i = 12, \ldots, 15$) (*as depicted in Fig.* 4) *as the underlying graph*; *the weighted graph with G_{16}* (*as depicted in Fig.* 4) *as the underlying graph in which the cycle C_5^w is of Type D.*

Theorem 4.5. *Let $G_w \in \mathcal{U}^*$ be a weighted unicyclic graph with girth* 4*. Then $i_+(G_w) = 3$ if and only if G_w is one of the following graphs*: *the weighted graphs with one of G_i's* ($i = 18, 19, \ldots, 28$) (*as depicted in Fig.* 5) *as the underlying graph*; *the weighted graphs with G_{17}* (*as depicted in Fig.* 5) *as the underlying graph in which the cycle C_4^w is of Type B*; *the weighted graphs with one of G_i's* ($i = 29, 30, \ldots, 34$) (*as depicted in Fig.* 5) *as the underlying graph in which the cycle C_4^w is of Type A.*

Proof. The sufficiency can be easily verified by Lemmas 2.6 and 3.2. Next we consider the necessity.

By Lemmas 2.6 and 3.2, the graphs with one of G_i's ($i = 35, 36, \ldots, 50$) (as depicted in Fig. 6) as the underlying graph have four positive eigenvalues. Let $G_w \in \mathcal{U}^*$ be a weighted graph with girth 4 and positive index 3. For convenience, denote by $G^\diamondsuit$ the underlying graph of $G_w - C_4^w$.

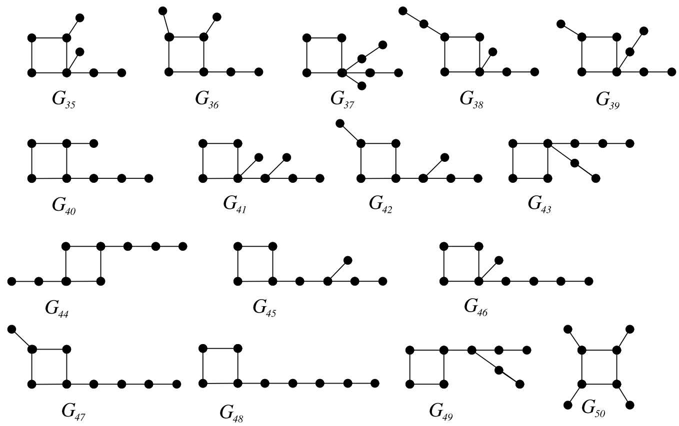

Fig. 6. Sixteen unweighted graphs excluded by $i_+(G_w) = 3$.

Case 1. $G^\diamondsuit$ is a set of isolated vertices.

G_w is the weighted graph with G_{18}, or G_{19} as the underlying graph.

Case 2. $G^\diamondsuit$ contains P_2, but no P_3, as an induced subgraph.

If $G^\diamondsuit = P_2$, G_w is the weighted graph with G_{17} as the underlying graph in which the cycle C_4^w is of Type B.

If $G^\diamondsuit$ is the union of an isolated vertex and P_2, G_w is the weighted graph with G_{20}, G_{21} or G_{22} as the underlying graph.

If $G^\diamondsuit$ is the union of two isolated vertices and P_2, G_w is the weighted graph with G_{23} or G_{24} as its underlying graph.

If $G^\diamondsuit$ is the union of more than two isolated vertices and P_2, any weighted graph has more than three positive eigenvalues since its underlying graph contains G_{35} or G_{36} as an induced subgraph.

If $G^\diamondsuit$ is two copies of P_2, G_w is the weighted graph with G_{29}, G_{30} or G_{31} as the underlying graph in which the cycle C_4^w is of Type A.

If $G^\diamondsuit$ is the union of some isolated vertices and two P_2's, any weighted graph has more than three positive eigenvalues since its underlying graph contains one of G_i's $(i = 35, 36, \dots, 39)$ as an induced subgraph.

If $G^\diamondsuit$ contains three P_2's as its induced subgraph, any weighted graph has more than three positive eigenvalues since its underlying graph contains one of G_i's $(i = 35, 36, 37)$ as an induced subgraph.

Case 3. $G^\diamondsuit$ contains P_3, but no P_4, as an induced subgraph.

If $G^\diamondsuit = P_3$, G_w is the weighted graph with G_{25} as its underlying graph.

If $G^\diamondsuit$ is the union of one isolated vertex and P_3, G_w is one of the following graphs: the weighted graphs with G_{26} or G_{27} as the underlying graph.

If $G^\diamondsuit$ is the union of two isolated vertices and P_3, G_w is the weighted graph with G_{28} as the underlying graph.

If $G^\diamondsuit$ is the union of more than two isolated vertices and P_3, any graph has more than three positive eigenvalues since it contains G_{40}, G_{41} or G_{42} as an induced subgraph.

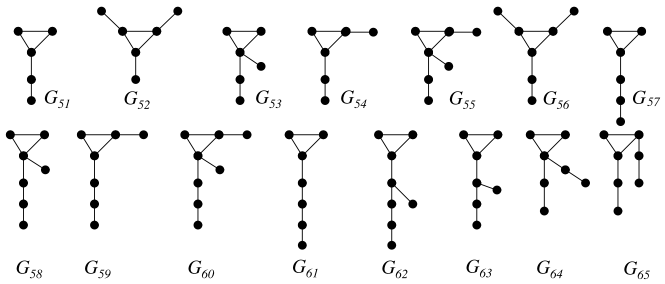

Fig. 7. Fifteen unweighted graphs in Theorem 4.6.

If $G^\diamondsuit$ contains the union of P_2 and P_3 as its induced subgraph, any graph has more than three positive eigenvalues since it contains G_{40}, G_{43} or G_{44} as an induced subgraph.

Case 4. $G^\diamondsuit$ contains P_4, but no P_5, as an induced subgraph.

G_w is the weighted graph with G_{32}, or G_{33} as the underlying graph in which the cycle C_4^w is of Type A. All other graphs have more than three positive eigenvalues since their underlying graphs contain one of G_i's ($i = 40, 41, \dots, 47$) as an induced subgraph.

Case 5. $G^\diamondsuit$ contains P_5 as an induced subgraph.

G_w is the weighted graph with G_{34} as the underlying graph in which the cycle C_4^w is of Type A. All other graphs have more than three positive eigenvalues since their underlying graphs contain one of G_i's ($i = 40, 41, \dots, 49$) as an induced subgraph. □

Similar to Theorem 4.5, we have

Theorem 4.6. *Let $G_w \in \mathcal{U}^*$ be a weighted unicyclic graph with girth 3. Then $i_+(G_w) = 3$ if and only if G_w is one of the following graphs: the weighted graphs with one of G_i's ($i = 52, 53, \dots, 60$) (as depicted in Fig. 7) as the underlying graph; the weighted graphs with G_{51} (as depicted in Fig. 7) as the underlying graph in which the cycle C_3^w is of Type C; the weighted graphs with one of G_i's ($i = 61, 62, \dots, 65$) (as depicted in Fig. 7) as the underlying graph in which the cycle C_3^w is of Type D.*

Theorem 4.7. *Let $G_w \in \mathcal{U}^*$ be a weighted unicyclic graph with rank 6 and girth k. Then $k = 3, 4, 5, 6,$ or 8. Furthermore,*

(1) *if $k = 3$, then G_w is one of the weighted graphs with one of G_i's ($i = 52, 53, \dots, 60$) (as depicted in Fig. 7) as the underlying graph;*
(2) *if $k = 4$, then G_w is one of the following graphs: the weighted graph with one of G_i's ($i = 18, 19, \dots, 28$) (as depicted in Fig. 5) as the underlying graph; the weighted graph with G_{17} (as depicted in Fig. 5) as the underlying graph in which the cycle C_4^w is of Type B; the weighted graph with one of G_i's ($i = 29, \dots, 34$) (as depicted in Fig. 5) as the underlying graph in which the cycle C_4^w is of Type A;*
(3) *if $k = 5$, then G_w is one of the weighted graphs with one of G_i's ($i = 12, \dots, 15$) (as depicted in Fig. 4) as the underlying graph;*

(4) *if* $k = 6$*, then* G_w *is one of the following graphs*: *the weighted graph with* G_3*,* G_4 *or* G_5 (*as depicted in* Fig. 2) *as the underlying graph*; *the weighted graph with* G_6 (*as depicted in* Fig. 2) *as the underlying graph in which the cycle* C_6^w *is of Type A*;
(5) *if* $k = 8$*, then* G_w *is* C_8^w *of Type A.*

Proof. By Lemma 2.12, $r(G_w) = 6$ if and only if $i_+(G_w) = i_-(G_w) = 3$ for any weighted unicyclic G_w. So it suffices to characterize the weighted unicyclic graphs with rank 6 among all weighted unicyclic graphs with three positive eigenvalues. That is to say, we eliminate graphs G_w with $i_-(G_w) \neq 3$ among all graphs with three positive eigenvalues. So the results follow from Lemma 4.1 and Theorems 4.2–4.6. □

It follows from Theorem 4.7 and Lemma 3.2 that the following result holds.

Corollary 4.8. *Let* $G \in \mathcal{U}^*$ *be an unweighted unicyclic graph with rank* 6 *and girth* k*. Then* $k = 3, 4, 5, 6,$ *or* 8*. Furthermore,*

(1) *if* $k = 3$*, then* G *is one of* G_i*'s* $(i = 52, 53, \dots, 60)$ (*as depicted in* Fig. 7);
(2) *if* $k = 4$*, then* G *is one of* G_i*'s* $(i = 18, 19, \dots, 34)$ (*as depicted in* Fig. 5);
(3) *if* $k = 5$*, then* G *is one of* G_i*'s* $(i = 12, \dots, 15)$ (*as depicted in* Fig. 4);
(4) *if* $k = 6$*, then* G *is one of* G_i*'s* $(i = 3, 4, 5)$ (*as depicted in* Fig. 2);
(5) *if* $k = 8$*, then* G *is* C_8*.*

Corollary 4.9. *Let* $\Gamma \in \mathcal{U}^*$ *be a unicyclic signed graph with rank* 6 *and girth* k*. Then* $k = 3, 4, 5, 6,$ *or* 8*. Furthermore,*

(1) *if* $k = 3$*, then* Γ *is one of the signed graph with one of* G_i*'s* $(i = 52, 53, \dots, 60)$ (*as depicted in* Fig. 7) *as the underlying graph*;
(2) *if* $k = 4$*, then* Γ *is one of the following graphs*: *the signed graph with one of* G_i*'s* $(i = 18, 19, \dots, 28)$ (*as depicted in* Fig. 5) *as the underlying graph*; *the unbalanced signed graph with* G_{17} (*as depicted in* Fig. 5) *as the underlying graph*; *the balanced signed graph with one of* G_i*'s* $(i = 29, \dots, 34)$ (*as depicted in* Fig. 5) *as the underlying graph*;
(3) *if* $k = 5$*, then* Γ *is one of the signed graphs with* G_{12}*,* G_{13}*,* G_{14} *or* G_{15} (*as depicted in* Fig. 4) *as the underlying graph*;
(4) *if* $k = 6$*, then* Γ *is one of the following graphs*: *the signed graph with* G_3*,* G_4 *or* G_5 (*as depicted in* Fig. 2) *as the underlying graph*; *the balanced signed graph with* G_6 (*as depicted in* Fig. 2) *as the underlying graph*;
(5) *if* $k = 8$*, then* Γ *is the balanced cycle* C_8*.*

It follows from Lemma 2.8 and Theorem 4.7 that the assertion holds.

Theorem 4.10. *Let* G_w *be a weighted unicyclic graph of order* n*. Then the rank of* G_w *is* 6 *if and only if* G_w *is obtained from one* G *of the graphs described in* Theorem 4.7

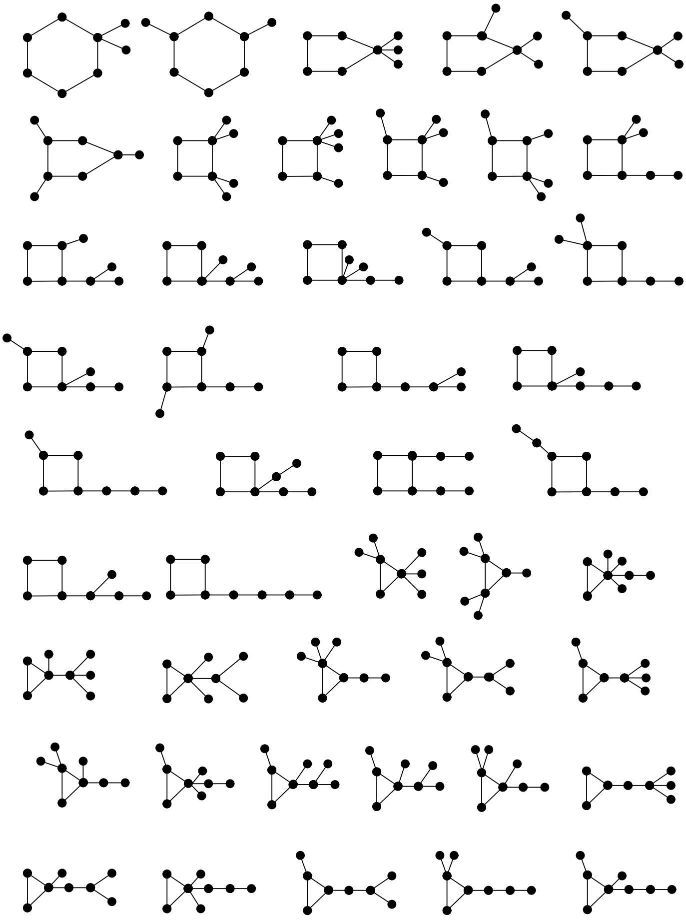

Fig. 8. 45 unweighted unicyclic graphs on eight vertices.

by attaching some appropriate pendant edges to some neighbor vertices of all pendant vertices of G.

Example 4.11. All unweighted unicyclic graphs of order 8 with rank 6 are C_8 and the 45 graphs depicted in Fig. 8. This result corresponds to the one obtained by Cvetković and Rowlinson [8].

5. Weighted unicyclic graphs with rank 2, 3, 5

Fan et al. [13] characterized the unicyclic signed graph of order n with rank 2, 3, 4, 5, respectively. In this section we shall consider the same question following the ideas in [13]. In Section 3, we characterized the weighted unicyclic graphs of order n with rank 4. Here we shall determine the weighted unicyclic graphs of order n with rank 2, 3, 5, respectively.

Theorem 5.1. *Let G_w be a weighted unicyclic graph of order n and C_k^w be the unique cycle in G_w with vertex set $\{v_1, v_2, \ldots, v_k\}$. Then:*

(1) $i_0(G_w) = n - 2$ $(r(G_w) = 2)$ *if and only if G_w is the weighted cycle C_4 which is of Type A.*
(2) $i_0(G_w) = n - 3$ $(r(G_w) = 3)$ *if and only if G_w is the cycle C_3 with arbitrary weights.*

Proof. It is obvious that the sufficiency for (1) or (2) holds by Lemma 3.2. Next we consider the necessity.

Assume that $i_0(G_w) = n - 2$. If G_w is a weighted cycle, then G_w is the weighted cycle C_4 which is of Type A by Lemma 3.2. Next assume that G_w contains pendant edges. Suppose there exists a vertex v_i in C_k^w such that it is saturated in $G_w\{v_i\}$. Without loss of generality, suppose $v_1 \in V(C_k^w)$ is saturated in $G_w\{v_1\}$. By Lemma 3.7, we have

$$\begin{aligned} i_0(G_w) &= i_0\big(G_w\{v_1\}\big) + i_0\big(G_w - G_w\{v_1\}\big) \\ &= n - 2m\big(G_w\{v_1\}\big) - 2m\big(G_w - G_w\{v_1\}\big). \end{aligned}$$

Since $m(G_w\{v_1\}) \geqslant 1$ and $m(G_w - G_w\{v_1\}) \geqslant 1$, $i_0(G_w) \leqslant n-4$. This is a contradiction.

Suppose there does not exist a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$. By Lemma 3.7,

$$\begin{aligned} i_0(G_w) &= i_0\big(G_w - C_k^w\big) + i_0\big(C_k^w\big) \\ &= n - k - 2m\big(G_w - C_k^w\big) + i_0\big(C_k^w\big). \end{aligned}$$

It yields that $i_0(C_k^w) = k + 2(m(G_w - C_k^w) - 1) \geqslant k \geqslant 3$ which is a contradiction.

Assume that $i_0(G_w) = n - 3$. If G_w is a weighted cycle, by Lemma 3.2, G_w is C_3^w with arbitrary weights. Assume that G_w contains at least one pendant edge. By the above discussion, if there exists a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$, then $i_0(G_w) \leqslant n - 4$ which is a contradiction. If there does not exist a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$, then $i_0(C_k^w) = k + 2m(G_w - C_k^w) - 3 \geqslant k - 1 \geqslant 2$. So only if $k = 3$, $i_0(G_w) = 2$. But $i_0(C_3^w) = 0$. This case cannot occur. □

The following results are immediate from Theorem 5.1.

Corollary 5.2. *Let G be an unweighted unicyclic graph of order n. Then:*

(1) $i_0(G) = n - 2$ $(r(G) = 2)$ *if and only if G is the cycle C_4.*
(2) $i_0(G) = n - 3$ $(r(G) = 3)$ *if and only if G is the cycle C_3.*

Corollary 5.3. *(See [13].) Let Γ be a unicyclic signed graph of order n. Then:*

(1) $i_0(\Gamma) = n - 2$ $(r(\Gamma) = 2)$ *if and only if Γ is the balanced cycle C_4.*
(2) $i_0(\Gamma) = n - 3$ $(r(\Gamma) = 3)$ *if and only if Γ is the cycle C_3.*

Let $H^1_{n,3}$ be an unweighted unicyclic graph obtained by joining a vertex of C_3 and the center of $K_{1,n-4}$, the star of order $n-3$.

Theorem 5.4. *Let G_w be a weighted unicyclic graph of order $n \geqslant 5$. Then $i_0(G_w) = n-5$ ($r(G_w) = 5$) if and only if G_w is the weighted graph with C_5, or $H^1_{n,3}$ as the underlying graph.*

Proof. It is obvious that the sufficiency holds by Lemmas 3.2 and 2.6.

Necessity: If G_w is a weighted cycle, by Lemma 3.2, G_w is C_5^w with arbitrary weights. Next assume that G_w contains at least one pendant edge. Suppose there exists a vertex $v_i \in V(C_k^w)$ which is saturated in $G_w\{v_i\}$. Then $i_0(G_w) = n - 2m(G_w\{v_i\}) - 2m(G_w - G_w\{v_i\})$. So $2m(G_w\{v_i\}) + 2m(G_w - G_w\{v_i\}) = 5$ which is a contradiction.

Assume that for any vertex $v \in V(C_k^w)$ it is not saturated in $G_w\{v\}$. By Lemma 3.2, we have $i_0(G_w) = n - k - 2m(G_w - C_k^w) + i_0(C_k^w)$. Hence

$$i_0\big(C_k^w\big) = k + 2m\big(G_w - C_k^w\big) - 5. \qquad (*)$$

Note that $i_0(C_k^w) = 0$, or 2. If $i_0(C_k^w) = 0$, then $k = 3$ and $m(G_w - C_k^w) = 1$ which implies $G_w - C_k^w$ is a star. Hence the underlying graph G of G_w is isomorphism to $H^1_{n,3}$. If $i_0(C_k^w) = 2$, we have $k \leqslant 5$. Then $k = 4$ by Lemma 3.2. From $(*)$, $2m(G_w - C_k^w) = 3$ which is a contradiction. This case cannot hold. □

The next results follow from Theorem 5.4.

Corollary 5.5. *(See [18].) Let G be a unicyclic graph of order n. Then $i_0(G) = n-5$ ($r(G) = 5$) if and only if G is the cycle C_5 or $H^1_{n,3}$.*

Corollary 5.6. *(See [13].) Let Γ be a unicyclic signed graph of order n. Then $i_0(\Gamma) = n-5$ ($r(\Gamma) = 5$) if and only if Γ is the following graphs: the cycle C_5; the signed graph with $H^1_{n,3}$ as the underlying graph.*

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