Spectral radius of graphs with given matching number

Lihua Feng\textsuperscript{a,b,}\textsuperscript{*}, Guihai Yu\textsuperscript{a}, Xiao-Dong Zhang\textsuperscript{b}

\textsuperscript{a} School of Mathematics, Shandong Institute of Business and Technology, 191 Binhaizhong Road, Yantai, Shandong 264005, PR China
\textsuperscript{b} Department of Mathematics, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, PR China

Received 25 May 2006; accepted 22 September 2006
Available online 7 November 2006
Submitted by R.A. Brualdi

Abstract

In this paper, we show that of all graphs of order \( n \) with matching number \( \beta \), the graphs with maximal spectral radius are \( K_2 \) if \( n = 2\beta \) or \( 2\beta + 1 \); \( K_{2\beta+1} \cup K_{n-2\beta-1} \) if \( 2\beta + 2 \leq n < 3\beta + 2 \); \( K_{\beta} \cap K_{n-\beta} \) or \( K_{2\beta+1} \cup K_{n-2\beta-1} \) if \( n = 3\beta + 2 \); \( K_{\beta} \cap K_{n-\beta} \) if \( n > 3\beta + 2 \), where \( K_t \) is the empty graph on \( t \) vertices.

© 2006 Elsevier Inc. All rights reserved.

AMS classification: 05C35; 05C50

Keywords: Graph; Matching number; Spectral radius

1. Introduction

Let \( G \) be a simple graph with vertex set \( V(G) \) and edge set \( E(G) \). The adjacency matrix of \( G \) is \( A(G) = (a_{ij}) \), where \( a_{ij} = 1 \) if \( ij \) is an edge of \( G \) and 0 otherwise. The eigenvalues of \( G \) are the eigenvalues of its adjacency matrix \( A(G) \). The largest eigenvalue of \( A(G) \) is called the spectral radius of \( G \) and denoted by \( \rho(G) \). It is well known that there is a unique positive eigenvector corresponding to \( \rho(G) \) whose entries sum to 1 if \( G \) is connected, we call this vector the Perron

\* Supported by National Natural Science Foundation of China (Nos. 10371075 and 10531070).
\* Corresponding author. Address: School of Mathematics, Shandong Institute of Business and Technology, 191 Binhaizhong Road, Yantai, Shandong 264005, PR China.
E-mail addresses: lihuafeng@sjtu.edu.cn (L. Feng), yuguihai@126.com (G. Yu), xiaodong@sjtu.edu.cn (X.-D. Zhang).
vector. Note that if we add edges to $G$, the spectral radius of $G$ increases strictly. We refer the readers to [5] for more details in spectral graph theory.

Two distinct edges in a graph $G$ are independent if they are not incident with a common vertex in $G$. A set of pairwise independent edges in $G$ is called a matching in $G$. A matching of maximum cardinality is a maximum matching in $G$. The matching number $\beta(G)$ (or just $\beta$, for short) of $G$ is the cardinality of a maximum matching of $G$. It is well known that $\beta(G) \leq \frac{n}{2}$ with equality if and only if $G$ has a perfect matching. Given a vertex subset $S$ of $G$, the subgraph induced by $S$ is denoted by $G[S]$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union $G_1 \cup G_2$ is defined to be $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join $G_1 \vee G_2$ of $G_1$ and $G_2$ is obtained from $G_1 \cup G_2$ by joining edges from each vertex of $G_1$ to each vertex of $G_2$. The components of a graph $G$ are its maximal connected subgraphs. Components of odd (even) order are called the odd (even) components. We denoted by $\overline{K}_t$ the graph on $t$ vertices with no edges. For other notation in graph theory, we follow from [3].

Brualdi and Solheid [4] proposed the following problem concerning the spectral radius of graphs: Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radius in this set and characterize the graphs in which the maximal spectral radius is attained. If $\mathcal{G}$ is the set of all connected graphs on $n$ vertices with $k$ cut vertices, Berman and Zhang [2] solved this problem. Liu et al. [9] studied this problem for $\mathcal{G}$ to be the set of all connected graphs on $n$ vertices with $k$ cut edges. Wu et al. [12] studied this problem for $\mathcal{G}$ to be the set of trees on $n$ vertices with $k$ pendant vertices. Moreover, if $\mathcal{G}$ is the set of trees on $n$ vertices with matching number $\beta$, there are also some known results, see for example [6–8,11] and the related references therein.

In this paper, we consider this problem when $\mathcal{G}$ is the set $\mathcal{G}_{n,\beta}$ of graphs on $n$ vertices with matching number $\beta$. The main result of this paper is the following:

**Theorem 1.1.** Let $\mathcal{G}_{n,\beta}$ be the set of graphs on $n$ vertices with matching number $\beta$. For any $G \in \mathcal{G}_{n,\beta}$, we have

1. If $n = 2\beta$ or $2\beta + 1$, then $\rho(G) \leq \rho(K_n)$ with equality if and only if $G = K_n$.
2. If $2\beta + 2 \leq n < 3\beta + 2$, then $\rho(G) \leq 2\beta$ with equality if and only if $G = K_{2\beta+1} \cup \overline{K}_{n-2\beta-1}$.
3. If $n = 3\beta + 2$, then $\rho(G) \leq 2\beta$ with equality if and only if $G = K_{3\beta+2} \cup \overline{K}_{n-3\beta-1}$.
4. If $n > 3\beta + 2$, then $\rho(G) \leq \frac{1}{2}(\beta - 1 + \sqrt{(\beta - 1)^2 + 4\beta(n - \beta)})$ with equality if and only if $G = K_{3\beta+2} \cup \overline{K}_{n-3\beta-1}$.

**2. Proof of Theorem 1.1**

Before the proof of the main result, we need some technical lemmas.

**Lemma 2.1** ([11] (also see [10]) (The Berge Formula)). Suppose $G$ is a graph on $n$ vertices with matching number $\beta$. Then there exists a set $S$ on $s$ vertices in $G$ such that $G - S$ has $q = n + s - 2\beta$ odd components.

**Lemma 2.2.** If $G$ be a graph with the maximal spectral radius in $\mathcal{G}_{n,\beta}$, then there exist positive odd numbers $n_1, \ldots, n_q$ such that
Proof. By Lemma 2.1, there exists a subset $S$ on $s$ vertices in $G$ such that $G - S$ has $q = n + s - 2\beta$ odd components. Let $G_1, G_2, \ldots, G_q$ be the odd components in $G - S$ with $|V(G_i)| = n_i \geq 1$ for $i = 1, \ldots, q$. Moreover, we may assume that $n_1 \leq n_2 \leq \cdots \leq n_q$. Clearly, $n \geq s + q = n + 2s - 2\beta$. Thus $s \leq \beta$.

If $G - S$ has no even components, then \( \bigcup_{i=1}^{q} V(G_i) = V(G) - S \). Further, since $G$ has the maximal spectral radius, we claim that (1) $G_i, 1 \leq i \leq q$ and $G[S]$ are complete graphs and (2) $G$ contains all edges joining each vertex in $S$ and each vertex in $G_i, i = 1, \ldots, q$. In fact, if either (1) or (2) does not hold, let $\tilde{G}$ be the graph obtained from $G$ by adding edges such that both (1) and (2) hold, then $\rho(\tilde{G}) > \rho(G)$. On the other hand, $\tilde{G}$ is still a graph on $n$ vertices with the matching number $\beta$. This contradicts to the fact that $G$ has the maximal spectral radius in $\mathcal{G}_n,\beta$.

If $G - S$ contains even components, let $C$ be the union of these even components. Then we add some edges to make $G \cup C$ to be a complete graph. In this way, we get a new graph $\tilde{G}$ and $\rho(G) \leq \rho(\tilde{G})$. On the other hand, it is easy to see that $\tilde{G}$ is still a graph on $n$ vertices with the matching number $\beta$. It is a contradiction. □

Lemma 2.3. If $G^*$ be a graph in $\mathcal{G}_n,\beta$ with the maximal spectral radius, then there exists a nonnegative number $q$ such that

\[
G^* = K_s \vee \left( K_{n_q} \bigcup K_{q-1} \right), \quad q = n + s - 2\beta, \quad n_q = 2\beta - 2s + 1.
\]

Proof. It follows from Lemma 2.2 that we need only to prove that $n_1 = n_2 = \cdots = n_{q-1} = 1, n_q = 2\beta - 2s + 1$. Let $x$ be the Perron vector of $A(G)$ corresponding to $\rho(G) = \rho$. From the symmetry of $G$, we can assume the eigencomponents of $x$ corresponding to the vertices in $K_{n_i}$ are $x_i, 1 \leq i \leq q$, the eigencomponents of $x$ corresponding to the vertices in $K_s$ are $y$. Then we have

\[
\rho x_1 = (n_1 - 1)x_1 + sy,
\rho x_2 = (n_2 - 1)x_2 + sy,
\ldots
\rho x_q = (n_q - 1)x_q + sy,
\rho y = (s - 1)y + \sum_{i=1}^{q} n_i x_i.
\]

Hence $\rho(G)$ satisfies the following equation:

\[
\lambda - s + 1 - \sum_{i=1}^{q} \frac{n_i s}{\lambda - n_i + 1} = 0,
\]

where $n_1 \leq n_2 \leq \cdots \leq n_q$ are odd integers.
Next we claim that if \( n_q - 1 \geq 3 \), then \( \rho(G) < \rho(G_0) \), where

\[
G_0 = K_s \bigcup \left( K_{n_q + 2} \setminus \bigcup_{i=1}^{q-2} K_{n_i} \right).
\]

Consider the following function:

\[
f(\delta, \lambda) = \frac{\lambda - s + 1}{s} - \sum_{i=1}^{q-2} \frac{n_i}{\lambda - n_i + 1} - \frac{n - n_q - \delta}{\lambda - (n_q - 1 - \delta) + 1} - \frac{n_q + \delta}{\lambda - (n_q + 1 + \delta) + 1},
\]

where \( \lambda \geq n_q - 1, 0 \leq \delta \leq 2 \). It is obvious that \( f(0, \rho(G)) = 0 \).

Taking derivative with respect to \( \delta \), we have, for \( \lambda \geq n_q - 1 \),

\[
\frac{df(\delta, \lambda)}{d\delta} = \frac{\lambda + 1}{(\lambda - n_q - 1 + 1 + \delta)^2} - \frac{\lambda + 1}{(\lambda - n_q + 1 - \delta)^2} = \frac{(\lambda - n_q - 1 + 1 + \delta)^2 \lambda - n_q + 1 - \delta^2}{(\lambda - n_q - 1 + 1 + \delta)^2 (\lambda - n_q + 1 - \delta)^2} < 0.
\]

Hence, \( f(\delta, \lambda) \) is strictly decreasing with respect to \( \delta \) for \( \lambda \geq n_q - 1 \). Note that \( \rho(G), \rho(G_0) \) are both at least \( n_q - 1 \). So we have \( f(2, \rho(G)) < 0 = f(0, \rho(G)) \).

For \( f(2, \lambda), \lambda \geq n_q - 1 \), taking derivative with respect to \( \lambda \), we have

\[
\frac{df(2, \lambda)}{d\lambda} = \frac{1}{s} + \sum_{i=1}^{q-2} \frac{n_i}{(\lambda - n_i + 1)^2} + \frac{n_q - 2}{(\lambda - n_q + 3)^2} + \frac{n_q + 2}{(\lambda - n_q - 1)^2} > 0.
\]

Hence \( f(2, \lambda) \) is increasing in \( \lambda \geq n_q - 1 \).

Since \( \rho(G_0) \) is the largest root of equation \( f(2, \lambda) = 0 \), so \( f(2, \rho(G)) < 0 = f(2, \rho(G_0)) \). Thus \( \rho(G) < \rho(G_0) \), that is, if we increase \( n_q \) by 2 while decrease \( n_q - 1 \) by 2, the spectral radius will increase, as claimed. Note that the resulting graph still has matching number \( \beta \).

By repeating the above procedure, the proof of Lemma 2.3 is completed. \( \square \)

Now we may present the proof of Theorem 1.1.

**Proof of Theorem 1.1.** (1) It is easy to see that \( \rho(G) \leq \rho(K_n) = n - 1 \) if \( n = 2\beta + 1 \) with equality if and only if \( G = K_n \).

In the remaining cases, we always assume that \( n \geq 2\beta + 2 \).

Note that from the proof of Lemma 2.2, \( 0 \leq s \leq \beta \). From Eq. (1) in Lemma 2.3, we know \( \rho(G^*) \) satisfies \( f(\lambda) = 0 \), where

\[
f(\lambda) = \lambda(\lambda - s + 1)(\lambda - 2\beta + 2s) - s(2\beta - 2s + 1)\lambda - s(n - 2\beta + s - 1)(\lambda - 2\beta + 2s)
\]

\[
= \lambda^3 - (2\beta - s - 1)\lambda^2 + (2\beta s - s^2 - 2\beta + 2s + sn)\lambda
\]

\[
+ 2s(\beta - s)(n + s - 2\beta + 1).
\]

It is easy to see that

\[
f(-1) = s(2\beta - 2s + 1)(n - 2\beta + s - 1) \geq 0;
\]

\[
f(0) = s(2\beta - 2s)(n - 2\beta + s - 1) \geq 0;\]
\[
f(2\beta - s) = -s^2(n + s - 2\beta + 1) \leq 0;
\]
\[
f(-\infty) < 0;
\]
\[
f(+\infty) > 0.
\]
Hence, if \( s = 0 \), the three roots of \( f(\lambda) \) are just \(-1, 0, 2\beta\). If \( s \neq 0 \), then \( 0 < s \leq \beta \) and the three roots of \( f(\lambda) \) lie in three intervals \((-\infty, -1), (0, 2\beta - s), (2\beta - s, +\infty)\). So we conclude that \( f(\lambda) \) has exactly one root \( \geq 2\beta - s \).

(2) If \( n > 3\beta + 2 \), by Lemma 2.3, we need just to verify that \( \rho(G^*) \leq \rho(H) \), where \( H = K_\beta \sqrt{K_{n-\beta}} \). A direct computation shows that \( \rho(H) \) satisfies \( g(\lambda) = 0 \), where
\[
g(\lambda) = \lambda^2 - (\beta - 1)\lambda - \beta(n - \beta).
\]
Note that if \( n > 3\beta + 2 \), then
\[
\rho(H) = \frac{1}{2} \left( \beta - 1 + \sqrt{\beta^2 - 4\beta(n - \beta)} \right) > 2\beta.
\]
After somewhat tedious computation, we get
\[
f(\lambda) = g(\lambda)(\lambda - \beta + s) + (\beta - s)((n + s - 2\beta - 1)(\lambda + 2s) - \beta(n - \beta)).
\]
Hence
\[
f(\rho(H)) = (\beta - s)((n + s - 2\beta - 1)(\rho(H) + 2s) - \beta(n - \beta))
\]
\[
\quad \geq (\beta - s)((n + s - 2\beta - 1)(2\beta + 2s) - \beta(n - \beta))
\]
\[
\quad \geq (\beta - s)((3\beta + 2 + s - 2\beta - 1)(2\beta + 2s) - \beta(3\beta + 2 - \beta))
\]
\[
\quad = 2(\beta - s)((\beta + s)(\beta + s + 1) - \beta(\beta + 1))
\]
\[
\quad \geq 0.
\]
The second inequality holds since \( n > 3\beta + 2 \). Hence, \( f(\rho(H)) \geq 0 \), this means \( \rho(G^*) \leq \rho(H) \), so we get this case. If \( \rho(G^*) = \rho(H) \), or \( f(\rho(H)) = 0 \), then \( s = \beta \). From Lemma 2.3, we have \( G^* = H \).

(3) If \( 2\beta + 2 \leq n < 3\beta + 2 \), note that in this case, \( \rho(H) < \rho(K_{2\beta+1} \cup K_{n-2\beta-1}) = 2\beta \), where \( H = K_\beta \sqrt{K_{n-\beta}} \). In this case,
\[
f(\lambda) = (\lambda - 2\beta) \left[ \lambda^2 + (s + 1)\lambda + (4\beta s - s^2 + 2s - sn) \right]
\]
\[
\quad + 2s \left[ 2\beta^2 + \beta - s(n + s - 2\beta - 1) \right].
\]
So we have
\[
f(2\beta) = 2s[2\beta^2 + \beta - s(n + s - 2\beta - 1)]
\]
\[
\quad \geq 2s[2\beta^2 + \beta - s(\beta + s + 1)] \quad \text{since} \quad n < 3\beta + 2
\]
\[
\quad = 2s(2\beta^2 - s\beta - s^2 + \beta - s)
\]
\[
\quad = 2s(\beta - s)(2\beta + s + 1)
\]
\[
\quad \geq 0.
\]
This means that \( \rho(G^*) \leq 2\beta \). If \( \rho(G^*) = 2\beta \), or \( f(2\beta) = 0 \), then from the above proof, we must have \( s = 0 \). By Lemma 2.3 again, we have \( G^* = K_{2\beta+1} \cup K_{n-2\beta-1} \).
(4) If $n = 3\beta + 2$, by modifying the above proofs, we can get that

$$f(2\beta) = 2s(\beta - s)(2\beta + s + 1) \geq 0.$$ 

Thus, $\rho(G^*) \leq 2\beta$. If $\rho(G^*) = 2\beta$, then $s = 0$ or $s = \beta$, which implies the result. □

Acknowledgments

The authors are grateful to the referee for his valuable comments, corrections and suggestions which lead to a great improvement of this paper. This work was supported by National Natural Science Foundation of China (Grant Nos. 10371075 and 10531070) and National Natural Science Foundation of Shanghai (Grant No. 06ZR14049).

References