Sharp Bounds for the Signless Laplacian Spectral Radius in Terms of Clique Number*

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Dedicated to Professors Abraham Berman, Moshe Goldberg, and Raphael Loewy in recognition of their important contributions to linear algebra and the linear algebra community.

Abstract

In this paper, we present a sharp upper and lower bounds for the signless Laplacian spectral radius of graphs in terms of clique number. Moreover, the extremal graphs which attain the upper and lower bounds are characterized. In addition, these results disprove the two conjectures on the signless Laplacian spectral radius in [P. Hansen and C. Lucas, Bounds and conjectures for the signless Laplacian index of graphs, Linear Algebra Appl., 432(2010) 3319-3336].

Key words: Signless Laplacian spectral radius; clique number, Turán graph.
MSC: 05C50, 05C35

1 Introduction

Throughout this paper, we only consider simple and undirected graphs. Let $G = (V(G),\ E(G))$ be a simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set

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$E(G)$. Let $A(G) = (a_{ij})$ be the $(0, 1)$ *adjacency matrix* of G with $a_{ij} = 1$ for v_i adjacent to v_j and 0 otherwise. Moreover, let $D(G) = diag(d(u), u \in V)$ be the diagonal matrix of vertex degrees $d(u)$ of G. Then $Q(G) = D(G) + A(G)$ is called *the signless Laplacian matrix* of G. The signless Laplacian matrix $Q(G)$ can be viewed as an operator on the space of functions $f : V(G) \to \mathcal{R}$ which satisfies

$$Q(G)f(u) = \sum_{v \sim u} (f(u) + f(v)),$$

where " $\sim$ " stands for the adjacency relation. The largest eigenvalue of $Q(G)$ is called the *signless Laplacian spectral radius* of G and denoted by $q_1(G)$, or for short q_1.

Recently, the signless Laplacian matrix of a graph has received increasing attention. For example, Desai and Rao [5] used the smallest eigenvalue of the signless Laplacian matrix of a connected graph to serve as a measure of how much a graph is close to bipartite, since 0 is the smallest eigenvalue of $Q(G)$ if and only if G is bipartite. Liu and Liu [9] presented lower and/or upper bounds for the clique number and the independence number in terms of the signless Laplacian eigenvalues. Oliveira et. al. [16] gave several upper and lower bounds for the signless Laplacian spectral radius. Zhang [20, 21] investigated the largest signless Laplacian spectral radius for a given degree graphic sequence. Cvetović and Simić [2]-[4] surveyed spectral graph theory based on the signless Laplacian matrix of a graph. Recently, Hansen and Lucas [8] proposed some conjectures on the signless Laplacian spectral radius.

Conjecture 1.1 *([8]) Let G be a connected graph on $n \geq 4$ vertices with signless Laplacian spectral radius q_1 and clique number ω. Then*

$$q_1 - \omega \leq \frac{3}{2}n - 4, \text{if } n \text{ is even}, \tag{1}$$

$$\frac{q_1}{\omega} \leq \frac{n}{2}. \tag{2}$$

The bound for (1) is attained by and only by the complement of a perfect matching when $n \geq 6$ is even. Moreover, when $n \geq 9$ is odd, $q_1 - \omega$ is maximum for and only for the complement of a perfect matching on $n - 3$ vertices and a triangle on three remaining vertices. The bound for (2) is attained by and only by the complete bipartite graph $K_{p,q}$.

Conjecture 1.2 *([8]) Let G be a connected graph on $n \geq 6$ vertices with signless Laplacian spectral radius q_1 and chromatic number χ. Then*

$$q_1 - \chi \leq \frac{3}{2}n - 4, \quad \text{if } n \text{ is even}. \tag{3}$$

The bound is attained by and only by the complement of a perfect matching when n is even. Moreover, when $n \geq 9$ is odd, $q_1 - \chi$ is maximum for and only for the complement of a perfect matching on $n-3$ vertices and a triangle on the three remaining vertices.

Hansen and Lucas [7] obtained some results on signless Laplacian related to clique number and chromatic number. For other results, see [2]-[4] and the references therein.

On the other hand, there are many *Turán-type extremal problems,* i.e., given a forbidden graph H, determine the maximal number of edges in a graph on n vertices that does not contain a copy of H. It states that among n-vertex graphs not containing a clique of size $t+1$, the complete t-partite graph $T_{n,t}$ with (almost) equal parts, which is called *Turán graph,* has the maximum number of edges. Spectral graph theory has similar Turán extremal problems which determine the largest (or smallest) eigenvalue of a graph not containing a subgraph H.

Nikiforov [10] proved a spectral extremal Turán theorem: let $\lambda(G)$ be the largest eigenvalues of the adjacency matrix of G not containing complete graph K_t of order t as a subgraph, then $\lambda(G) \leq \lambda(T_{n,t-1})$ with equality if and only if $G = T_{n,t-1}$; the same result has been proved previously by Guiduli [6], but his proof was made public only after the publication of [10]. Further, Nikiforov explicitly advocated the study of general Turán problems in many publications (see [10]-[15]). For instance, he determined [13] the maximum spectral radius of graphs without paths of given length and presented [15] a comprehensive survey on these topics. In addition, Sudakov et al. [17] presented a generalization of Turán Theorem in terms of Laplacian eigenvalues.

Motivated by these conjectures and Turán-type extremal problems, we investigate in this paper the extremal graphs with maximal or minimal signless Laplacian spectral radius among all graphs of order n with given clique number, which may be regarded as a part of spectral extremal theory. The main result of this paper reads:

Theorem 1.3 *Let G be a connected graph of order n with clique number $\omega \geq 2$. Then*

$$q_1(G) \leq \frac{(3\omega-4)k + 3r - 2 + \sqrt{k^2\omega^2 + [(2r+4)\omega - 8r]k + (r-2)^2}}{2}, \tag{4}$$

where $n = k\omega + r, 0 \leq r < \omega$. Moreover, equality holds in (4) if and only if G is complete bipartite graph for $\omega = 2$ and Turán graph $T_{n,\omega}$ for $\omega \geq 3$.

2 Proof of Theorem 1.3

In order to prove Theorem 1.3, we need more notation and preliminary results. Two vertices in a graph G are called *duplicate* if they have precisely the same neighbors. In the operation of *duplication of a vertex u to a vertex v,* all edges incident to vertex u are deleted and then all edges between u and vertices adjacent to v are added. Clearly, after this duplication, u and v are duplicate, and duplication in a graph not containing K_{t+1} preserves this property. The *complement* $\overline{G}$ of a simple graph $G = (V(G), E(G))$ is the simple graph with vertex set $V(G)$, two vertices being adjacent in $\overline{G}$ if and only if they are not adjacent in G. For two disjoint graphs G and H, the *union* $G + H$ of G and H is the graph with vertex set $V(G)\bigcup V(H)$ and edge set $E(G)\bigcup E(H)$; the *join* $G \bigtriangledown H$ of G and H is the graph obtained from the union $G + H$ by joining each vertex of G to each vertex of H.

Let f be a nonnegative function on vertex set $V(G)$ of G. Then f is also regarded as a nonnegative vector corresponding to vertex set $V(G)$. For any vertex u, the *weight* $w_G(u)$ of vertex u with respect to f in G is defined as

$$w_G(u) = Q(G)f(u) = \sum_{v\sim u}(f(u) + f(v)) \tag{5}$$

if u is not isolated vertex and 0 otherwise, where " $\sim$ " stands for adjacency relation. We adapt a variation of Zykov's proof of Turán's theorem, which was also used in [6].

Lemma 2.1 *Let G be a simple graph of order n not containing K_{t+1}. Then there exists a simple graph G_1 such that $G_1 = \overline{K_p} \bigtriangledown H$ and $q_1(G) \le q_1(G_1)$, where H is a simple graph not containing K_t and $1 \le p \le n-1$. Moreover, equality holds if and only if $G = G_1$.*

Proof. Without loss of generality, we assume that G is connected, since $q_1(G)$ is an increasing function with respect to adding edges. Let f be the unit positive eigenvector on $V(G)$ corresponding to the eigenvalue $q_1(G)$ of G such that $q_1(G) = \langle f,\ Q(G)f\rangle$ and $\langle f,\ f\rangle = 1$. Let $w_G(u) = \max\{w_G(v):\ v \in V(G)\}$. Denote by V_1 the set of all neighbors of vertex u and $V_2 = V(G)\backslash(V_1\bigcup\{u\})$ with $|V_2| = p-1$. Now we construct a new graph G_1 obtained from G by a duplication of each vertex $v \in V_2$ to vertex u. Then G_1 still does not contain K_{t+1} and the induced subgraph $G[V_1]$ does not contain K_t. Hence G_1 can be written as $G_1 = \overline{K_p} \bigtriangledown H$, where $H = G[V_1]$ does not contain K_t.

On the other hand, for any $v \in V_1$, any vertex adjacent to v in G must be adjacent to v in G_1. Then

$$w_G(v) = \sum_{xv \in E(G)} (f(v) + f(x)) \leq \sum_{xv \in E(G_1)} (f(v) + f(x)) = w_{G_1}(v) \tag{6}$$

with equality holding if and only if v is adjacent to each vertex in V_2 in G. For any vertex $v \in V_2$,

$$q_1(G)f(u) = Q(G)f(u) = w_G(u) \geq w_G(v) = Q(G)f(v) = q_1(G)f(v). \tag{7}$$

Hence $f(u) \geq f(v)$. In addition,

$$w_G(u) = q_1(G)f(u) = \sum_{xu \in E(G)} (f(u) + f(x)) = d_G(u)f(u) + \sum_{x \in V_1} f(x)$$

and $q_1(G) > d_G(v)$ for any $v \in V(G)$ (see [16]). Therefore, for any $v \in V_2$,

$$\begin{aligned} w_{G_1}(v) &= \sum_{xv \in E(G_1)} (f(v) + f(x)) = d_{G_1}(v)f(v) + \sum_{x \in V_1} f(x) \\ &= d_G(u)f(v) + w_G(u) - d_G(u)f(u) \\ &= w_G(v) + (w_G(u) - w_G(v)) - d_G(u)(f(u) - f(v)) \\ &= w_G(v) + (q_1(G) - d_G(v))(f(u) - f(v)) \\ &\geq w_G(v). \end{aligned}$$

Moreover, $w_{G_1}(u) = w_G(u)$. Then

$$\langle f, Q(G_1)f \rangle = \sum_{v \in V(G_1)} f(v)w_{G_1}(v) \geq \sum_{v \in V(G)} f(v)w_G(v) = q_1(G).$$

Hence by Rayleigh quotient, $q_1(G_1) \geq q_1(G)$ with equality holding if and only if $w_{G_1}(v) = w_G(v)$ for all $v \in V(G) = V(G_1)$, which implies $G = G_1$. ■

Lemma 2.2 *Let G be a simple graph not containing K_{t+1} and R be another simple graph. Then there exists a simple graph G_1 such that $G_1 = \overline{K_p} \bigtriangledown H$ and $q_1(R \bigtriangledown G) \leq q_1(R \bigtriangledown G_1) = q_1(R \bigtriangledown \overline{K_p} \bigtriangledown H)$, where H is a simple graph not containing K_t and $1 \leq p \leq n-1$.*

Proof. Let f be unit positive eigenvector of $Q(R \bigtriangledown G)$ corresponding to $q_1(R \bigtriangledown G)$. Let

$$w_{R \bigtriangledown G}(v) \equiv Q(R \bigtriangledown G)f(v) = \sum_{xv \in E(R \bigtriangledown G)} (f(v) + f(x)) = q_1(R \bigtriangledown G)f(v)$$

for any $v \in V(R \bigtriangledown G)$, and let $w_{R\bigtriangledown G}(u) = \max\{w_{R\bigtriangledown G}(v) : \ v \in V(G)\}$. Therefore, as in (7),

$$q_1(R \bigtriangledown G)f(u) = \sum_{xu \in E(R\bigtriangledown G)} (f(u) + f(x)) = w_{R\bigtriangledown G}(u) \geq w_{R\bigtriangledown G}(v) = q_1(R \bigtriangledown G)f(v)$$

which implies $f(u) \geq f(v)$ for any $v \in V(G)$. Denote by V_1 the set of all neighbors of vertex u in $V(G)$ and $V_2 = V(G) \backslash (V_1 \bigcup \{u\})$ with $|V_2| = p - 1$. Now we construct a new graph G_1 obtained from G by a duplicatation of each vertex $v \in V_2$ to vertex u. Then G_1 still does not contain K_{t+1} and the induced subgraph $G[V_1]$ does not contain K_t. Hence G_1 can be written as $G_1 = \overline{K_p} \bigtriangledown H$, where $H = G[V_1]$ does not contain K_t.

Clearly, for any vertex $v \in V(R)$, $w_{R\bigtriangledown G_1}(v) = w_{R\bigtriangledown G}(v)$. For any vertex $v \in V_1$, any neighbor of v in $R \bigtriangledown G$ is also its neighbor in $R \bigtriangledown G_1$. So for any $v \in V_1$, $w_{R\bigtriangledown G_1}(v) \geq w_{R\bigtriangledown G}(v)$ with equality if and only if v is adjacent to each vertex in V_2 in G. For any vertex $v \in V_2$, by simple calculations, as in the previous proof, we have

$$\begin{aligned}
w_{R\bigtriangledown G_1}(v) &= \sum_{xv \in E(G_1)} (f(v) + f(x)) + \sum_{x \in V(R)} (f(v) + f(x)) \\
&= \sum_{x \in V_1} ((f(u) + f(x)) - (f(u) - f(v))) + \sum_{x \in V(R)} ((f(u) + f(x)) - (f(u) - f(v))) \\
&= \sum_{x \in V_1} (f(u) + f(x)) + \sum_{x \in V(R)} (f(u) + f(x)) - (|V_1| + |V(R)|)(f(u) - f(v)) \\
&= w_{R\bigtriangledown G}(u) - (|V(R)| + d_G(u))(f(u) - f(v)) \\
&= w_{R\bigtriangledown G}(v) + (w_{R\bigtriangledown G}(u) - w_{R\bigtriangledown G}(v)) - (|V(R)| + d_G(u))(f(u) - f(v)) \\
&= w_{R\bigtriangledown G}(v) + (q_1(R \bigtriangledown G)f(u) - q_1(R \bigtriangledown G)f(v)) - (|V(R)| + d_G(u))(f(u) - f(v)) \\
&= w_{R\bigtriangledown G}(v) + (q_1(R \bigtriangledown G) - (|V(R)| + d_G(u)))(f(u) - f(v)) \\
&\geq w_{R\bigtriangledown G}(v),
\end{aligned}$$

since $q_1(R\bigtriangledown G) > |V(R)| + d_G(u)$ by Perron-Frobenius theorem. Moreover, $w_{R\bigtriangledown G_1}(u) = w_{R\bigtriangledown G}(u)$. Therefore

$$\begin{aligned}
q_1(R \bigtriangledown G) &= \langle f, \ Q(R \bigtriangledown G)f \rangle = \sum_{v \in V(R\bigtriangledown G)} f(v) w_{R\bigtriangledown G}(v) \\
&\leq \sum_{v \in V(R\bigtriangledown G_1)} f(v) w_{R\bigtriangledown G_1}(v) = \langle f, \ Q(R \bigtriangledown G_1)f \rangle \\
&\leq q_1(R \bigtriangledown G_1).
\end{aligned}$$

Moreover, equality holds if and only if f is an eigenvector of $Q(R \triangledown G_1)$, which implies $w_{R\triangledown G}(v) = w_{R\triangledown G_1}(v)$ for all $v \in V(R \triangledown G)$. So equality holds if and only if $G = G_1$. Hence the assertion holds. ■

The following lemma on the signless Laplacian spectral radius of a complete $t-$partite graph is well-known (see, for example, [1] or [19]).

Lemma 2.3 *([1],[19]) Let G be a complete $t-$partite graph on n vertices. If $n = tk + r, 0 \leq r < t$, then $q_1(G) = n$ for $t = 2$, and*

$$q_1(G) \leq \frac{(3t-4)k + 3r - 2 + \sqrt{t^2k^2 + [(2r+4)t - 8r]k + (r-2)^2}}{2} \tag{8}$$

for $t \geq 3$. Moreover, equality holds if and only if G is Turán graph $T_{n,t}$ which is the complete t-partite graph on n vertices in which the partite sets are of size k or $k+1$.

Corollary 2.4 *For any $2 \leq t \leq n-1$,*

$$q_1(T_{n,t}) < q_1(T_{n,t+1}).$$

Proof. It follows directly from Lemma 2.3 with a simple calculation. ■

Now we are ready to present a proof of Theorem 1.3.

Proof. Let $\omega(G) = t$ and f be the positive eigenvector of $Q(G)$ corresponding to $q_1(G)$ with $\langle f, f \rangle = 1$. We consider the following two cases.

Case 1: $t = 2$. Then by [3], G is bipartite and $q_1(G) \leq n$ with equality if and only if G is complete bipartite graph. Hence (4) holds with equality if and only if G is complete bipartite graph.

Case 2: $t \geq 3$. Then G does not contain K_{t+1} as a subgraph. By Lemma 2.1, there exists a graph $G_1 = \overline{K_{n_1}} \triangledown H_1$ such that H_1 does not contain K_t and $q_1(G) \leq q_1(G_1)$. Moreover, equality holds if and only if $G = G_1$. Since $\overline{K_{n_1}}$ is a simple graph and H_1 does not contain K_t, by Lemma 2.2, there exists a graph $G_2 = \overline{K_{n_1}} \triangledown \overline{K_{n_2}} \triangledown H_2$ such that H_2 does not contain K_{t-1} and $q_1(G_1) \leq q_2(G_2)$. Moreover, equality holds if and only if $G_2 = G_1$. If H_2 does not contain any edges, then $H_2 = \overline{K_{n_3}}$. Hence $G_2 = \overline{K_{n_1}} \triangledown \overline{K_{n_2}} \triangledown \overline{K_{n_3}}$ is complete 3-partite graph and $q_1(G) \leq q_1(G_1) \leq q_1(G_2) = q_1(\overline{K_{n_1}} \triangledown \overline{K_{n_2}} \triangledown \overline{K_{n_3}})$. If H_2 contains at least one edge, then by Lemma 2.2, there exists a graph G_3 such that $G_3 = \overline{K_{n_1}} \triangledown \overline{K_{n_2}} \triangledown \overline{K_{n_3}} \triangledown H_3$ and $q_1(G_2) \leq q_1(G_3)$ with H_3 not containing K_{t-2}, since H_2 does not contain K_{t-1} and $\overline{K_{n_1}} \triangledown \overline{K_{n_2}}$ is a simple graph. By repeated use of Lemma 2.2, there exists a series of graphs $G_1, \cdots, G_s$ such that

$G_i = \overline{K_{n_1}} \bigtriangledown \overline{K_{n_2}} \bigtriangledown \cdots \bigtriangledown \overline{K_{n_i}} \bigtriangledown H_i$ and $q_1(G) \leq q_1(G_1) \leq \cdots \leq q_1(G_i)$ with equality if and only if $G_i = G$, where H_i does not contain K_{t+1-i} and $i = 1, \cdots, s \leq t$. Moreover, $H_s = \overline{K_{n_s}}$. Therefore, G_s is a complete s-partite graph. Further, by Lemma 2.3 and Corollary 2.4,

$$\begin{aligned} q_1(G) &\leq q_1(G_1) \leq \cdots \leq q_1(G_s) \leq q_1(T_{n,s}) \leq q_1(T_{n,t}) \\ &= \frac{(3t-4)k + 3r - 2 + \sqrt{t^2k^2 + [(2r+4)t - 8r]k + (r-2)^2}}{2} \end{aligned}$$

with equality if and only if $G = G_1 = \cdots = G_s = T_{n,s} = T_{n,t}$. This completes the proof. ■

Remark: From Theorem 1.3, we are able to deduce Turán theorem for $t \geq 3$.

Corollary 2.5 *Let G be a connected graph of order n not containing K_{t+1}. If $t \geq 3$ and $n = kt + r, 0 \leq r < t$, then*

$$|E(G)| \leq |E(T_{n,t})| = \frac{t^2 - t}{2}k^2 + (t-1)rk + \frac{r(r-1)}{2}.$$

Moreover, equality holds if and only if $G = T_{n,t}$.

Proof. Clearly $|E(T_{n,t})| = \frac{t^2-t}{2}k^2 + (t-1)rk + \frac{r(r-1)}{2}$. By Rayleigh's quotient, it is easy to see that $q_1(T_{n,t}) \geq \frac{4|E(T_{n,t})|}{n}$. Then $\frac{nq_1(T_{n,t})}{4} - |E(T_{n,t})| \geq 0$ and

$$\begin{aligned} \varepsilon &\equiv \frac{nq_1(T_{n,t})}{4} - |E(T_{n,t})| \\ &= -\frac{1}{8}\{t^2k^2 + [(2r+2)t - 4r]k + r^2 - 2r - \\ &\quad (tk + r)\sqrt{t^2k^2 + [(2r+4)t - 8r]k + (r-2)^2}\} \\ &\geq 0. \end{aligned}$$

On the other hand, let

$$\varphi(x) \equiv 4x^2 + \{t^2k^2 + [(2r+2)t - 4r]k + r^2 - 2r\}x + (r-t)rk^2 + (r-t)rk.$$

Then $\varphi(x) = 0$ has a root

$$x_1 = -\frac{1}{8}\{t^2k^2 + [(2r+2)t - 4r]k + r^2 - 2r - (tk+r)\sqrt{t^2k^2 + [(2r+4)t - 8r]k + (r-2)^2}\} = \varepsilon,$$

i.e., ε is a nonnegative root of $\varphi(x) = 0$. Further, since

$$\varphi(0) = (r-t)rk^2 + (r-t)rk \leq 0$$

and

$$\varphi(1) = (t^2 - rt + r^2)k^2 + [(r+2)t + r^2 - 4r]k + (r-1)^2 > 0,$$

equation $\varphi(x) = 0$ has only two roots: one is negative and the other is nonnegative which lies in $[0, 1)$. Then $0 \le \varepsilon < 1$, i.e., $\lfloor \varepsilon \rfloor = 0$, where $\lfloor a \rfloor$ is the largest integer not greater than a. Hence $\lfloor \frac{nq_1(T_{n,t})}{4} \rfloor = |E(T_{n,t})|$. Therefore, by Theorem 1.3,

$$|E(G)| \le \lfloor \frac{nq_1(G)}{4} \rfloor \le \lfloor \frac{nq_1(T_{n,t})}{4} \rfloor = |E(T_{n,t})|.$$

So the assertion holds. ■

The following result has been proved in [1] and [19].

Corollary 2.6 *([1], [19]) Let G be a connected simple graph of order n with chromatic number $\chi \ge 3$. If $n = k\chi + r, 0 \le r < \chi$, then*

$$q_1(G) \le \frac{(3\chi - 4)k + 3r - 2 + \sqrt{k^2\chi^2 + [(2r+4)\chi - 8r]k + (r-2)^2}}{2}$$

with equality if and only if G is $T_{n,\chi}$.

Proof. The assertion follows from $\omega(G) \le \chi(G)$ and Theorem 1.3. ■

The following corollary gives some conditions under which Conjectures 1.1 and 1.2 hold, or need not hold.

Corollary 2.7 *Let G be a simple connected graph of order $n \ge 10$ with clique number ω.*

(i). If $\omega \le 4$ or $\omega \ge \lceil \frac{n}{2} \rceil$, then $q_1(G) \le \frac{3n}{2} + \omega - 4$. Moreover, equality holds if and only if G is Turán graph $T_{n,4}$ and $n = 4k$ or Turán graph $T_{n,k}$ and $n = 2k$. In other words, Conjectures 1.1 and 1.2 hold.

(ii). If $5 \le \omega < \lceil \frac{n}{2} \rceil$, then $q_1(T_{n,\omega}) > \frac{3n}{2} + \omega - 4$. In other words, Conjectures 1.1 and 1.2 generally do not hold.

(iii). $\frac{q_1(G)}{\omega} \le \frac{n}{2}$.

Proof. If $\omega = 2$, then $q_1(G) \le n$ and $q_1(G) \le n \le \frac{3n}{2} + 2 - 4$. So (i) holds.

If $\omega = 3$ and $n = 3k + r$ with $0 \le r < 3$, then by (4),

$$\begin{aligned} q_1(G) &\le \frac{5k + 3r - 2 + \sqrt{9k^2 + (12 - 2r)k + (r-2)^2}}{2} \\ &\le \frac{5k + 3r - 2 + 3k + 2}{2} \\ &\le \frac{9k + 3r - 2}{2} = \frac{3n}{2} + \omega - 4. \end{aligned}$$

So (i) holds.

If $\omega = 4$, by (4), we have

$$q_1(G) \leq \frac{8k + 3r - 2 + \sqrt{16k^2 + 16k + (r-2)^2}}{2} \leq \frac{8k + 3r - 2 + (4k+2)}{2} = \frac{3n}{2} + \omega - 4$$

with equality if and only if $n = 4k$. Hence (i) holds.

Now assume that $\omega \geq \lceil \frac{n}{2} \rceil$. If $n = 2\omega$, then by (4),

$$q_1(G) \leq \frac{2(3\omega - 4) - 2 + \sqrt{4\omega^2 + 8\omega + 4}}{2} = 4\omega - 4 = \frac{3n}{2} + \omega - 4.$$

Moreover, equality if and only if $n = 2\omega$. If $n \neq 2\omega$, then $n = \omega + r$ with $r < \omega$ and

$$\begin{aligned} q_1(G) &\leq \frac{(3\omega - 4) + (3r - 2) + \sqrt{\omega^2 + (2r+4)\omega - 8r + (r-2)^2}}{2} \\ &< \frac{(3\omega - 4) + (3r - 2) + 2\omega - 2}{2} \\ &= \frac{3n}{2} + \omega - 4. \end{aligned}$$

Hence (i) holds.

(ii). Suppose that $5 \leq \omega < \lceil \frac{n}{2} \rceil$. Let $n = k\omega + r$, $0 \leq r < \omega$. Then $r \neq 0$ or $k > 2$. Hence by Theorem 1.3 and some calculations, it is easy to verify that

$$\begin{aligned} q_1(T_{n,\omega}) &= \frac{(3\omega - 4)k + 3r - 2 + \sqrt{\omega^2 k^2 + [(2r+4)\omega - 8r]k + (r-2)^2}}{2} \\ &\geq \frac{(3\omega - 4)k + 3r - 2 + (k\omega + 2)}{2} \\ &= \frac{3n}{2} + \frac{(\omega - 4)k}{2} \\ &\geq \frac{3n}{2} + \omega - 4. \end{aligned}$$

Moreover, if $q_1(T_{n,\omega}) = \frac{3n}{2} + \omega - 4$, then $k = 2$ and $r = 0$ which is impossible. So (ii) holds.

(iii). If $\omega = 1$, then $q_1(G) = 0 < \frac{n\omega}{2}$. If $\omega \geq 2$, then by [3], $q_1(G) \leq n = \frac{n\omega}{2}$. If $\omega = 3$, the assertion follows from (1). If $\omega \geq 4$, then by [3], $q_1(G) \leq 2n - 2 < \frac{n\omega}{2}$. Hence (iii) holds. ∎

Remark: In fact, Theorem 1.3 also confirms the following conjecture of Hansen and Lucas [7].

Corollary 2.8 *([7]) Let G be a graph of order n with clique number ω. Then $q_1(G) \le 2n(1-\frac{1}{\omega})$. Moreover, if $\omega \neq 2$, the upper bound is sharp if and only if G is a complete regular ω-partite graph.*

Proof. If $\omega = 1$, then $q_1(G) = 0$ and the assertion holds. Now assume that $\omega \ge 2$ and $n = k\omega + r$ with $0 \le r < \omega$. Then $(r-2)^2 \le (r+2-\frac{4r}{\omega})^2$ with equality if and only if $r = 0$. Hence by (4),

$$\begin{aligned} q_1(G) &\le \frac{(3\omega-4)k+3r-2+\sqrt{k^2\omega^2+[(2r+4)\omega-8r]k+(r-2)^2}}{2} \\ &\le \frac{(3\omega-4)k+3r-2+\sqrt{k^2\omega^2+2k\omega(r+2-\frac{4r}{\omega})+(r+2-\frac{4r}{\omega})^2}}{2} \\ &= \frac{(3\omega-4)k+3r-2+k\omega+r+2-\frac{4r}{\omega}}{2} \\ &= \frac{4k\omega+4r-4k-\frac{4r}{\omega}}{2} \\ &= 2n(1-\frac{1}{\omega}). \end{aligned}$$

So the assertion holds. ■

3 Further Results

In this section, we begin with a lower bound for the signless Laplacian spectral radius of a graph in terms of clique number.

Theorem 3.1 *Let G be a connected graph of order n with clique number ω.*

(i). If $\omega = 2$, then $q_1(G) \ge 2 + 2\cos\frac{\pi}{n}$ with equality if and only if G is a path of order n.

(ii). If $\omega \ge 3$, then

$$q_1(G) \ge q_1(Ki_{n,\omega}), \tag{9}$$

where $Ki_{n,\omega}$ is the kite graph of order n which is obtained by joining one vertex of a complete graph K_ω to one end vertex of a path $P_{n-\omega}$ with a bridge. Moreover, equality holds if and only if G is the kite $Ki_{n,\omega}$.

Proof. If $\omega = 2$, (i) follows from Lemma 1 in [8]. Now assume that $\omega \ge 3$. Since $q_1(G)$ is an strictly increasing function with respect to adding edges for a connected graph, there exists a connected graph G_1 of order n obtained from G by deleting some edges such that $\omega(G_1) = \omega$ and any proper subgraph of G_1 is disconnected or the clique number fewer than ω. Hence $q_1(G) \geq q_1(G_1)$ with equality if and only if $G_1 = G$. By repeated use of Theorem 2.2 in [3], $q_1(G_1) \geq q_1(Ki_{n,\omega})$ with equality if and only if $G_1 = Ki_{n,\omega}$. Hence the assertion holds. ■

Corollary 3.2 *Let G be a connected graph of order n with clique number $\omega \geq 3$. Then*

$$q_1(G) \geq \frac{2\omega - 1 + \sqrt{4\omega^2 - 12\omega + 17}}{2} \tag{10}$$

with equality if and only if $G = Ki_{n,n-1}$ and $\omega = n - 1$.

Proof. By Theorem 3.1, $q_1(G) \geq q_1(Ki_{n,\omega})$. Since the line graph of $Ki_{\omega+1,\omega}$ is the induced subgraph of the line graph of $Ki_{n,\omega}$, it is easy to see that $q_1(Ki_{n,\omega}) \geq q_1(Ki_{\omega+1,\omega})$ with equality if and only if $\omega = n - 1$. Hence by [8], we have

$$q_1(Ki_{n,\omega}) \geq q_1(Ki_{\omega+1,\omega}) = \frac{2\omega - 1 + \sqrt{4\omega^2 - 12\omega + 17}}{2}.$$

Hence the assertion holds. ■

Remark: Oliveira et.al. [16] presented several sharp upper bounds for the signless Laplacian spectral radius of a graph in terms of vertex degrees and 2-average degree. Let G be a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ and $m_i = \frac{\sum_{v_jv_i \in E(G)} d_j}{d_i}$ for $i = 1, \cdots, n$. Part results can be stated as follows:

$$q_1(G) \leq \max_{1 \leq i \leq n} \{ \ d_i + \sqrt{d_i m_i} \ \} \tag{11}$$

and

$$q_1(G) \leq \max_{1 \leq i \leq n} \left\{ \ \frac{d_i + \sqrt{d_i^2 + 8d_i m_i}}{2} \ \right\}. \tag{12}$$

Liu and Liu [9] gave an upper bound in terms of the largest degree and clique number, i.e.,

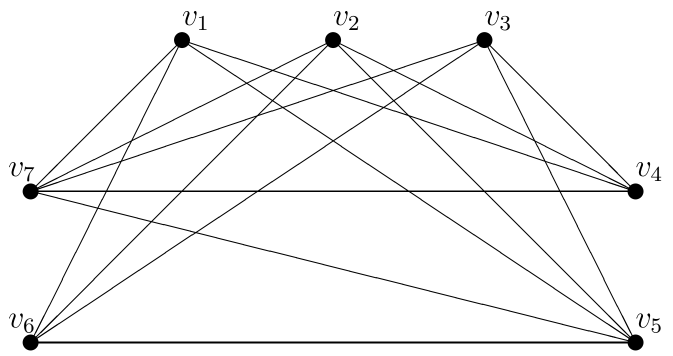
$$q_1(G) \leq n + d_1 - \frac{n}{\omega(G)}. \tag{13}$$

Yu et. al. [18] obtained the following upper bound in terms of degree sequence, i.e.,

$$q_1(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\}. \tag{14}$$

In general, these bounds (4), (11), (12) (13), (14) are not comparable. We present two examples to illustrate that our bounds are best in some cases.

Example 3.3 *Let G_1 be Turán graph $T_{10,3}$ of order 10 and G_2 be a graph of order 7 as follows:*



G_2

Then we have following results

	$q_1(G)$	*(4)*	*(11)*	*(12)*	*(13)*	*(14)*
$T_{10,3}$	*13.2915*	*13.2915*	*13.7082*	*13.6119*	*13.6667*	*13.5826*
G_2	*8.7417*	*9.2749*	*9.5826*	*9.4462*	*9.6667*	*8.8284*

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References

[1] G. X.Cai, Y. Z. Fan, The signless Laplacian spectral radius of graphs with given chromatic number, *Math. Appl. (Wuhan),* **22(1)** (2009) 161-167.

[2] D. Cvetković and S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian, I, *Publ. Inst. Math. (Beograd)(N.S.),* **85**(99)(2009), 19-33.

[3] D. Cvetković and S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian, II, *Linear Algebra Appl.,* 432(2010) 2257-2272.

[4] D. Cvetković and S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian, III, *Appl. Anal. Discrete Math,* **4**(2010), 156-166.

[5] M. Desai and V. Rao, A characterization of the smallest eigenvalue of a graph, *J. Graph Theory,* 18(1994) 181-194.

[6] B. Guiduli, *Spectral extrema for graphs*, Ph.D thesis, University of Chicago, 1998.

[7] P. Hansen and C. Lucas, An inequality for the signless Laplacian index of a graph using the chromatic number, *Graph Theory Notes N. Y.,* **57**(2009), 39-42.

[8] P. Hansen and C. Lucas, Bounds and conjectures for the signless Laplacian index of graphs, *Linear Algebra Appl.,* **432**(2010) 3319-3336.

[9] J.-P. Liu and B.-L. Liu, The maximum clique and the signless Laplacian eigenvalues, *Czechoslovak Math. J.,* 58(133) (2008) 1233-1240.

[10] V. Nikiforov, Bounds on graph eigenvalues II, *Linear Algebra Appl.,* **427**(2007) 183-189.

[11] V. Nikiforov, A spectral Erdös-Stone-Bollobás theorem, *Combin. Probab. Comput.,* **18**(2009) 455-458.

[12] V. Nikiforov, A contribution to the Zarankiewicz problem. *Linear Algebra Appl.,* **432**(2010) 1405-1411.

[13] V. Nikiforov, The spectral radius of graphs without paths and cycles of specified length, *Linear Algebra Appl.,* **432**(2010) 2243-2256.

[14] V. Nikiforov, Turàn's theorem inverted. *Discrete Math.* **310**(2010), 125-131.

[15] V. Nikiforov, Some new results in extremal graph theory, arXiv: 1107.1121v1 [math.CO], 6 July, 2011.

[16] C. S. Oliveira, L. S. de Lima, N. M. M. de Abreu and P. Hansen, Bounds on the index of the signless Laplacian of a graph, *Discrete Appl. Math.,* **158**(2010), 355-360.

[17] B. Sudakov, T. Szabo and V. Vu, A generalization of Turán's theorem, *J. Graph Theory,* 49(2005) 187-195.

[18] G.-L. Yu, Y.-R. Wu and J.-L. Shu, Sharp bounds on the signless Laplacian spectral radii of graphs, *Linear Algebra Appl.,* **434**(2011) 683-687.

[19] G.-L. Yu, Y.-R. Wu and J.-L. Shu, Signless Laplacian spectral radii of graphs with given chromatic number, *Linear Algebra Appl.,* **435**(2011) 1813-1822.

[20] X.-D. Zhang, The Laplacian spectral radii of trees with degree sequences, *Discrete Math.,* 308(2008) 3143-3150.

[21] X.-D. Zhang, The signless Laplacian spectral radius of graphs with given degree sequences.*Discrete Appl. Math.,* **157**(2009) 2928-2937.