Kotzig frames and circuit double covers

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\textbf{A B S T R A C T}

A cubic graph \(H\) is called a Kotzig graph if \(H\) has a circuit double cover consisting of three Hamilton circuits. It was first proved by Goddyn that if a cubic graph \(G\) contains a spanning subgraph \(H\) which is a subdivision of a Kotzig graph then \(G\) has a circuit double cover. A spanning subgraph \(H\) of a cubic graph \(G\) is called a Kotzig frame if the contracted graph \(G/H\) is even and every non-circuit component of \(H\) is a subdivision of a Kotzig graph. It was conjectured by Häggkvist and Markström (Kotzig Frame Conjecture, JCTB 2006) that if a cubic graph \(G\) contains a Kotzig frame, then \(G\) has a circuit double cover. This conjecture was verified for some special cases: it is proved by Goddyn if a Kotzig frame has only one component, by Häggkvist and Markström (JCTB 2006) if a Kotzig frame has at most one non-circuit component. In this paper, the Kotzig Frame Conjecture is further verified for some families of cubic graphs with Kotzig frames \(H\) of the following types: (i) a Kotzig frame \(H\) has at most two components; or (ii) the contracted graph \(G/H\) is a tree if parallel edges are identified as a single edge. The first result strengthens the theorem by Goddyn. The second result is a further generalization of the first result, and is a partial result to the Kotzig Frame Conjecture for frames with multiple Kotzig components.

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1. Introduction

Let \(G = (V, E)\) be a graph with vertex set \(V\) and edge set \(E\). A circuit is a connected 2-regular graph. An even subgraph (or a cycle) is a graph such that the degree of each vertex is even. A bridge (or, cut-edge) of a graph \(G\) is an edge \(e \in E(G)\) whose removal increases the number of components of \(G\) (that is, a bridge \(e\) is not contained in any circuit of \(G\)).

Graphs considered in this paper may contain loops or parallel edges. However, most of our graphs are bridgeless. The following open problem has been recognized as one of the central problems in graph theory.

\textbf{Conjecture 1.1} (\textit{Circuit Double Cover Conjecture Szekeres} [12], \textit{Seymour} [11]). Every bridgeless graph \(G\) has a family \(C\) of circuits such that every edge of \(G\) is contained in precisely two members of \(C\).

Since an even subgraph is the union of a set of edge-disjoint circuits, the circuit double cover problem is equivalent to the even subgraph (cycle) double cover. An even subgraph double cover \(\mathcal{F}\) of a graph \(G\) is called a \(k\)-even subgraph double cover (or \(k\)-cycle double cover) if \(|\mathcal{F}| \leq k\).

Comprehensive surveys about progress to this notoriously hard problem can be seen in papers [9] by Jaeger [8] by Jackson, etc., or the books [4, 5, 14].

In this paper, we study a special approach to the conjecture, which was initially started in [6], and, further generalized in [7].

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Lemma 2.2. Let $G$ be a cubic graph containing a Kotzig frame with at most two components. Then $G$ has a $Kotzig$ coloring of $H$.

Kotzig graphs were originally studied in [10]. Well-known examples of Kotzig graphs are $3K_2$, $K_4$, Möbius ladders $M_{2k+1}$, Heawood graph and dodecahedron.

Theorem 1.3 [Goddyn [6], and also Häggkvist and Markström [7]]. If a cubic graph $G$ contains a spanning subgraph $H$ which is a subdivision of a Kotzig graph, then $G$ has a 6-even subgraph double cover.

A 2-factor $F$ of a cubic graph is even if every component of $F$ is of even length. If a cubic graph $G$ has an even 2-factor, then the graph $G$ has many nice properties: $G$ is 3-edge-colorable, $G$ has a circuit double cover, etc. The following concepts were introduced in [7] as a generalization of even 2-factors.

Definition 1.4. Let $G$ be a cubic graph. A spanning subgraph $H$ of $G$ is called a frame of $G$ if the contracted graph $G/H$ is an even graph.

Definition 1.5. Let $G$ be a cubic graph. A frame $H$ of $G$ is called a Kotzig frame of $G$ if, for each non-circuit component $H_i$ of $H$, the suppressed graph $G/H_i$ is a Kotzig graph.

The following conjecture was proposed in [7] as a generalization of Theorem 1.3.

Conjecture 1.6 (Häggkvist and Markström [7]). Every cubic graph containing a Kotzig frame has a circuit double cover.

In [7], Conjecture 1.6 was verified for some special cases.

Theorem 1.7 (Häggkvist and Markström [7]). If a cubic graph $G$ contains a Kotzig frame with at most one non-circuit component, then $G$ has a 6-even subgraph double cover.

In this paper, Conjecture 1.6 is further verified for some other families of cubic graphs, in which a Kotzig frame may contain more than one non-circuit components.

Theorem 1.8. Let $G$ be a cubic graph containing a Kotzig frame with at most two components. Then $G$ has a 6-even subgraph double cover.

Theorem 1.9. Let $G$ be a cubic graph and $H$ be a Kotzig frame of $G$ such that the contracted graph $G/H$ is a tree if parallel edges are identified as a single edge. Then $G$ has a 6-even subgraph double cover.

2. Notation, terminology and basic lemmas

For most standardized notation and terminology, we follow [1–3,13], or [14].

Let $G$ be a graph and $H_1$, $H_2$ be two vertex disjoint subgraphs of $G$. The set of edges with one end in $H_1$ and another in $H_2$ is denoted by $[H_1, H_2]$.

Let $G$ be a graph. The suppressed graph of $G$, denoted by $\overline{G}$, is the graph obtained from $G$ by replacing every maximal induced path by a single edge.

Definition 2.1. Let $H$ be a bridgeless subgraph of a cubic graph $G$. A mapping $c : E(H) \to Z_3$ is called a parity 3-edge-coloring of $H$ if, for each vertex $v \in H$ and each $\mu \in Z_3$,

\[
|c^{-1}(\mu) \cap E(v)| \equiv |E(v) \cap E(H)| \pmod{2}.
\]

It is obvious that if $H$ itself is cubic, then a parity 3-edge-coloring is a proper 3-edge-coloring (traditional definition).

The following lemma has been used frequently in many circuit covering problems.

Lemma 2.2. If a cubic graph has an even 2-factor $C$, then $G$ has a 3-even subgraph double cover $\mathcal{F}$ such that $C \in \mathcal{F}$.

3. Outline of the proofs

For a cubic graph $G$ with a spanning Kotzig graph $H$ (see Fig. 1), let $\{C_{12}, C_{01}, C_{02}\}$ be a circuit double cover of $H$ consisting of three Hamilton circuits, and let $M = E(G) - E(H)$ (a matching). One can decompose $M$ into three parts $\{M_{12}, M_{01}, M_{02}\}$ such that $M_{ij}$ consists of chords of the circuit $C_{ij}$. Note that $C_{ij}$ corresponds to a Hamilton circuit in the suppressed cubic graph $G_{ij} = \overline{C_{ij}} \cup M_{ij}$. Therefore, by Lemma 2.2, let $\mathcal{F}_{ij}$ be a 3-even subgraph double cover of $C_{ij}$ containing $C_{ij}$. So, $\bigcup_{|ij| \in \{0,1,2\}} \mathcal{F}_{ij} - \{C_{12}, C_{02}, C_{01}\}$ is a 6-even subgraph double cover of $G$. (See Fig. 2.) This is the outline of the proof of Theorem 1.3 [6,7].
The main idea of the above discussion could be further extended to Kotzig frames with more than one component. (See Fig. 3.) Let \( \{C_{12}^\mu, C_{01}^\mu, C_{02}^\mu\} \) be a 3-Hamilton circuit double cover of a component \( H_\mu \) of \( H \). If we were able to decompose the matching \( M = E(G) - E(H) \) into three parts \( \{M_{12}, M_{01}, M_{02}\} \) such that \( C_{ij} = \cup \mu C_{ij}^\mu \) corresponds to an even 2-factor of \( G_{ij} = \left( \cup \mu C_{ij}^\mu \right) \cup M_{ij} \) (see Fig. 4), then, with the same argument as above, we would have obtained a 6-even subgraph double cover of \( G \).
This can be considered as an approach to the conjecture. However, there are two major barriers in this approach: in order to have each $C_j$ as an even 2-factor in $G_j$, we need

1. a proper Kotzig coloring in each component $H_\mu$,
2. a proper partition of the matching $M$.

Here $C_j = c^{-1}(i) \cup c^{-1}(j)$ where $c : E(H) \mapsto Z_3$ is a parity 3-edge-coloring of $E(H)$ such that $c$ is Kotzig in $\overline{H_\mu}$ for every component $H_\mu$ of $H$.

There are three types of edges of $M$:

(i)-Chords of some circuit $C_j$ (we may simply add them into $M_j$ since the existence of chords does not have any impact on an even 2-factor $C_j$). (Example: the edge $e_0$ of $G_{12}$ in Fig. 4.)

(ii) Edges between different components $H_\mu$ and $H_\nu$. Assume that $H$ has two components, $H_0$ and $H_1$. For an edge $e = xy$ with $x \in V(H_0)$ incident with two $\alpha$-colored edges, and $y \in V(H_1)$ incident with two $\beta$-colored edges, if $\alpha \neq \beta$, there is no choice that this edge must be added into $M_{\alpha,\beta}$ (the second type of edges). (Example: the edges $e_1$, $e_4$, $e_5$ and $e_6$ in Fig. 4.)

How about those edges $e$ with $\alpha = \beta$ (the third type of edges)? (Example: the edges $e_2$ and $e_3$ in Fig. 4.) It can be added into either $M_{\alpha,\alpha+1}$ or $M_{\alpha-1,\alpha}$. How to distribute this type of edges so that $C_j$ becomes an even 2-factor of $G_j$? Except for the first type of edges, $M$-edges must be distributed to each $M_j$ so that each of them is of even size. In this paper, Lemma 4.1 resolves this distribution problems.

4. Lemmas

As we mentioned in Section 3, the following technical lemma plays a key role in this paper which resolves the problems of the partition of $M$ and the Kotzig coloring in each Kotzig component.

Lemma 4.1. Let $[X, Y]$ be a bipartition of the vertex set of $K_{3,3}$ that $X = \{x_0, x_1, x_2\}$, $Y = \{y_0, y_1, y_2\}$. Let $w : E(K_{3,3}) \mapsto \{0, 1\}$ such that the total weight $w(K_{3,3}) = \sum_{e \in E(K_{3,3})} w(e)$ is even. Then,

(a) There is a permutation $\pi$ of $Z_3$, and we re-label the vertex set $X = \{x_1, x_2, x_3\}$ as follows,

\[ x_i \leftrightarrow x_{\pi(i)} \]

(b) Under the new labeling of $X$, $K_{3,3}$ has a circuit cover

\[ \mathcal{D} = \{D_0 = x_0 y_0 x_1 y_1 x_0, D_1 = x_1 y_1 x_2 y_2 x_1, D_2 = x_2 y_2 x_0 y_0 x_2\} \]

and a weight

\[ w_{\alpha} : E(D_\alpha) \mapsto \{0, 1\} \]

for each $\alpha \in Z_3$ such that

\[ \sum_{e \in E(D_\alpha)} w_{\alpha}(e) \equiv 0 \pmod{2} \] (1)

(\text{that is, every $D_\alpha$ is of even total weight under $w_{\alpha}$}),

(b-2) for each $e \in E(K_{3,3})$,

\[ w(e) = \sum_{\alpha \in Z_3} w_{\alpha}(e) \] (2)

where $w_{\alpha}(e) = 0$ if $e \notin E(C_{\alpha})$ (that is, $\{w_0, w_1, w_2\}$ is a decomposition of the original weight $w$).

Proof. Case 1. $w^{-1}(1)$ contains a matching $\{e_1, e_2\}$. Then there is permutation $\pi$ of $Z_3$ such that $\{e_1, e_2\} = \{x_\alpha y_\alpha, x_{\alpha+1} y_{\alpha+1}\}$. Without loss of generality, consider $\alpha = 0$.

For the circuit $D_1 = x_1 y_1 x_2 y_2 x_1$, let

\[ w_1(e) = w(e) \text{ if } e \neq x_1 y_1, \]

and

\[ w_1(e) = w_1(y_1 x_2) + w_1(x_2 y_2) + w_1(y_2 x_1) \pmod{2} \text{ if } e = x_1 y_1. \]

For the circuit $D_0 = x_0 y_0 x_1 y_1 x_0$, let

\[ w_0(e) = w(e) \text{ if } e \neq x_1 y_1, x_0 y_0, \]

\[ w_0(e) = w(e) - w_1(e) \text{ if } e = x_1 y_1, \]

and

\[ w_0(e) = w_0(y_0 x_1) + w_0(x_1 y_1) + w_0(y_1 x_0) \pmod{2} \text{ if } e = x_0 y_0. \]
For the circuit $D_2 = x_2y_2x_0y_0x_2$, let
\[
\begin{align*}
w_2(e) &= w(e) & \text{if } e \neq x_2y_2, x_0y_0, \\
w_2(e) &= 0 & \text{if } e = x_2y_2,
\end{align*}
\]
and
\[
\begin{align*}
w_2(e) &= w(e) - w_0(e) & \text{if } e = x_0y_0.
\end{align*}
\]

Case 2. $w^{-1}(1)$ contains no matching of size 2. Since the total weight $w(K_{3,3})$ is even, there are two cases: $w^{-1}(1) = \emptyset$ or $w^{-1}(1) = \{e'_1, e'_2\}$. The first case is trivial: we may simply let $w_\alpha : E(D_\alpha) \mapsto \{0\}$ for every $\alpha$. So, we only consider the second case.

There is a permutation $\pi$ of $Z_3$ such that $\{e'_1, e'_2\} = \{x_\beta y_\beta, x_\gamma y_\gamma\}$ or $= \{x_\beta y_\beta, x_\gamma y_\gamma\}$ with $\beta \neq \gamma$. Without loss of generality, consider
\[
\begin{align*}
w^{-1}(1) &= \{e'_1, e'_2\} = \{x_0y_0, x_0y_1\}.
\end{align*}
\]

For the circuit $D_0 = x_0y_0x_1y_1x_0$, let
\[
\begin{align*}
w_1(e) &= w(e) & \text{for every } e \in E(D_0).
\end{align*}
\]

For each circuit $D_\alpha = x_\alpha y_\alpha x_{\alpha+1}y_{\alpha+1}x_\alpha$ with $\alpha = 1, 2$, let
\[
\begin{align*}
w_\alpha(e) &= 0 & \text{for every } e \in E(D_\alpha).
\end{align*}
\]

It is easy to check that, in both cases, the total weight $w_\alpha$ of each circuit $D_\alpha$ is even, and the sum of $w_\alpha$'s is the original given weight $w$. \qed

Let $H_1$ be a component of a Kotzig frame $H$. If $H_1$ is not a circuit, let $c_j : E(H_1) \mapsto Z_3$ be a Kotzig coloring of the suppressed cubic graph $H_j$. The coloring $c_j$ is also considered as an edge coloring of $H_j$ in a natural way (although, it may not be a proper coloring of $H_j$). If $H_1$ is a circuit, then $c_j(E(H_1)) \subseteq Z_3$, that is, $H_j$ is uniformly colored with one color. That is, we are considering a parity 3-edge-coloring for a Kotzig frame $H$.

We prove a slightly stronger statement, which will not only imply Theorem 1.8, but also be applied in the proof of Theorem 1.9.

**Lemma 4.2.** Let $H$ be a Kotzig frame of $G$ with two components $H_0$ and $H_1$. Let $c : E(H) \mapsto Z_3$ be a parity 3-edge-coloring of $H$, such that, in each component $H_j$, $c$ is a Kotzig coloring of $H_j$. Then

(a) There is a permutation $\pi$ of $Z_3$, and we re-label the coloring $c$ of $H_0$ as follows,
\[
c(e) \leftarrow \pi(c(e))
\]
for each $e \in E(H_0)$;

(b) Under the new parity coloring $c$, let $c_{i,j} = c^{-1}(i) \cup c^{-1}(j)$;

(c) And there is a partition $\{I_{0,1}, I_{0,2}, I_{1,2}, I_{1,0}, I_{0,2}, J_{1,2}\}$ of $M = E(G) - E(H)$ such that $I_{i,j}$ is the set of chords of $C_{i,j}$, and $J_{i,j}$ is of even size and joining two components of $C_{i,j}$.

**Proof.** Based on the coloring of $c$ on $H_0$ and $H_1$, $M = E(G) - E(H)$ has a partition
\[
\{E_{i,(0,\alpha),(1,\beta)} : i, j \in \{0, 1\}, \alpha, \beta \in Z_3\}
\]
where an edge $e \in E_{i,(0,\alpha),(1,\beta)}$ if $e = uv$ and one endvertex $u \in V(H_i)$ incident with two $\alpha$-colored edges, and another endvertex $v \in V(H_j)$ incident with two $\beta$-colored edges.

(Here, we are to apply Lemma 4.1.) Consider an auxiliary graph $K_{3,3}$ with a bipartition $\{X, Y\}$ of the vertex set, where
\[
X = \{x_\alpha : \alpha \in Z_3\}, \quad Y = \{y_\alpha : \alpha \in Z_3\},
\]
and each vertex $x_\alpha$ corresponds to the edge subset $c^{-1}(\alpha) \cap H_0$, each vertex $y_\alpha$ corresponds to the edge subset $c^{-1}(\alpha) \cap H_1$, and each edge $x_\alpha y_\beta$ of $K_{3,3}$ corresponds to the edge subset $E_{(0,\alpha),(1,\beta)}$ of $M$.

Let $\omega : E(K_{3,3}) \mapsto \{0, 1\}$ that
\[
\begin{align*}
\omega(x_\alpha y_\beta) &= |E_{(0,\alpha),(1,\beta)}| \pmod 2.
\end{align*}
\]

By Lemma 4.1, there is a permutation $\pi$ on the vertex set $X$ and Eqs. (1) and (2) are satisfied. We re-label the coloring $c$ in $E(H_0)$ according to the permutation $\pi$:
\[
c(e) \leftarrow \pi(c(e))
\]
for each $e \in E(H_0)$. With the revised coloring $c$, all related labelings are adjusted accordingly:
\[
\begin{align*}
x_\alpha &\leftarrow x_{\pi(\alpha)}, \\
E_{(0,\alpha),(1,\beta)} &\leftarrow E_{(0,\pi(\alpha)),(1,\beta)}, \\
E_{(0,\alpha),(0,\beta)} &\leftarrow E_{(0,\pi(\alpha)),(0,\pi(\beta))}.
\end{align*}
\]
Hence, by Eqs. (1) and (2) (under the new coloring c and new labeling in X), the auxiliary graph $K_{3,3}$ has a set of three 4-circuits

$$\mathcal{D} = \{D_0 = x_0y_0x_1y_1, D_1 = x_1y_1x_2y_2, D_2 = x_2y_2x_0y_0\}$$

and a binary weight

$$w_{\alpha} : E(D_{\alpha}) \mapsto \{0, 1\}$$

such that

(1) by Eq. (2), for each $e = x_\alpha y_\beta \in E(K_{3,3})$,

$$|E_{(0,\alpha),(1,\beta)}| \equiv w(x_\alpha y_\beta) = \sum_{\gamma \in Z_3} w_{\gamma}(e) \pmod{2} \quad (3)$$

where $w_{\gamma}(e) = 0$ if $e \notin E(D_{\gamma})$ (that is, $\{w_0, w_1, w_2\}$ is a decomposition of the original weight $w$);

(2) by Eq. (1), for each $\alpha \in Z_3$,

$$\sum_{e \in E(D_{\alpha})} w_{\alpha}(e) \equiv 0 \pmod{2} \quad (4)$$

(that is, every $D_{\alpha}$ is of even total weight under $w_{\alpha}$).

We are to distribute all edges of $M$ into $\{I_0, I_1, I_2, J_0, J_1, J_2\}$. There are three types of edges in $M$

(i) Edges of $E_{(i,\alpha),(i,\beta)}$ for $i = 0, 1$ (they are chords of some circuit of $c^{-1}(\alpha) \cup c^{-1}(\beta)$);

(ii) Edges of $E_{(0,\alpha),(1,\beta)}$ with $\alpha \neq \beta$;

(iii) Edges of $E_{(0,\alpha),(1,\beta)}$.

Type (i) edges: for each $\alpha \in Z_3$, let

$$E_{(0,\alpha),(0,\alpha)} \cup E_{(0,\alpha),(0,\alpha+1)} \cup E_{(1,\alpha),(1,\alpha)} \cup E_{(1,\alpha),(1,\alpha+1)} = I_{\alpha,\alpha+1}$$

(these are chords of some circuit of $c^{-1}(\alpha) \cup c^{-1}(\alpha + 1)$, and the addition of these chords does not have any impact on an even 2-factor). (Example: the edge $e_0$ of $G_{Z_3}$ in Fig. 4)

Types (ii) and (iii) edges are linking edges between $H_0$ and $H_1$ (it is $[H_0, H_1]$), they are to be distributed according to Eq. (3) so that the number of $M$-edges between two circuits of $c^{-1}(\alpha) \cup c^{-1}(\alpha + 1)$ is even.

Note that, edges of type (ii) are covered by precisely one member of $\mathcal{D}$. Hence, for each $\alpha \in Z_3$,

$$E_{(0,\alpha),(1,\alpha+1)} \cup E_{(0,\alpha+1),(1,\alpha)}$$

are added into $I_{\alpha,\alpha+1}$. (Example: the edges $e_1, e_4, e_5$ and $e_6$ in Fig. 4)

Note that, edges of type (iii) are covered by precisely two members of $\mathcal{D}$. Hence, for each $\alpha \in Z_3$, $E_{(0,\alpha),(1,\alpha)}$ will be distributed to $J_{\alpha+1,\alpha}$ and $J_{\alpha-1,\alpha}$ according to the weights $w_{\alpha}$ and $w_{\alpha-1}$ given in Lemma 4.1. Let $\epsilon = w_{\alpha-1}(x_\alpha y_\alpha)$. Then add $e$ edges of $E_{(0,\alpha),(1,\alpha)}$ into $J_{\alpha-1,\alpha}$ and add the remaining into $J_{\alpha+1,\alpha}$. (Example: the edges $e_2$ and $e_3$ in Fig. 4)

By Eq. (4), the total weight $w_{\epsilon}$ of the circuit $D_{\epsilon}$ is even. Hence, in each $G_{\epsilon,\alpha+1}$, the set $J_{\alpha,\alpha+1}$, consisting of edges between two components of $C_{\alpha,\alpha+1} = c^{-1}(\alpha) \cup c^{-1}(\alpha + 1)$, is of even size. $\square$

5. Proofs of main theorems

Proof of Theorem 1.8. Let $c$ be a parity 3-edge-coloring of $H$ such that $c$ is Kotzigin in $H_j$ for each $j = 0, 1$. By Lemma 4.2, the coloring can be modified so that $C_{i,j} = c^{-1}(i) \cup c^{-1}(j)$ corresponds to an even 2-factor of $C_{i,j} = C_{i,j} \cup M_{i,j}$. By Lemma 2.2, each $G_{i,j}$ has a 2-even subgraph cover that covers its $M$-edges twice and the even 2-factor $C_{i,j}$ once. Hence, their union is a 6-even subgraph double cover. $\square$

Proof of Theorem 1.9. We prove a stronger statement for the purpose of induction:

$(\ast)$ Let $G$ be a cubic graph and $H$ be a Kotzigin frame of $G$ such that the contracted graph $G/H$ is a tree if parallel edges are identified as a single edge. Then there is a parity 3-edge-coloring $c$ of $H$ such that

(1) $c$ is Kotzigin in $H_j$ for every component $H_j$ of $H$, let $C_{i,j} = c^{-1}(i) \cup c^{-1}(j)$;

(2) there is a partition $\{M_{0,1}, M_{0,2}, M_{1,2}\}$ of $M = E(G) - E(H)$;

(3) $C_{i,j}$ corresponds to an even 2-factor of $C_{i,j} \cup M_{i,j}$.

Proof of $(\ast)$. Induction on the number of components of $H$ (it is true if $H$ has only one component).

Let $T$ be the tree obtained from the contracted graph $G/H$ by identifying each parallel edges as a single edge. Let $v_0$ be a degree one vertex of $T$ with the unique neighbor $v_1$, and let $v_1$ correspond to a component $H_1$ of $H$. Here, $N(H_0) \subseteq V(H_1)$.

Note that $H^* = H - H_0$ remains as a Kotzigin frame of $G^* = G - V(H_0)$. Let $M^* = E(G^*) - E(H^*) = M \cap E(G^*)$. By induction, $H^*$ has a parity coloring $c^* : E(H^*) \mapsto Z_2$ such that the restriction of $c^*$ to each $H_j (j \geq 1)$ is Kotzigin, and $M^*$ has a partition $\{M^*_{0,1}, M^*_{0,2}, M^*_{1,2}\}$ such that $C_{i,j} = (c^*)^{-1}(i) \cup (c^*)^{-1}(j)$ is an even 2-factor of $C_{i,j} = C_{i,j} \cup M_{i,j}$. 


The statement is proved by applying Lemma 4.2: the parity coloring \( c^* \) can be extended to include \( H_0 \) (without any change in \( H^* \)), and the partition of \( M^* = \{ M^*_0, M^*_1, M^*_2 \} \) can be extended to a partition of \( M = \{ M_0, M_1, M_2 \} \) by simply letting

\[
M_{i,j} = I_{i,j} \cup J_{i,j} \cup M^*_{i,j}
\]

where \( I_{i,j} \) and \( J_{i,j} \) are defined in Lemma 4.2 as subsets of \( M - M^* \). Again, by Lemma 4.2, \( C_{i,j} = c^{-1}(i) \cup c^{-1}(j) \) corresponds to an even 2-factor of \( \overline{C_{i,j}} \cup \overline{M_{i,j}} \) since \( \vert I_{i,j} \vert \) is even. So (\( \ast \)) is proved.

\[
\text{Theorem 1.9 is proved by applying Lemma 2.2 to each } G_{i,j}. \quad \square
\]

The following conjecture is an improvement of Theorem 1.9 and is a special case of Conjecture 1.6.

**Conjecture 5.1.** If \( G \) has a Kotzig frame \( H \) such that \( G/H \) is bipartite, then \( G \) has a circuit double cover.

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