Jacket transform eigenvalue decomposition

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Abstract

In this paper we give necessary and sufficient conditions for a Jacket decomposability. Some properties of Jacket matrices are also proved. Jacket matrices are interesting for an eigenvalue decomposition and signal processing because they are easily invertible.
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1. Introduction

Structured matrices such as the Toeplitz, Hankel, Hadamard, Jacket, etc. matrices, play an important role in signal processing. The Hadamard transform and its generalizations such as the center weighted Hadamard transform have been used for audio and video coding because of the high practical value of these transformations for representing signal and images [1].

Jacket matrices were introduced by Lee in 1989 in relation with the center weighted Hadamard transform [2,3]. They are closely related to various famous mathematical objects like Turyn and Butson type Hadamard matrices, orthogonal designs, etc., which have numerous applications to many mathematical and theoretical physics problems. The class of Jacket matrices contains the classes of Hadamard and complex Hadamard matrices [4]. It has a large overlap, but does not coincide, with the class of generalized Hadamard matrices [5,6]. Jacket matrices have also applied to solving various problem in signal processing and coding theory (for details see [7]). Jacket transform and reversed jacked transform based on the Jacket matrices have also demonstrated a high potential for applications.

Eigenvalue decomposition is another intensively used method of solving numerous problem in signal processing. In general, methods based on matrices with easily computable inverse matrix are widely used in practice. Jacket matrices possess this property.

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In this paper we address the problem of finding necessary and sufficient conditions for a Jacket decomposability. The paper is organized as follows. In the next section we present the necessary notions and prove some new properties of Jacket matrices. The obtained results concerning the eigenvalue decomposition are presented in Section 3.

2. Jacket matrix properties

Let $\mathbb{F}$ be a field (finite or infinite). Let $A = (a_{kl})$ be an $n \times n$ matrix whose entries are nonzero elements of $\mathbb{F}$. Denote by $A^*$ the transpose matrix of the element-wise inverses matrix of $A$, that is, $A^* = (a_{kl}^*)$, with $a_{kl}^* = a_{lk}^{-1}$.

**Definition 1.** Matrix $A$ is called Jacket matrix if $AA^* = A^*A = nI_n$, where $I_n$ is the $n \times n$ identity matrix over $\mathbb{F}$, or equivalently

$$
\sum_{j=1}^{n} a_{kj}a_{lj}^{-1} = \begin{cases} n & k = l, \\ 0 & k \neq l. \end{cases}
$$

Examples of Jacket matrices are the following matrices of order $n = 2$ and $3$, respectively:

$$
A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \omega^2 + \omega + 1 = 0.
$$

Numerous examples can be found in [7]. Also, it is easy to check that the Vandermonde matrix of $n$th roots of unity

$$
W = W(x_1, x_2, \ldots, x_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}, \quad x_j^n = 1, \ 1 \leq j \leq n
$$

is a Jacket matrix, too. $W$ is called DFT matrix since it is the matrix of the Discrete Fourier Transformation (DFT). In the case when the field $\mathbb{F}$ is the field of complex numbers, $x_k = e^{2\pi i k / n}$.

The transpose matrix $A^*$ and the Kronecker product $A \otimes B$ of two Jacket matrices are Jacket matrices, too. The inverse of a Jacket matrix, $A^{-1} = \frac{1}{n}A^*$, is also a Jacket matrix and its easy calculation makes Jacket matrices very useful for eigenvalue decomposition.

**Definition 2.** A matrix $A$ is called Jacket diagonalizable (or decomposable), if it can be represented as a product of three matrices, $A = TKT^{-1}$, where $T$ is a Jacket matrix and $K$ is a diagonal matrix. The process of diagonalization itself is referred to as Jacket decomposition.

**Definition 3.** Two Jacket matrices are called equivalent if one can be obtained from the other by permuting rows and columns and by multiplying rows and columns by nonzero element of the field.

Therefore, two matrices $A$ and $B$ are equivalent, denoted by $A \sim B$, if and only if

$$
A = PDBEQ,
$$

where $P$ and $Q$ are permutation matrices, and $D$ and $E$ are diagonal matrices.

In partial, any Jacket matrix is equivalent to a Jacket matrix whose first row and first column are all-one vectors, i.e., to a matrix of the form:

$$
A = \begin{pmatrix} 1 & 1 \\ 1^t & B \end{pmatrix}, \quad (3)
$$

where $1$ is all-one row-vector.
Theorem 1. If $A$ is a Jacket matrix of the form $(3)$ over a field $\mathbb{F}$ and $B = (b_{ij})$ has dimension $n$, then the elements of any row of $B$ form the set of roots of
\[ X^n + X^{n-1} + a_2X^{n-2} + \cdots + a_{n-2}X^2 + (-1)^nt(X + 1) = 0 \]
for a suitable $t$.

Proof. Consider the $k$th row of $B$ (i.e., $(k + 1)$th row of $A$) and the first row of $A$. It immediately follows from (1) that
\[ \sum_{j=1}^{n} a_{kj} = -1 \quad \text{and} \quad \sum_{j=1}^{n} \frac{1}{a_{kj}} = -1. \]

The second equation can be rewritten in the form
\[ \sigma_{n-1}(a_{k1}, a_{k2}, \ldots, a_{kn}) = -a_{k1}a_{k2} \cdots a_{kn} = -\sigma_n(a_{k1}, a_{k2}, \ldots, a_{kn}) = -t, \]
where $\sigma_i(a_{k1}, a_{k2}, \ldots, a_{kn})$ is the $i$th elementary symmetric polynomial of $a_{k1}, a_{k2}, \ldots, a_{kn}$. Therefore (according to Vieta's formulas), $a_{kj}$, $j = 1, \ldots, n$, are roots of the polynomial
\[ X^n - (-1)X^{n-1} + a_2X^{n-2} + \cdots - (-1)^{n-1}tX + (-1)^nt. \]

Theorem 2. A Vandermonde matrix
\[
W(x_1, x_2, \ldots, x_n) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\ldots & \ldots & \ldots & \ldots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix}
\]
where $x_j \in \mathbb{F}$, $x_j \neq 0$, is Jacket matrix if and only if $x_1, x_2, \ldots, x_n$ are roots of a polynomial of the form $X^n - a^n$.

Proof (Necessity (‘only if’)). Dividing any row of $W(x_1, x_2, \ldots, x_n)$ by the element in the first column it can be transform into the Vandermonde matrix with only 1s in the first column, namely
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
y_1 & y_2 & \cdots & y_{n-1} \\
y_1^2 & y_2^2 & \cdots & y_{n-1}^2 \\
\ldots & \ldots & \ldots & \ldots \\
y_1^{n-1} & y_2^{n-1} & \cdots & y_{n-1}^{n-1}
\end{pmatrix}
\]

According to Theorem 1 the elements $y_1, y_2, \ldots, y_{n-1}$ are roots of a polynomial
\[
y^{n-1} + y^{n-2} + b_2y^{n-3} + \cdots + (-1)^{n-1}b(Y + 1) = 0
\]
for a suitable $b$. On the other hand, the definition of Jacket matrices gives the next equalities for the power sums:
\[ S_k = y_1^k + y_2^k + \cdots + y_{n-1}^k = -1, \quad k = 1, 2, \ldots, n - 1. \]
The identities between the coefficients of a polynomial and the sums of $k$th powers of its roots are given by Newton–Girard formulas which in our case look as follows:
\[ S_1 + 1 = 0, \]
\[ S_2 + S_1 + 2b_2 = 0, \]
Recall that a circulant matrix is said to be of the following form:

Let

Proof. Therefore, $S_k = -1$ for $k = 1, 2, \ldots, n - 1$, we obtain a triangular linear system, which gives immediately

Therefore, $y_1, y_2, \ldots, y_{n-1}$ are roots of the polynomial

i.e., they are $n$th roots of unity different of unity. Hence $y_k^n = (x_k/x_1)^n = 1$ and then

$$x_1^n = x_2^n = \cdots = x_n^n = a^n.$$  

Sufficiency. As we mentioned above, it is well known that any Vandermonde matrix whose entries are $n$th roots of unity (i.e., matrix of the form (4)) is a Jacket matrix. Multiplying the $k$th row by $a^{k-1}$ we obtain a Vandermonde matrix with the desired properties. □

3. Eigenvalue decomposition

We begin this section with proving the following simple statement, which enables us having a set of Jacket decomposable matrices to enlarge easily it.

Theorem 3. If $A$ and $B$ are Jacket decomposable matrices, then their Kronecker product $A \otimes B$ is also Jacket decomposable.

Proof. Let $A = TA_1T^{-1}$ and $B = UA_2U^{-1}$, where $T$ and $U$ are Jacket matrices, and $A_1$ and $A_2$ are diagonal. Then

$$A \otimes B = (TA_1T^{-1}) \otimes (UA_2U^{-1}) = (T \otimes U)(A_1 \otimes A_2)(T^{-1} \otimes U^{-1}).$$

$A_1 \otimes A_2$ is a diagonal matrix, and $T \otimes U$ is a Jacket matrix. Since $T^{-1} \otimes U^{-1} = (T \otimes U)^{-1}$ we can rewrite the above equality in the form

$$A \otimes B = (T \otimes U)(A_1 \otimes A_2)(T \otimes U)^{-1}.$$  

Therefore, $A \otimes B$ is Jacket decomposable. □

Recall that a circulant matrix is said to be a matrix of the following form:

$$
C = \begin{pmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
    c_{n-1} & c_0 & a_0 & \cdots & c_{n-2} \\
    c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\
    & \ddots & \ddots & \ddots & \ddots \\
    c_1 & c_2 & c_3 & \cdots & c_0
\end{pmatrix}.
$$

Circulant matrices are a special kind of Toeplitz matrices and play important role in numerical analysis and graph theory.

The interest in circulant matrices is based on the following well known their property:

$$W^{-1}CW = \text{diag}(f(x_1), f(x_2), \ldots, f(x_n)), \quad (5)$$

$$S_3 + S_2 + b_2S_1 + 3b_3 = 0,$$

$$S_k + S_{k-1} + b_2S_{k-2} + \cdots + b_{k-1}S_1 + kb_k = 0,$$

$$S_{n-2} + S_{n-3} + \cdots + b_{n-3}S_1 + (n - 2)(-1)^{n-1}b = 0,$$

$$S_{n-1} + S_{n-2} + \cdots + (-1)^{n-1}bS_1 + (n - 1)(-1)^{n-1}b = 0.$$
where \( f(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \) and \( W \) is the DFT matrix. It is obviously that \( f(x_1), f(x_2), \ldots, f(x_n) \) are the eigenvalues of \( C \).

It is interesting that the inverse assertion is also true, that is, the following theorem holds:

**Theorem 4.** A matrix \( A = (a_{ij}) \) is diagonalizable by the Vandermonde matrix \( W \) (as a matrix of similarity transformation), if and only if \( A \) is a circulant matrix.

**Proof.** *(Sufficiency)* As we mentioned above, this fact can be found in any book on numerical analysis or linear algebra, and we omit it.

*(Necessity (‘only if’ part))* Let

\[
A = WAW^{-1}, \quad \text{where } A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).
\]

Denote \( f_k(x) = a_{k1} + a_{k2}x + \cdots + a_{kn}x^{n-1}, \ k = 1, 2, \ldots, n. \)

It is easy to check that

\[
AW = \begin{pmatrix}
f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\
f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\
f_3(x_1) & f_3(x_2) & \cdots & f_3(x_n) \\
\vdots & \vdots & \ddots & \vdots \\
f_n(x_1) & f_n(x_2) & \cdots & f_n(x_n)
\end{pmatrix}
\]

On the other hand

\[
WA = \begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\lambda_1x_1 & \lambda_2x_2 & \cdots & \lambda_nx_n \\
\lambda_1x_1^2 & \lambda_2x_2^2 & \cdots & \lambda_nx_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1x_1^{n-1} & \lambda_2x_2^{n-1} & \cdots & \lambda_nx_n^{n-1}
\end{pmatrix}
\]

Since \( AW = WA \), we get

\[
\begin{align*}
f_1(x_j) &= \lambda_j, \\
f_2(x_j) &= \lambda_jx_j, \\
\vdots \\
f_k(x_j) &= \lambda_jx_j^{k-1}, \quad j = 1, 2, \ldots, n \\
\vdots \\
f_n(x_j) &= \lambda_jx_j^{n-1}.
\end{align*}
\]

Multiplying the first equation by \( x_j^{k-1} \) and subtracting the result from the \( k \)th equation for any \( j = 1, 2, \ldots, n \) we obtain

\[
(a_{k1} - a_{1,n-k+2}) + (a_{k2} - a_{1,n-k+3})x_j + \cdots + (a_{kn} - a_{1,n-k+1})x_j^{n-1} = 0.
\]

But this means that the vector

\[
((a_{k1} - a_{1,n-k+2}), (a_{k2} - a_{1,n-k+3}), (a_{k3} - a_{1,n-k+4}), \ldots, (a_{kn} - a_{1,n-k+1}))
\]

is a solution of the homogenous linear system:

\[
W \begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_n
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]
Since \( W \) is nonsingular matrix, this system has only the trivial (zero) solution. Thus
\[
a_{k,1} - a_{1,n-k+2} = a_{k,2} - a_{1,n-k+3} = a_{k,3} - a_{1,n-k+4} = \cdots = a_{k,n} - a_{1,n-k+1} = 0.
\]
Therefore, the \( k \)th row of \( A, k = 1, 2, \ldots, n \), is a right shift by \( k - 1 \) positions of the first row and hence, \( A \) is a circulant. \( \square \)

Now we will proceed to a more general case of Jacket decomposition \( A = TAT^{-1} \). We shall consider the case when \( T \) is equivalent to DFT matrix \( W \), that is, when
\[
T = PDWEQ,
\] (6)
where \( P \) and \( Q \) are permutation matrices, and \( D \) and \( E \) are diagonal matrices.

**Theorem 5.** A matrix \( A = (a_{ij}) \) is diagonalizable by a Jacket matrix which is equivalent to the DFT matrix \( W \), if and only if
\[
A = PBP^{-1},
\]
where \( P \) is a permutation matrix and \( B \) has the form
\[
B = \begin{pmatrix}
c_1 & c_2d_1d_2^{-1} & c_3d_1d_2^{-1} & \cdots & c_nd_1d_n^{-1} \\
c_nd_2d_1^{-1} & c_1 & c_2d_2d_3^{-1} & \cdots & c_{n-1}d_2d_n^{-1} \\
c_{n-1}d_3d_1^{-1} & c_nd_2d_3^{-1} & c_1 & \cdots & c_{n-2}d_3d_n^{-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_2d_n^d_1^{-1} & c_3d_n^d_2^{-1} & c_4d_n^d_3^{-1} & \cdots & c_1
\end{pmatrix}
\] (7)

**Proof.** By (6) we get
\[
A = PDWEAQ^{-1}E^{-1}W^{-1}D^{-1}P^{-1}.
\]
Since \( Q \) is a permutation matrix, the matrix \( A_1 = QAQ^{-1} \) is also diagonal and differs from \( A \) only in the order of diagonal elements. Then all three matrices \( E, A_1 \) and \( E^{-1} \), as diagonal matrices, commute with each other. Hence \( EA_1E^{-1} = A_1EE^{-1} = A_1 \). Therefore,
\[
A = PDWA_1W^{-1}D^{-1}P^{-1} = PDCD^{-1}P^{-1}.
\]
According to Theorem 4 matrix \( C = WA_1W^{-1} \) is circulant and let its first row be \( (c_1, c_2, \ldots, c_n) \). Let \( D = \text{diag}(d_1, d_2, \ldots, d_n) \). Then
\[
A = PBP^{-1},
\]
where \( B = DCD^{-1} \) has the form (7), that is, the element in \( i \)th row and \( j \)th column is multiplied by \( d_id_j^{-1} \). \( \square \)

**Example 1.** Consider the Hankel matrix (on the left side)
\[
H = \begin{pmatrix}
a & b & c & d & e \\
b & c & d & e & f \\
c & d & e & f & g \\
d & e & f & g & h \\
e & f & g & h & i
\end{pmatrix}, \quad H_1 = \begin{pmatrix}
e & d & c & b \\
f & e & d & c \\
g & f & e & d \\
h & g & f & e \\
i & h & g & f
\end{pmatrix}
\]
with \( ai = df \) and \( hb = cg \). By a permutation of columns \( H \) can be transformed into the matrix \( H_1 \) on the right side above. By straightforward calculations we obtain the elements of the corresponding circulant:
\[
c_1 = e, \quad c_2 = dx, \quad c_3 = cy^2, \quad c_4 = g/y^2, \quad c_5 = f/x, \quad \text{where} \ x = \sqrt{f/a}, \ y = \sqrt{g/b}.
\]
Let \( \sigma \) be a matrix equivalence, i.e., it can be realized by multiplying from the left or right side by a product of diagonal and permutation matrices. Let \( \sigma A \) denote the left action of \( \sigma \). Respectively, we shall write \( A\sigma \) when we consider \( \sigma \) as a right equivalence. It is straightforward to see that Jacket inverse satisfies \( (\sigma A)^* = A^*\sigma^{-1} \).
Theorem 6. If $A$ and $B$ are Jacket decomposable matrices and $r$ is a matrix equivalence, then the matrix

$$ C = \begin{pmatrix} A + \sigma B \sigma^{-1} & A - \sigma B \sigma^{-1} \\ A - \sigma B \sigma^{-1} & A + \sigma B \sigma^{-1} \end{pmatrix} $$

is also Jacket decomposable.

Proof. Let $A = U A_1 U^{-1}$ and $B = V A_2 V^{-1}$, where $U$ and $V$ are Jacket matrices, and $A_1$ and $A_2$ are diagonal. Then

$$ C = \begin{pmatrix} U A_1 U^{-1} + \sigma V A_2 V^{-1} \sigma^{-1} & U A_1 U^{-1} - \sigma V A_2 V^{-1} \sigma^{-1} \\ U A_1 U^{-1} - \sigma V A_2 V^{-1} \sigma^{-1} & U A_1 U^{-1} + \sigma V A_2 V^{-1} \sigma^{-1} \end{pmatrix} $$

$$ = \begin{pmatrix} U & \sigma V \\ U & -\sigma V \end{pmatrix} \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \begin{pmatrix} U^{-1} & U^{-1} \\ V^{-1} \sigma^{-1} & -V^{-1} \sigma^{-1} \end{pmatrix}.$$ 

It is easy to check that

$$ \begin{pmatrix} U & \sigma V \\ U & -\sigma V \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} U^{-1} & U^{-1} \\ V^{-1} \sigma^{-1} & -V^{-1} \sigma^{-1} \end{pmatrix}. $$

The multiplier $\frac{1}{2}$ will be included in the eigenvalue matrix. \qed

References