In this paper, it is discovered that the statistical property of the consensus and synchronization of the small-world networks, that is, the Cheeger constant, is a major determinant to measure the convergence rate of the consensus and synchronization of the small-world networks. Further, we give a mathematical rigorous estimation of the lower bound for the algebraic connectivity of the small-world networks, which is much larger than the algebraic connectivity of the regular circle. This result explains why the consensus problems on the small-world network have an ultrafast convergence rate and how much it can be improved. Moreover, it also characterizes quantitatively what kind of the small-world networks can be synchronized. © 2010 American Institute of Physics.

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I. INTRODUCTION

The consensus problem for a network dynamics means to reach a common agreement of all agents in a network, where the agents act in coordination with the others through an information network. It has been widely investigated through interdisciplinary approaches. The underlying topological structure of networks plays an important role in the analysis of convergence speed for the consensus problem whose theoretical framework was proposed and solved by Olfati-Saber and Murray. Recently much attention has been focused on the consensus problem on the small-world networks, which was introduced by Watts and Strogatz (WS) (Ref. 52) as a model of real world complex networks. In the small-world networks, since any agents can be reached from another by a short path, based on an intuitive judgment, this fact can lead to a faster spreading of information between agents and then results to faster consensus on the small-world networks.

From mathematical view, the measurement of convergence speed of solving the consensus problems in networks is used by the algebraic connectivity of a network, which is the second smallest eigenvalue of its Laplacian matrix. There are some strongly numerical evidences which support the conjecture that the network dynamics on the small-world networks would display enhanced global coordination compared to the regular lattices. In the recent paper of Olfati-Saber, it is observed that the algebraic connectivity of the small-world networks could be increased dramatically by more than 1000 times. The question arises how can we measure or estimate the increment mathematically and quantitatively? And what statistical property determines the algebraic connectivity? This paper will answer these related questions.
maps which coupled on arrays and networks have attracted a lot of interest.\textsuperscript{20,22,23,27,28,51,45,44,43,53} It is also discovered that the synchronization problem is closely related with spectrum of the underlying topological networks.\textsuperscript{14,48,13}

In Ref. 44, Pecora and Carroll showed that a general criterion for the synchronized states of both chaotic and limited circle systems was given by the negativity of the master stability function, which is maximum Floquet or Lyapunov exponents for the generic variational equation. Further, Barahona and Pecora\textsuperscript{3} calculated the critical values $\alpha_1$ and $\alpha_2$ of the master stability function for some identical oscillator systems in coupled networks such that the system is synchronizeable in $(\alpha_1, \alpha_2)$. Hence, they defined that a network is synchronizeable if $\lambda_{\max}/\lambda_2 < \alpha_2/\alpha_1 = \beta$. Moreover, Fink \textit{et al.}\textsuperscript{18} showed that the stability surface that governs the synchronization of a large class of arrays of identical oscillators can be probed with a simple array of just three identical oscillators.

The synchronization problem in small-world networks has been carefully studied.\textsuperscript{19,29,52,40,36,40} It has been observed that the ability to achieve the synchronization is greatly enhanced by making the network to become a small-world network. The statistical result is really interesting and has been explained.\textsuperscript{1} Generally, it was believed that the enhancement was due to short average distance.\textsuperscript{43} However, we will show that it is much more related with other properties such as the Cheeger constant. Barahona and Pecora\textsuperscript{3} discovered in the small-world network that the critical value $\lambda_{\max}/\lambda_2$ obeys sublogarithmic, and applied perturbation analysis to give an estimation of the synchronizeable small-world network with expectation of the second smallest Laplacian eigenvalue to derive the synchronizeable maximum $k^\ast$ and corresponding average shortcut per node $s^\ast_{\text{sync}}$. Our method can arrive an estimation for the small-world networks in sense of distribution, which is available for all pairs of $k, s$. This shows rigorously that synchronization in the small-world networks will be more easily reached than the regular circle. Also we determine the class of the small-world networks which is Barahona–Pecora’s synchronizeable.

Synchronization for delayed coupling and synchronization on the scale-free networks are also interesting topics; one can refer to Refs. 13, 31, and 49. The discrete synchronization problems without\textsuperscript{18,19,1} and with delays\textsuperscript{2} are also fascinating problems and they are also closely related to the spectral property of underlying topological structure.

In this paper, we apply an estimation of the Cheeger constant to give a rigorous spectral analysis for the Laplacian matrices of the small-world networks. The main results are as follows.

\textbf{Theorem 1.1:} Let $S(n, c, k)$ be the small-world network with $n$ nodes, which is a union of an Erdős–Rényi random graph $G(n, c/n)$ and a $2k$ regular cycle. Then the algebraic connectivity of $S(n, c, k)$ is almost surely bounded below by

\begin{equation}
\frac{k^2 c^2 \log \log n}{2(k+1)^3 \log^3 n}.
\end{equation}

Olfati and Saber\textsuperscript{40} defined

\begin{equation}
\gamma_2(n, c, k) = \frac{\lambda_2(n, c, k)}{\lambda_2(n, 0, k)}
\end{equation}

to be the algebraic connectivity gain of $S(n, c, k)$.

\textbf{Theorem 1.2:} The algebraic connectivity gain of the small-world network $S(n, c, k)$ follows almost surely inequality

\begin{equation}
\gamma_2(S(n, c, k)) \geq \frac{3kc^2 n^2 \log \log n}{2(k+1)^3(2k+1)^2 \log^3 n}.
\end{equation}

As $n \to \infty$, the algebraic connectivity gain increases in the order of $n^2$ nearly. This result explains why the convergence speed for the consensus problems on the small-world network is
ultrafast. The approach method is similar to that of Durrett.\textsuperscript{11} While considering the convergence speed of a random walk on the small-world network, Durrett\textsuperscript{11} obtained a similar result for the normalized Laplacian.

According to synchronicity, we can also obtain a similar result.

**Theorem 1.3:** For a given Pecora–Carroll type system and the critical value $\beta$ for Pecora–Carroll associated master stability function, let $S(n,c,k)$ be the small-world network with $n$ nodes, which is a union of an Erdös–Rényi random graph $G(n,c/n)$ and a $2k$ regular cycle. Then $S(n,c,k)$ is almost surely synchronizable if $ck/(k+1) \geq 4 \log^2 n/\beta^{1/2} \log \log n$.

This paper is organized as follows. In Sec. II, we give the proofs of the main results of the paper. In Sec. III, we discuss and investigate the relationship between the theoretical results and some numerical results. In this way, we explain why the convergence rate for the consensus problem on the small-world networks from the mathematical and experiment views.

**II. SPECTRAL ANALYSIS OF SMALL-WORLD NETWORKS**

In this section, we will study the network dynamics whose information exchange network is called small-world network. The small-world network is a model with two important characteristics: the clustering effect and the small-world phenomenon, which was originally introduced by WS in their famous paper.\textsuperscript{52} For technical reason, we will adopt the NW model which was introduced by Newman and Watts,\textsuperscript{39,38} which is the union of an Erdös–Rényi random graph and a $2k$ regular lattice. It is known that the NW model and the WS model are, essentially, the same. The NW model can be illustrated as follows (Fig. 1).

Erdös and Rényi introduced the concept of random graph nearly 50 years ago. A random graph $G(n,p)$ is a graph of $n$ nodes and edges between nodes occur independently with probability $p$. This random network model has been intensively studied (see Ref. 5 and references therein). A $2k$ regular lattice $C(n,k)$ with $n$ nodes can be constructed from a cycle of $n$ nodes and connected each node to its $2k$ nearest neighbors.

The algebraic connectivity $\lambda_2$ of the small-world network is used to measure the convergence rate of the consensus problem for the network dynamics over the small-world network. Olfati-Saber\textsuperscript{40} discovered that the algebraic connectivity of a small-world network can be made more than 1000 times greater than a $2k$ regular lattice from numerical experiment.

Let $S(n,c,k)$ be a small-world network that is the union of a random graph $G(n,c/n)$ and a $2k$ regular lattice $C(n,k)$. Denote by $\lambda_2(n,c,k)$ the algebraic connectivity of $S(n,c,k)$. If $c=0$, then $\lambda_2(n,0,k)$ is the algebraic connectivity of the $2k$ regular lattice $C(n,k)$. The relationship between parameter $c$ and the average shortcut per node $s$ is $c=2s$.\textsuperscript{37,38}

Similarly, we show that the synchronicity problem depends on the algebraic connectivity and the largest eigenvalue of the Laplacian matrix. Further, we show that the increment of synchronizability\textsuperscript{3} of $S(n,c,k)$ is much larger than that of the regular lattice $C(n,k)$.

It was believed that the increment of the consensus and synchronizability\textsuperscript{43} is due to short average distance. Also other statistical properties\textsuperscript{1} have been investigated. By the graph theory, it
is discovered that the algebraic connectivity is much more related with the Cheeger constant than the invariant of the short average distance. Let $G=(V,E)$ be a graph with $n$ nodes and the adjacency matrix $A=(a_{ij})$, where $a_{ij}=1$ for $v_i$ link to $v_j$ and 0 for otherwise. The Cheeger constant of a graph $G$ is defined by

$$\chi(G) = \min \left\{ \frac{\sum_{i \in S, j \notin S} a_{ij}}{|S|}; S \subset V, 0 < |S| \leq \frac{n}{2} \right\},$$

where the minimum is taken over all nonempty subsets $S$ of $V$ satisfying $|S| \leq n/2$. It is a statistical property and $\sum_{i \in S,j \notin S} a_{ij}$ is the number of edges from subset $S$ to its complement. The number $\chi(G)$ is also called isoperimetric number of the graph $G$, which is useful in a number of areas including recursive algorithms, network design, etc. The related results may be referred to Refs. 6 and 7. In fact the large Cheeger constant also leads a short diameter, so the large Cheeger constant is the determinant statistical property of the small-world networks for increment of consensus and synchronization. From Ref. 4 or Ref. 34 we have the following result.

**Lemma 2.1:** [Mohar (Ref. 34)] Let $G$ be a graph with $n$ nodes. Denote by $\lambda_2(G)$ and $\chi(G)$ the algebraic connectivity and the Cheeger constant of $G$, respectively. If $\Delta(G)$ is the maximum degree, then

$$\lambda_2(G) \geq \frac{\chi(G)^2}{2\Delta(G)}.$$  \hspace{1cm} (5)

Inequality (5) gives the relation between the algebraic connectivity and the Cheeger constant for any networks. In order to establish the lower bound for the algebraic connectivity of the small-world networks, we need to determine the Cheeger constant and the maximum degree for the small-world networks. For the maximum degree $\Delta(S(n,c,k))$, we have the following classical result (Ref. 5, p. 77).

**Lemma 2.2:** [Bollobás (Ref. 5)] Let $G(n,c/n)$ be a random graph and $\Delta(G(n,c/n))$ be the maximum degree. Then as $n \to \infty$, almost surely, we have

$$\Delta\left(G\left(n, \frac{c}{n}\right)\right) = (1 + o(1)) \frac{\log n}{\log \log n}.$$  \hspace{1cm} (6)

Since

$$\Delta\left(G\left(n, \frac{c}{n}\right)\right) \leq \Delta(S(n,c,k)) \leq \Delta\left(G\left(n, \frac{c}{n}\right)\right) + 2k,$$  \hspace{1cm} (7)

(6) yields the following:

$$\Delta(S(n,c,k)) = (1 + o(1)) \frac{\log n}{\log \log n} \text{ a.s.}$$  \hspace{1cm} (8)

From (8), we establish the value of the maximum degree of the small-world networks. Further, we consider the Cheeger constant of the small-world networks.

**Lemma 2.3:** Let $S(n,c,k) = C(n,k) \cup G(n,c/n)$ be the small-world network, where $c$ is a positive constant. Then almost surely, we have

$$\chi(S(n,c,k)) \geq \frac{k \Delta}{2(k+1) \log n}.$$  \hspace{1cm} (9)

**Proof:** In fact we will prove the following result that for any $\delta > 4$, $\chi(S(n,c,k)) \geq 2k \Delta / \delta(k+1) \log n$, which is stronger than (9). We consider the following four cases.
Case 1: Let $e(S, S')$ stand for the number of edges from node set $S$ to its complement $S'$ in $V=\{1 \cdots n\}$ and $|S|=s$ with $0<s \leq (1+k)\delta \log n/c$. Then

$$\frac{e(S, S')}{s} \geq \frac{2k}{s} \geq \frac{2ck}{(1+k)\delta \log n}.$$  \hspace{1cm} (10)

Case 2: Let $r$ be the number of separate sets on the lattice $|S|=s$ with $n/2 \geq s > (1+k)\delta \log n/c$ and $r \geq cs/(1+k)\delta \log n$. Then

$$\frac{e(S, S')}{s} \geq \frac{2r}{s} \geq \frac{2c}{\left(1+\frac{1}{k}\right)\delta \log n}.$$  \hspace{1cm} (11)

Case 3: Let $r'$ be the number of separate sets on the lattice whose $2k$ neighborhood does not intersect node set $S$, $|S|=s$ with $n/2 \geq s > (1+k)\delta \log n/c$ and $r' \geq cs/(1+k)\delta \log n$. Then

$$\frac{e(S, S')}{s} \geq \frac{2kr}{s} \geq \frac{2c}{\left(1+\frac{1}{k}\right)\delta \log n}.$$  \hspace{1cm} (12)

Case 4: Let $|S|=s$ with $n/2 \geq s > (1+k)\delta \log n/c$ such that $r < cs/(1+k)\delta \log n$ and $r' < cs/(1+k)\delta \log n$. Then the number of sets satisfying this condition is at most

$$\binom{n}{r'}\binom{s}{r}(2k)^{r'-r'} \leq \exp(r'\log(n/r')) + r \log(s/r) + (r'-r')\log(2k))$$

$$\leq \exp((r'+r)\log n + (r-r')\log(2k)) = \exp(es),$$

where $e \leq (1+1/k)c/(1+1/k)\delta = c/\delta$. Moreover, for a given set $S$ and $S'$, let an arbitrary $\eta > 0$ by the Chernoff’s bound,\(^{11}\) we have

$$\Pr\left(e(S, S') \leq \frac{\eta cs}{2}\right) \leq \Pr\left(e(S, S') \leq \frac{\eta cs(n-s)}{n}\right)$$

$$\leq \exp\left(-\left(1-\eta\right)^2\frac{cs(n-s)}{n}\right) \leq \exp(-(1-\eta)^2cs/4).$$

Hence, let $F=\{s,n/2 \geq s > (1+k)\delta \log n/c\}$. We have

$$\Pr\left(\exists S,r < \frac{cs}{(1+\frac{1}{k})\delta \log n},r' < \frac{cs}{(1+k)\delta \log n},e(S, S') \leq \frac{\eta cs}{2}\right| F < \exp\left(-\left(1-\eta\right)^2c/4 - \epsilon\right)s),$$

and set $\eta < 1-2/\sqrt{\delta}$, which implies $(1-\eta)^2c/4-\epsilon < 0$. Let $F'=n\geq s > (1+k)\delta \log n/c$. Then
By Theorem 1.1 and (13), the algebraic connectivity gain of the small-world networks has the lower bound so that almost surely,

\[ \gamma_2(S) \geq \frac{3kc^2n^2 \log \log n}{2(k+1)^2(2k+1)\pi^2 \log^3 n}. \]

Hence, we prove Theorem 1.2.

Now we consider the synchronization problem. In order to prove Theorem 1.3, we need the following result.

**Lemma 2.4:** Let \( G(n,c,n) \) be a random graph and \( \Delta(G) \) be the maximum degree. Then as \( n \to \infty \), almost surely, we have

\[ \lambda_{\max} \leq 2\Delta(G). \]

**Proof:** Since \( \lambda_{\max} \) is an eigenvalue, let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) be a corresponding eigenvector of \( \lambda_{\max} \) and \( |x_i| = \max_j |x_j|, j = 1, \ldots, n \). Denote by \( L = \{i,j\} \) the Laplacian matrix. Then \( |\lambda_{\max} x_i| = |\sum_{j=1}^n \sum_{i,j} L_{ij} |x_j||x_i| \leq d_{i} |x_i| + \sum_{j \neq i} |x_j| \leq 2d_i |x_i| \). Hence, \( \lambda_{\max} \leq 2d_{\max} \leq 2\Delta \).

Now we are ready to characterize the size \( n \) of the synchronizable small-world network.

**Theorem 1.3:** Given a Pecora–Carroll type system and the critical value \( \beta \) for Pecora–Carroll associated master stability function, let \( S(n,c,k) \) be the small-world network with \( n \) nodes, which
is a union of an Erdős–Rényi random graph $G(n,c/n)$ and a $2k$ regular cycle. Then $S(n,c,k)$ is almost surely synchronizable if $ck/k+1 \geq 4 \log^2 n / \beta^{1/2} \log \log n$.

**Proof:** By Lemmas 2.4 and 2.2, and Theorem 1.1, we have $16(k+1)^2 \log^4 n / k^2 c^2 \log \log^2 n \leq \beta$. Hence, it is synchronizable if $ck/k+1 \geq 4 \log^2 n / \beta^{1/2} \log \log n$ and we finish the proof.

Theorem 1.3 shows that although the small-world networks can increase synchronizability, we should not expect it to be Pecora–Carroll synchronizable character for large amount of oscillators when the average shortcut per node is constant. 53

### III. SIMULATION RESULTS

The mathematical results require the condition $\log n \gg 1$. However, the numerical simulation shows that our lower bound also fits well for small $n$.

In Fig. 2, we run tests on two networks with parameters $n=500, k=1$ and $n=1000, k=2$ each with 100 samples. The connected probability $p=c/n$ with $c$ is chosen between 0.05 and 1. The dashed line represents our low bound in inequality (9) and the solid line is the mean value of algebraic connectivity for the small-world networks for different $p$, while every dot stands for a single realization. The parallel reflects a consistent tendency and the deviation comes from theoretical estimation (Fig. 3).

As to the synchronization bound, we compare tests on networks with parameters $n=300,500,1000$ and $k=1,2,3,5,10$, as Ref. 3 for $s_{\text{sync}}=\lambda_{\text{max}}/\lambda_2$. The constraint on $k$ is due to requirement of our method that $k$ and $c$ cannot be too large. The simulation also shows evidently consistent tendency.

### IV. CONCLUSION

In this paper, we show that the determinant measurement related to the consensus and synchronization improvement of the small-world networks is the Cheeger constant. We also present a lower bound for the algebraic connectivity and the algebraic connectivity gain of the small-world networks. These results explain why the convergence rate of solving consensus problems in the small-world networks is ultrafast, which was observed by Olfati-Saber 40 by simulation. We also prove the synchronizability increment in the small-world networks and determine the class of the small-world networks which is Barahona–Pecora synchronizable.
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