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Complete multipartite graphs are determined by their distance spectra ☆

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In this paper, we prove that the complete multipartite graphs are determined by their distance spectra, which confirms the conjecture proposed by Lin, Hong, Wang and Shu (2013) [7], although it is well known that the complete multipartite graphs cannot be determined by their adjacency spectra.

1. Introduction

In this paper, we only consider simple and undirected connected graphs. Let $G = (V(G), E(G))$ be a graph of order n with vertex set $V(G)$ and edge set $E(G)$. Let $d_G(v)$ and $N_G(v)$ denote the degree and neighbors of a vertex v, respectively. $D(G) = (a_{uv})_{n\times n}$

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denotes the *distance matrix* of G with $a_{uv} = d_G(u,v)$, where $d_G(u,v)$ is the distance between vertices u and v. The largest eigenvalue of $D(G)$, denoted by $\lambda(G)$, is called the *distance spectral radius* of G. The research for distance matrix can be dated back to the paper [6,5], which presents an interesting result that the determinant of the distance matrix of trees with order n is always $(-1)^{n-1}(n-1)2^{n-2}$, independent of the structure of the tree. Recently, the distance matrix of a graph has received increasing attention. For example, Liu [8] characterized the graphs with minimal spectral radius of the distance matrix in three classes of simple connected graphs with fixed vertex connectivity, matching number and chromatic number, respectively. Zhang [10] determined the unique graph with minimum distance spectral radius among all connected graphs with a given diameter. Bose, Nath and Paul [1] characterized the graph with minimal distance spectral radius among all graphs with the fixed number of pendent vertices.

Denote by $Sp(D(G))$ the set of all eigenvalues of $D(G)$ including the multiplicity. Two graphs G and G' are called *D-cospectral* if $Sp(D(G)) = Sp(D(G'))$. A graph G is determined by its D-spectra if $Sp(D(G)) = Sp(D(G'))$ implies $G \cong G'$. There are three excellent surveys [2–4] on that which graphs can be determined by their spectra. Lin et al. [7] characterized all the connected graphs with smallest eigenvalue being -2 and proposed the following conjecture:

Conjecture 1.1. *(See [7].) Let $G = K_{n_1,n_2,\dots,n_k}$ be a complete k-partite graph. Then G is determined by its D-spectrum.*

Moreover, they proved that the conjecture is true for $k = 2$. On the other hand, it is well known that complete multipartite graphs cannot be determined by their spectra. For example, $K_{1,4}$ is not determined by its adjacency spectra, since $K_{1,4}$ and the union of cycle of order 4 and an isolated vertex have the same spectrum but are not isomorphic. In this paper, we will give a positive answer to Conjecture 1.1.

2. Main results

Before presenting the proof of the conjecture, we need the following theorems and lemmas.

Theorem 2.1. *(See [7].) Let $D(G)$ be the distance matrix of a connected graph G. Then the smallest eigenvalue of $D(G)$ is equal to -2 with multiplicity $n-k$ if and only if G is a complete k-partite graph for $2 \leqslant k \leqslant n-1$.*

Theorem 2.2. *(See [7].) Let $G = K_{n_1,\dots,n_k}$ be a complete k-partite graph. Then the characteristic polynomial of $D(G)$ is*

$$P_D(\lambda) = (\lambda+2)^{n-k}\left[\prod_{i=1}^{k}(\lambda - n_i + 2) - \sum_{i=1}^{k} n_i \prod_{j=1,\, j\neq i}^{k}(\lambda - n_j + 2)\right].$$

On symmetric function, we follow notations in Chapter 7 of the book *Enumerative Combinatorics* [9]. For any given nonnegative vectors $a = (a_1, \ldots, a_k)$, $x = (x_1, \ldots, x_k)$, denote by $x^a = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$.

Definition 2.3. Let $x = (x_1, x_2, \ldots, x_k) \in R^k$. $f(x)$ is called the symmetric function of degree d if

$$f(x_1, \ldots, x_k) = \sum_{(a_1,\ldots,a_k)\in\mathcal{V}} c_{a_1,\ldots,a_k} x_1^{a_1} \ldots x_k^{a_k},$$

where $\mathcal{V}$ is the set of k dimensional nonnegative integer vectors, $c_{a_1,\ldots,a_k} \in R$ and $f(x)$ satisfies the following conditions:

(a) $f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k)$, $1 \leqslant i < j \leqslant k$;
(b) $d = \max_{a\in\mathcal{V}}\{a_1 + a_2 + \cdots + a_k\}$.

Let

$$\psi_t(x) = \sum_{j=1}^{k} x_j^t.$$

It is well known that

Theorem 2.4. *(See [9].) For any symmetric function $f(x)$, there exists a unique function $F(x) = \sum_{a\in\mathcal{V}} c_a x^a$, where $\mathcal{V}$ is the subset of k dimensional nonnegative integer vectors, such that $f(x) = F(\psi_1(x), \psi_2(x), \ldots, \psi_k(x))$.*

Lemma 2.5. *Let*

$$\begin{aligned} g(\lambda) &:= \prod_{i=1}^{k}(\lambda - n_i + 2) - \sum_{i=1}^{k} n_i \prod_{j=1,\, j\neq i}^{k} (\lambda - n_j + 2) \\ &:= \sum_{i=0}^{k} \xi_i \lambda^{k-i} \end{aligned}$$

with $\sum_{i=1}^{k} n_i = n$. Then there exists a unique function $F^{(i)}(y) = \sum_{a\in\mathcal{V}_i} c_a y^a$ with $y = (y_1, \ldots, y_i)$ such that

$$\xi_i = F^{(i)}\bigl(\psi_1(n_1, \ldots, n_k), \psi_2(n_1, \ldots, n_k), \ldots, \psi_i(n_1, \ldots, n_k)\bigr), \quad i = 1, \ldots, k,$$

where $\mathcal{V}_i$ is the subset of i dimensional nonnegative integer vectors. Moreover, there are some nonzero constants c_i such that $F^{(i)}(y_1, y_2, \ldots, y_i) - c_i y_i$ is a function on variables $y_1, \ldots, y_{i-1}$ for $i = 1, \ldots, k$, i.e., $F^{(i)}(y_1, y_2, \ldots, y_i) = c_i y_i + G^{(i)}(y_1, \ldots, y_{i-1})$.

Proof. Clearly, $\xi_0 = 1$, and $\xi_1 = -2\psi_1(n_1, \dots, n_k) + 2k$. For $1 < i \leqslant k$, it follows from the definition that

$$\begin{aligned}
\xi_i &= (-1)^i \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant k} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_i} - 2) \\
&\quad - \sum_{l=1}^{k} n_l (-1)^{i-1} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{i-1} \leqslant k,\, l \notin \{j_1, \dots, j_{i-1}\}} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2) \\
&= (-1)^i \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant k} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_i} - 2) \\
&\quad + (-1)^i \sum_{l=1}^{k} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{i-1} \leqslant k,\, l \notin \{j_1, \dots, j_{i-1}\}} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2)(n_l - 2) \\
&\quad + 2(-1)^i \sum_{l=1}^{k} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{i-1} \leqslant k,\, l \notin \{j_1, \dots, j_{i-1}\}} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2) \\
&= (-1)^i (i+1) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant k} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_i} - 2) \\
&\quad + (-1)^i 2(k+1-i) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{i-1} \leqslant k} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2).
\end{aligned}$$

The last equation holds because the element $(n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2)(n_{j_i} - 2)$ appears in $\sum_{l=1}^{k} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{i-1} \leqslant k,\, l \notin \{j_1, \dots, j_{i-1}\}} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2)(n_l - 2)$ if and only if for some $t \in \{1, 2, \dots, i\}$, $l = j_t$, moreover, $(n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2)$ appears in $\sum_{l=1}^{k} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{i-1} \leqslant k,\, l \notin \{j_1, \dots, j_{i-1}\}} (n_{j_1} - 2)(n_{j_2} - 2) \dots (n_{j_{i-1}} - 2)$ if and only if for any $t \in \{1, 2, \dots, i-1\}$, $l \neq j_t$. Hence ξ_i is a symmetric function of degree i on the variables $(n_1, \dots, n_k)$ for $i = 1, \dots, k$. Then, by Theorem 2.4, there exists a unique function $F^{(i)}(y) = \sum_{a \in \mathcal{V}_i} c_a y^a$ on $y = (y_1, \dots, y_i)$ such that

$$\xi_i = F^{(i)}\bigl(\psi_1(n_1, \dots, n_k), \psi_2(n_1, \dots, n_k), \dots, \psi_i(n_1, \dots, n_k)\bigr).$$

Moreover, the coefficient in single term of ξ_i with maximum degree i is $(-1)^i(i+1)$. Hence the coefficient of $F(y) = \sum_{a \in \mathcal{V}_i} c_a y^a$ on variable y_i is a constant. Therefore the assertion holds. $\square$

Lemma 2.6. *Let*

$$\begin{aligned}
g(\lambda) &:= \prod_{i=1}^{k} (\lambda - n_i + 2) - \sum_{i=1}^{k} n_i \prod_{j=1,\, j \neq i}^{k} (\lambda - n_j + 2) \\
&:= \sum_{i=0}^{k} \xi_i \lambda^{k-i}
\end{aligned}$$

with $\sum_{i=1}^{k} n_i = n$. Then there exists a unique function $H^{(i)}(y) = \sum_{a \in \mathcal{V}_i} c_a y^a$ such that $\psi_i(n_1, \ldots, n_k) = H^{(i)}(\xi_1(n_1, \ldots, n_k), \xi_2(n_1, \ldots, n_k), \ldots, \xi_i(n_1, \ldots, n_k))$, where $\mathcal{V}_i$ is the subset of i dimensional nonnegative integer vectors, $i = 1, \ldots, k$.

Proof. We prove the assertion by the induction on i. For $i = 1$, clearly, there exists a function $H^{(1)}(y) = H^{(1)}(y_1) = -\frac{1}{2}y_1 + k$ such that $\psi_1(n_1, \ldots, n_k) = -\frac{1}{2}\xi_1 + k$ since $\xi_1 = -2\psi_1 + 2k$. Assume that the assertion holds for $i = 1, \ldots, t$. In other words, there exist $H^{(1)}(y), \ldots, H^{(t)}(y)$ such that $\psi_j(n_1, \ldots, n_k) = H^{(j)}(\xi_1, \ldots, \xi_j)$ for $j = 1, \ldots, t$. For $i = t + 1$, by Lemma 2.5, there exists a unique function $F^{(t+1)}(y) = \sum_{a \in \mathcal{V}_{t+1}} c_a y^a$ with $y = (y_1, \ldots, y_{t+1})$ such that

$$\begin{aligned}\xi_{t+1}(n_1, \ldots, n_k) &= F^{(t+1)}\big(\psi_1(n_1, \ldots, n_k), \ldots, \psi_{t+1}(n_1, \ldots, n_k)\big) \\ &= c_{t+1}\psi_{t+1}(n_1, \ldots, n_k) + G^{(t+1)}\big(\psi_1(n_1, \ldots, n_k), \ldots, \psi_t(n_1, \ldots, n_k)\big).\end{aligned}$$

So

$$\begin{aligned}\psi_{t+1}(n_1, \ldots, n_k) = -\frac{1}{c_{t+1}}\big(&\xi_{t+1}(n_1, \ldots, n_k) \\ &- G^{(t+1)}\big(\psi_1(n_1, \ldots, n_k), \ldots, \psi_t(n_1, \ldots, n_k)\big)\big).\end{aligned}$$

Then there exists a unique function $H^{(t+1)}(y) = \sum_{a \in \mathcal{V}} c_a y^a$ such that $\psi_{t+1}(n_1, \ldots, n_k) = H^{(i)}(\xi_1(n_1, \ldots, n_k), \xi_2(n_1, \ldots, n_k), \ldots, \xi_{t+1}(n_1, \ldots, n_k))$. So the assertion holds. □

Lemma 2.7. *For any given vector $z = (z_1, z_2, \ldots, z_k)$, if the equations system*

$$\psi_t(x) = z_t, \quad i.e., \sum_{i=1}^{k} x_i^t = z_t, \ t = 1, \ldots, k$$

with unknown variables $x_1, \ldots, x_k$ has two nonnegative solutions with $x = (x_1, \ldots, x_k) = (a_1, \ldots, a_k)$ and $x = (b_1, \ldots, b_k)$, then there is a permutation σ such that $a = (a_1, \ldots, a_k) = (b_{\sigma(1)}, \ldots, b_{\sigma(n)})$.

Proof. We prove the assertion by the induction on k. For $k = 2$, it is easy to see that the system $x_1 + x_2 = z_1$, $x_1^2 + x_2^2 = z_2$ with unknown variables x_1, x_2 has two solutions with $(x_1, x_2) = (\frac{z_1 + \sqrt{4z_2 - z_1^2}}{2}, \frac{z_1 - \sqrt{4z_2 - z_1^2}}{2})$ and $(x_1, x_2) = (\frac{z_1 - \sqrt{4z_2 - z_1^2}}{2}, \frac{z_1 + \sqrt{4z_2 - z_1^2}}{2})$. Hence the assertion holds. Assume that the assertion holds for the number less than k. Moreover, assume that the equations system

$$\psi_t(x) = z_t, \quad i = 1, \ldots, k$$

with unknown variables $x_1, \ldots, x_k$ has two nonnegative solutions with $x = (x_1, \ldots, x_k) = (a_1, \ldots, a_k)$ and $x = (b_1, \ldots, b_k)$. Then we claim that there exist $1 \leqslant i, j \leqslant k$ such that

$a_i = b_j$. In fact, otherwise, without loss of generality, we can assume that $a_1 = \cdots = a_p > a_{p+1} \geqslant \cdots \geqslant a_k$ and $a_1 > b_1$. Then $\psi_t(a) = \psi_t(b)$ for $t = 1, \dots, k$. Since $\psi_t(x)$ is symmetric function, by Theorem 2.4, there exists a unique function $F_t(x)$ such that $\psi_t(x) = F_t(\psi_1, \dots, \psi_k)$ for all $t \geqslant 1$. Hence $\psi_t(a) = \psi_t(b)$ holds for all $t \geqslant 1$. Then

$$p + \sum_{i=p+1}^{k} \left(\frac{a_i}{a_1}\right)^t = \sum_{i=1}^{k} \left(\frac{b_i}{a_1}\right)^t,$$

for all $t \geqslant 1$. Therefore

$$p + \sum_{i=p+1}^{k} \lim_{t\to\infty} \left(\frac{a_i}{a_1}\right)^t = \lim_{t\to\infty} \sum_{i=1}^{k} \left(\frac{b_i}{a_1}\right)^t,$$

which implies that $p = 0$. It is impossible. So the claim holds. Therefore there exist $1 \leqslant i, j \leqslant k$ such that $a_i = b_j$. Moreover, it is easy to see that the equations system $\sum_{l=1}^{k-1} x_l^t = z_t - a_i^t$ for $t = 1, \dots, k-1$ has two solutions $(x_1, \dots, x_{k-1}) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k)$ and $(x_1, \dots, x_{k-1}) = (b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_k)$. By the induction hypothesis, there exists a permutation σ_1 such that $(b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_k) = (a_{\sigma_1(1)}, \dots a_{\sigma_1(i-1)}, a_{\sigma_1(i+1)}, \dots, a_{\sigma_1(k)})$. Hence the assertion holds. □

Now we are ready to present our main theorem.

Theorem 2.8. *Let $G = K_{n_1,n_2,\dots,n_k}$ be a complete k-partite graph with $\sum_{i=1}^{k} n_i = n$. Then G is determined by its D-spectrum.*

Proof. Let G' be any simple graph with $Sp(D(G')) = Sp(D(G))$. Since G is complete k-partite graph, by Theorem 2.1, G' is complete k-partite graphs, since the smallest eigenvalue of $D(G)$ and $D(G')$ is equal to -2 with multiplicity $n-k$. Hence assume that $G' = K_{p_1,\dots,p_k}$ with $\sum_{i=1}^{k} p_i = n$. Let the function

$$\begin{aligned} f_i(x_1, \dots, x_k) :=& (-1)^i (i+1) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant k} (x_{j_1} - 2)(x_{j_2} - 2) \dots (x_{j_i} - 2) \\ &+ (-1)^i 2(k+1-i) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{i-1} \leqslant k} (x_{j_1} - 2)(x_{j_2} - 2) \dots (x_{j_{i-1}} - 2), \end{aligned}$$

for $i = 1, \dots, k$. By Lemma 2.6, there exists a unique function $H^{(i)}(y)$ with $y = (y_1, \dots, y_i)$ such that

$$\psi_i(x) = \sum_{j=1}^{k} x_j^i = H^{(i)}\big(f_1(x_1, \dots, x_k), \dots, f_i(x_1, \dots, x_k)\big)$$

for $i = 1, \dots, k$. Since the coefficients of the characteristic polynomial of distance matrices of $D(G)$ and $D(G')$ are equal, we have $g_G(\lambda) = g_{G'}(\lambda)$. Hence $f_i(n_1, \dots, n_k) = f_i(p_1, \dots, p_k) := \theta_i$. Therefore

$$\psi_i(n_1, \dots, n_k) = H^{(i)}\big(f_1(n_1, \dots, n_k), \dots, f_i(n_1, \dots, n_k)\big) = H^{(i)}(\theta_1, \dots, \theta_k)$$

and

$$\psi_i(p_1, \dots, p_k) = H^{(i)}\big(f_1(p_1, \dots, p_k), \dots, f_i(p_1, \dots, p_k)\big) = H^{(i)}(\theta_1, \dots, \theta_k)$$

for $i = 1, \dots, k$. Hence the equations system $\psi_i(x_1, \dots, x_k) = H^{(i)}(\theta_1, \dots, \theta_k)$, $i = 1, \dots, k$ with unknown variables $x_1, \dots, x_k$ has two solutions $(x_1, \dots, x_k) = (n_1, \dots, n_k)$ and $(x_1, \dots, x_k) = (p_1, \dots, p_k)$. By Lemma 2.7, there exists a permutation σ such that $(n_1, \dots, n_k) = (p_{\sigma(1)}, \dots, p_{\sigma(k)})$. Hence $G' = K_{p_1, \dots, p_k} = K_{n_1, \dots, n_k}$. Hence the assertion holds. □

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