Asymptotic incidence energy of lattices

Jia-Bao Liu\textsuperscript{a,b}, Xiang-Feng Pan\textsuperscript{a,*}

\textsuperscript{a} School of Mathematical Sciences, Anhui University, Hefei 230601, PR China
\textsuperscript{b} Department of Public Courses, Anhui Xinhua University, Hefei 230088, PR China

HIGHLIGHTS

- We propose the incidence energy per vertex problem for lattice systems.
- The explicit asymptotic values of $\mathcal{E}(G)$ for various lattices are obtained.
- We deduce $\mathcal{E}(G)$ of many types of lattices is independent of various boundary conditions.

ABSTRACT

The energy of a graph $G$ arising in chemical physics, denoted by $\mathcal{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of $G$. As an analogue to $\mathcal{E}(G)$, the incidence energy $\mathcal{I}(G)$, defined as the sum of the singular values of the incidence matrix of $G$, is a much studied quantity with well known applications in chemical physics. In this paper, based on the results by Yan and Zhang (2009), we propose the incidence energy per vertex problem for lattice systems, and present the closed-form formulae expressing the incidence energy of the hexagonal lattice, triangular lattice, and $3\times4^2$ lattice, respectively. Moreover, we show that the incidence energy per vertex of lattices is independent of the toroidal, cylindrical, and free boundary conditions. In particular, the explicit asymptotic values of the incidence energy in these lattices are obtained by utilizing the applications of analysis approach with the help of calculational software.

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1. Introduction

Lattices have several attractive features that make them interesting candidates for use in matter physics. The resistance distances of lattices were well studied on the basis of electrical network theory in Refs. [1–3]. A general problem of interest in physics, chemistry and mathematics is the calculations of various energies of lattices [2,4–6], since the energies can be used to estimate the total $\pi$-electron energy in conjugated hydrocarbons [7]. Another energy application is dimer problem in statistical physics, the problem considers the molecular free energy per dimer, which mimics the adsorption of diatomic molecules on a surface [4]. Historically in lattice statistics, the hexagonal lattice, triangular lattice, and $3\times4^2$ lattice have attracted the most attention [1,4,8–11].

Throughout the paper all graphs considered are simple and undirected. Let $A(G)$ be the adjacency matrix, the eigenvalues of $G$ are the eigenvalues of its adjacency matrix $A(G)$ [12]. These eigenvalues, arranged in a non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$. In 1976 Gutman introduced the concept of energy $\mathcal{E}(G)$ [7] for a simple graph $G$, which is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. 

* Corresponding author. Tel.: +86 551 63861313.
E-mail address: xfpan@ahu.edu.cn (X.-F. Pan).

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Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denote by $|V(G)|$ and $|E(G)|$ the numbers of vertices and edges, respectively. The degree of a vertex $v_i$ is the number of edges incident to $v_i$ in a graph $G$. Let $I(G)$ be the (vertex–edge) incidence matrix of the graph $G$. The $(i, j)$-entry of $I(G)$ is 1 if $v_i$ is incident with $e_j$ and 0 otherwise. (In what follows, we use the unit matrix of order $n$ to be denoted by $E_n$ to avoid confusion with the incidence matrix.) Inspired by Nikiforov’s idea [13], in 2009 Jooyandeh et al. [14] introduced the concept of incidence energy $\mathcal{E}(G)$ of a graph $G$, defining it as the sum of the singular values of the incidence matrix $I(G)$, that is, $\mathcal{E}(G) = \sum_{i=1}^{n} \sigma_i$, where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of the incidence matrix $I(G)$. As an analogue to $\mathcal{E}(G)$, the incidence energy $\mathcal{E}(G)$ is a novel topological index, for more work on $\mathcal{E}(G)$, the readers are referred to Refs. [14–17] and recent articles [18–23].

Let $D(G)$ be the diagonal matrix of vertex degrees of $G$, the Laplacian eigenvalues of $G$ is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix is $Q(G) = D(G) + A(G)$. It is well known that $L(G)$ and $Q(G)$ are symmetric and positive semidefinite, then we denote the eigenvalues of $L(G)$ and $Q(G)$ by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0$, respectively. It is well known that the spectra of $L(G)$ and $Q(G)$ coincide if and only if the graph $G$ is bipartite [24,25].

Another well known fact is the identity [12]: $I(G)I(G)^t = A(G) + D(G)$, i.e., $I(G)I(G)^t = Q(G)$. Its immediate consequence is that $\sigma_i = \sqrt{q_i}$. Therefore,

$$\mathcal{E}(G) = \sum_{i=1}^{n} \sqrt{q_i}.$$  

For the convenience to following description, we use this expression of the incidence energy $\mathcal{E}(G)$.

The Laplacian-energy-like invariant of $G$, $\mathcal{LLE}(G)$ for short, is defined as $\mathcal{LLE}(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$. The concept of $\mathcal{LLE}(G)$ was first introduced by J. Liu and B. Liu ([26], 2008), where it showed that $\mathcal{LLE}(G)$ has similar features as the graph energy $\mathcal{E}(G)$ [27].

These indexes have attracted extensive attention due to its wide applications in physics, chemistry, graph theory, etc. [1,2,6]. In Ref. [4] the energy $\mathcal{E}(G)$ of some lattices was studied. It is an interesting problem to investigate the incidence energy of some lattices with various boundary conditions. Motivated by results above, we consider the problem of computation of the $\mathcal{E}(G)$ for the lattices with various conditions in the paper.

The rest of the paper is organized as follows. In Section 2, we propose the asymptotic incidence energy of square lattices and give the related explanations. We provide a detailed derivation of the asymptotic incidence energy change due to edge deletion in Section 3. The asymptotic incidence energy of hexagonal, triangular, and $3^44^2$ lattices with various boundary conditions are investigated in Section 4. We present concluding remarks and conclude the paper in Section 5.

### 2. Asymptotic incidence energy of square lattices

At the beginning of this section, we first introduce some notations in graph theory, which will be used in following discussion.

Given graphs $G$ and $H$ with vertex sets $U$ and $V$, the Cartesian product $G \square H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \square H$ is the Cartesian product $U \times V$; and any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \square H$ if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $u$ is adjacent to $v$ in $G$ [28].

Let $P_m \square P_n$, $P_m \square C_n$, and $C_m \square C_n$ denote the square lattices with free, cylindrical and toroidal boundary conditions, respectively, where $P_m$ and $C_n$ denote the path and the cycle with $n$ vertices. Obviously, $P_m \square P_n$ is a sequence of spanning subgraphs of the sequence $P_m \square C_n$ of finite graphs, and $P_m \square C_n$ is a sequence of spanning subgraphs of the sequence $C_m \square C_n$ of finite graphs. Particularly,

$$\lim_{m,n \to \infty} \frac{|\{v \in V(P_m \square P_n) : d_{P_m \square P_n}(v) = d_{C_m \square C_n}(v)\}|}{|V(C_m \square C_n)|} = 1,$$

that is, almost all vertices of $C_m \square C_n$ and $P_m \square C_n$ (resp. $C_m \square C_n$ and $P_m \square P_n$) have the same degrees.

Let $G_1, G_2$ be graphs with adjacency matrices $A_1, A_2$ degree matrices $D_1, D_2$ and signless Laplacians $Q_1, Q_2$, respectively. Then $Q_1 = A_1 + D_1$, $Q_2 = A_2 + D_2$. It is known that $G_1 \square G_2$ has the adjacency matrix $A(G_1 \square G_2) = A_1 \otimes A_2 + E_1 \otimes A_2$, where $A \otimes E$ denotes the tensor product of two matrices $A$ and $E$, and $E_1, E_2$ are identity matrices with the same order as $G_1, G_2$, respectively. In a quite analogous manner, $G_1 \square G_2$ has the signless Laplacian $Q(G_1 \square G_2) = (A_1 + D_1) \otimes E_2 + E_1 \otimes (A_2 + D_2)$, and if $q_1(G_1), q_1(G_2)$ are $Q$-eigenvalues of $G_1, G_2$, then the signless Laplacian eigenvalues of $G_1 \square G_2$ are all possible sums $q_1(G_1) + q_1(G_2)$, as noted in Ref. [25]. It is well known that the singular Laplacian eigenvalues of a path $P_n$ and a cycle $C_n$ are $2 - 2\cos\frac{\pi i}{n}$ ($i = 0, 1, \ldots, n - 1$) [25] (See page 25) and $2 + 2\cos\frac{\pi i}{n}$ ($j = 0, 1, \ldots, n - 1$) [29] (See page 3323), respectively.

Consequently, the signless Laplacian eigenvalues of $P_m \square P_n$ (resp. $P_m \square C_n$ and $C_m \square C_n$) are $4 - 2\cos\frac{\pi i}{m} - 2\cos\frac{\pi j}{n}$, $i = 0, 1, \ldots, m - 1$; $j = 0, 1, \ldots, n - 1$ (resp. $4 - 2\cos\frac{\pi i}{m} + 2\cos\frac{\pi j}{n}$, $i = 0, 1, \ldots, m - 1$; $j = 0, 1, \ldots, n - 1$ and $4 + 2\cos\frac{\pi i}{m} + 2\cos\frac{\pi j}{n}$, $i = 0, 1, \ldots, m - 1$; $j = 0, 1, \ldots, n - 1$).
Hence the asymptotic incidence energy per vertex of $P_m \square P_n$, $P_m \square C_n$, and $C_m \square C_n$ are defined as

1. \[ \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{I}(P_m \square P_n)}{|V(P_m \square P_n)|} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sqrt{4 - 2 \cos \frac{i\pi}{m} - 2 \cos \frac{j\pi}{n}} \]
   \[ = \int_0^1 \int_0^1 \sqrt{4 - 2 \cos \pi x - 2 \cos \pi y} \, dx \, dy \]
   \[ = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sqrt{4 - 2 \cos x - 2 \cos y} \, dx \, dy \approx 1.9162; \]

2. \[ \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{I}(P_m \square C_n)}{|V(P_m \square C_n)|} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sqrt{4 - 2 \cos \frac{i\pi}{m} + 2 \cos \frac{2\pi j}{n}} \]
   \[ = \int_0^1 \int_0^1 \sqrt{4 - 2 \cos \pi x + 2 \cos \frac{2\pi y}{m}} \, dx \, dy \]
   \[ = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{4 - 2 \cos x + 2 \cos \frac{2\pi y}{m}} \, dx \, dy \approx 1.9162; \]

3. \[ \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{I}(C_m \square C_n)}{|V(C_m \square C_n)|} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sqrt{4 + 2 \cos \frac{2\pi i}{m} + 2 \cos \frac{2\pi j}{n}} \]
   \[ = \int_0^1 \int_0^1 \sqrt{4 + 2 \cos 2\pi x + 2 \cos \frac{2\pi y}{m}} \, dx \, dy \]
   \[ = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{4 + 2 \cos x + 2 \cos \frac{2\pi y}{m}} \, dx \, dy \approx 1.9162. \]

The numerical integration value in last line is calculated with the software MATLAB. The equality of the three asymptotic energies for the free, cylindrical and toroidal boundary conditions does not require numerical calculations to be established. In fact, it follows immediately from the analytic expressions. However, we use computer software MATLAB in order to obtain the above explicit numerical integration values. Hence, $P_m \square P_n$, $P_m \square C_n$ and $C_m \square C_n$, have the same asymptotic incidence energy $\mathcal{I}(P_m \square P_n) = \mathcal{I}(P_m \square C_n) = \mathcal{I}(C_m \square C_n) \approx 1.9162mn$ as $m, n$ approach infinity.

**Remark 2.1.** The asymptotic incidence energy $\mathcal{I}(G)$ of square lattices is independent on the three boundary conditions, i.e., the free, cylindrical and toroidal boundary conditions.

The phenomenon above is not accidental. We propose a detailed derivation of the asymptotic incidence energy change due to edge deletion and the related explanations in Section 3.

### 3. Graph asymptotic incidence energy change due to edge deletion

Let us recall some results first. The authors of Ref. [1] proved the following theorem.

**Theorem 3.1 ([1]).** Let $G$ be a graph on $n$ vertices with $m$ edges, then

\[ \sqrt{2m} \leq \mathcal{I}(G) \leq \sqrt{2mn}. \]

Moreover, the left equality holds if and only if $m \leq 1$. On the other hand the right equality holds if and only if $m = 0$.

According to Theorem 3.1, noticing that an $r$-regular graph has $m = \frac{rn}{2}$ edges, one can easily get the following result.

**Lemma 3.2.** Let $G$ be an $r$-regular graph on $n$ vertices with $m$ edges, then

\[ \sqrt{2m} \leq \mathcal{I}(G) \leq 2m\sqrt{\frac{1}{r}} \leq 2m. \]

Furthermore, the right equality holds if and only if $r = 1$.

If $H$ is a subgraph of $G$, then $G - E(H)$ denotes the subgraph obtained from $G$ by deleting all edges in $H$. The authors of Refs. [30,31] first investigated how the energy of a graph changes when edges are removed. They found the following Lemma 3.3.

**Lemma 3.3 ([30,31]).** Let $H$ be an induced subgraph of a graph $G$. Then

\[ \mathcal{E}(G) - \mathcal{E}(H) \leq \mathcal{E}(G - E(H)) \leq \mathcal{E}(G) + \mathcal{E}(H). \]
In addition, the authors of Ref. [4] obtained the following result.
\[ |\varepsilon(G) - \varepsilon(H)| \leq \varepsilon(G - E(H)) \leq \varepsilon(G) + \varepsilon(H). \]

With a similar approach, one can prove the following result.

**Theorem 3.5.** Suppose \( \{G_n\} \) and \( \{H_n\} \) are two sequences of graphs such that
\[
\lim_{n \to \infty} \frac{\Delta(G_n, H_n)}{\mathcal{I}(G_n)} = 0.
\]
Then
\[
\lim_{n \to \infty} \frac{\mathcal{I}(H_n)}{\mathcal{I}(G_n)} = 1.
\]

**Proof.** Let \( F_n \) be the subgraph induced by \( E(G_n) \cap E(H_n) \). Note that
\[
\frac{\mathcal{I}(H_n)}{\mathcal{I}(G_n)} - 1 = \frac{\mathcal{I}(H_n) - \mathcal{I}(G_n)}{\mathcal{I}(G_n)} = \frac{\mathcal{I}(H_n) - \mathcal{I}(F_n) + \mathcal{I}(F_n) - \mathcal{I}(G_n)}{\mathcal{I}(G_n)} \leq \frac{\mathcal{I}(G_n) - \mathcal{I}(F_n)}{\mathcal{I}(G_n)} + \frac{\mathcal{I}(H_n) - \mathcal{I}(F_n)}{\mathcal{I}(G_n)}.
\]
Based on Lemmas 3.2 and 3.4,
\[
|\mathcal{I}(G_n) - \mathcal{I}(F_n)| \leq |\varepsilon(G_n - E(F_n))| \leq 2 |E(G_n)| - 2 |E(F_n)|,
\]
\[
|\mathcal{I}(H_n) - \mathcal{I}(F_n)| \leq |\varepsilon(H_n - E(F_n))| \leq 2 |E(H_n)| - 2 |E(F_n)|.
\]
Consequently,
\[
\frac{\mathcal{I}(H_n)}{\mathcal{I}(G_n)} - 1 \leq \frac{2 \Delta(G_n, H_n)}{\mathcal{I}(G_n)}.
\]
This implies the theorem. \( \Box \)

**Theorem 3.6.** Let \( \{G_n\} \) be a sequence of finite simple graphs with bounded average degree such that
\[
\lim_{n \to \infty} |V(G_n)| = \infty, \quad \lim_{n \to \infty} \frac{\mathcal{I}(G_n)}{|V(G_n)|} = h \neq 0.
\]
Let \( \{H_n\} \) be a sequence of spanning subgraphs of \( \{G_n\} \) such that
\[
\lim_{n \to \infty} \frac{|v \in V(H_n) : d_{H_n}(v) = d_{G_n}(v)|}{|V(G_n)|} = 1.
\]
Then
\[
\lim_{n \to \infty} \frac{\mathcal{I}(H_n)}{|V(G_n)|} = h.
\]
That is, \( G_n \) and \( H_n \) have the same asymptotic incidence energy.

A direct sequence of **Theorem 3.6** is that \( P_n \sqcup P_n \), \( P_n \sqcup C_n \), and \( C_n \sqcup C_n \) have the same asymptotic incidence energy which is shown in the introduction. More generally, by **Theorem 3.6**, we have

**Corollary 3.7.** Suppose \( G_i = P_n \) or \( G_i = C_n \), \( i = 1, 2, \ldots, k \), and \( k \) is a constant. If \( n \) is sufficiently large, then the asymptotic incidence energy of the \( k \)-dimensional lattices
\[
\mathcal{I}(G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k) \approx \frac{\pi^k}{\pi^k} \int_0^\pi \int_0^\pi \cdots \int_0^\pi \sqrt{\sum_{i=1}^k (2 + 2 \cos 2\pi i)} \, dx_1 \, dx_2 \cdots dx_k.
\]
Remark 3.8. A simple method has been presented, which can be used to calculate the asymptotic incidence energy of a graph with bounded average degree based on Theorem 3.6. Let \(\{G_n\}\) be a sequence of finite simple graphs with bounded average degree. Obviously, it is not easy to calculate its asymptotic incidence energy directly. However, we can find a graph \(H_n\) with bounded average degree, which satisfies \(|V(G_n)| = |V(H_n)|\) and almost all vertices of \(G_n\) and \(H_n\) have the same degrees. If we can compute the asymptotic incidence energy of \(H_n\) directly, then we can obtain the asymptotic incidence energy of \(G_n\) by Theorem 3.6.

4. Asymptotic incidence energy of some lattices

In this section, we obtain the asymptotic formulae of incidence energies of hexagonal, triangular, and \(3^3.4^2\) lattices with toroidal, cylindrical, and free boundary conditions.

4.1. The hexagonal lattice

Our notation for the hexagonal lattices follows Refs. [2,4,11]. The hexagonal lattices with toroidal, cylindrical and free boundary conditions, denoted by \(H^t(n, m)\), \(H^c(n, m)\), and \(H^f(n, m)\) are illustrated in Fig. 1, respectively, where \((a_1, b_1), (a_2, b_2), \ldots, (a_{m+1}, b_{m+1}); (a_1, c_1^1), (c_1, c_2^1), (c_2, c_3^1), \ldots, (c_{n-1}, c_n^1), (c_n, b_{m+1})\) are edges in \(H^t(n, m)\), and \((a_1, b_1), (a_2, b_2), \ldots, (a_{m+1}, b_{m+1})\) are edges in \(H^c(n, m)\) (see Fig. 1(b)). The hexagonal lattice \(H^f(n, m)\) with free boundary condition is obtained by deleting edges \((a_1, b_1), (a_2, b_2), \ldots, (a_{m+1}, b_{m+1})\) from \(H^c(n, m)\) (see Fig. 1(c)).

Theorem 4.1. For the hexagonal lattices \(H^t(n, m)\), \(H^c(n, m)\) and \(H^f(n, m)\) with toroidal, cylindrical, and free boundary conditions,

\[
\lim_{m \to \infty, n \to \infty} \frac{\mathcal{J}(H^t(n, m))}{2(m+1)(n+1)} = \lim_{m \to \infty, n \to \infty} \frac{\mathcal{J}(H^c(n, m))}{2(m+1)(n+1)} = \lim_{m \to \infty, n \to \infty} \frac{\mathcal{J}(H^f(n, m))}{2(m+1)(n+1)}
\]

\[
= \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{3 + \sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} \, dx \, dy
\]

\[
+ \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{3 - \sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} \, dx \, dy
\]

\[\approx 1.6437.
\]

that is, the hexagonal lattices \(H^t(n, m)\), \(H^c(n, m)\) and \(H^f(n, m)\) with toroidal, cylindrical, and free boundary conditions have the same asymptotic incidence energy \(\mathcal{J}(H^t(n, m)) = \mathcal{J}(H^c(n, m)) = \mathcal{J}(H^f(n, m)) \approx 3.2875(m+1)(n+1)\).

Proof. By definitions of \(H^t(n, m)\), \(H^c(n, m)\) and \(H^f(n, m)\), one knows that \(H^c(n, m)\) and \(H^f(n, m)\) are spanning subgraphs of \(H^t(n, m)\). Furthermore, almost all vertices of \(H^t(n, m)\), \(H^c(n, m)\) and \(H^f(n, m)\) have degree 3. Hence, by Theorem 3.6,

\[
\lim_{m \to \infty, n \to \infty} \frac{\mathcal{J}(H^t(n, m))}{2(m+1)(n+1)} = \lim_{m \to \infty, n \to \infty} \frac{\mathcal{J}(H^c(n, m))}{2(m+1)(n+1)} = \lim_{m \to \infty, n \to \infty} \frac{\mathcal{J}(H^f(n, m))}{2(m+1)(n+1)}
\]

It suffices to prove that

\[
\lim_{m \to \infty, n \to \infty} \frac{\mathcal{J}(H^t(n, m))}{2(m+1)(n+1)} = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{3 + \sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} \, dx \, dy
\]

\[
+ \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{3 - \sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} \, dx \, dy
\]

\[\approx 1.6437.
\]

The Laplacian eigenvalues of \(H^t(m, n)\) can be found in Ref. [2]. Note that \(H^t(m, n)\) is 3-regular lattices, the signless Laplacian matrix \(Q(H^t(m, n))\) of \(H^t(m, n)\) is similar to the block diagonal matrix whose diagonal blocks are

\[
B_{ij} = \begin{cases} 3 & \text{if } i = j = 0, 1, \ldots, n; j = 0, 1, \ldots, m, \\
1 + \omega_{n+1}^{i} + \omega_{m+1}^{j} & \text{otherwise},
\end{cases}
\]

where \(\omega_k = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m}\), \(i = 0, 1, \ldots, n; j = 0, 1, \ldots, m\).

Hence the signless Laplacian eigenvalues of \(H^t(m, n)\) are

\[
3 \pm \sqrt{3 + 2\cos \frac{2\pi i}{m+1} + 2\cos \frac{2\pi j}{n+1} + 2\cos \left( \frac{2\pi i}{m+1} + \frac{2\pi j}{n+1} \right)}, \quad i = 0, 1, \ldots, m; j = 0, 1, \ldots, n.
\]
Let \( a_i = \frac{2\pi i}{m+1}, \beta_j = \frac{2\pi j}{n+1}, i = 0, 1, \ldots, m; j = 0, 1, \ldots, n \). By the definition of the incidence energy, the incidence energy of \( H^i(m, n) \) can be expressed as

\[
\mathcal{E}(H^i(m, n)) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sqrt{3 - \sqrt{3 + 2\cos(\alpha_i) + 2\cos(\beta_j) + 2\cos(\alpha_i + \beta_j)}} \\
+ \sum_{i=0}^{m} \sum_{j=0}^{n} \sqrt{3 + \sqrt{3 + 2\cos(\alpha_i) + 2\cos(\beta_j) + 2\cos(\alpha_i + \beta_j)}}.
\]

The result given below follows immediately from the above equality.

\[
\lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{E}(H^i(m, n))}{2(m+1)(n+1)} = \frac{1}{2} \int_{0}^{1} \int_{0}^{1} \sqrt{3 + \sqrt{3 + 2\cos(2\pi x) + 2\cos(2\pi y) + 2\cos(2\pi(x+y))}} \, dx \, dy \\
+ \frac{1}{2} \int_{0}^{1} \int_{0}^{1} \sqrt{3 - \sqrt{3 + 2\cos(2\pi x) + 2\cos(2\pi y) + 2\cos(2\pi(x+y))}} \, dx \, dy \\
= \frac{1}{8\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \sqrt{3 + \sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} \, dx \, dy \\
+ \frac{1}{8\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \sqrt{3 - \sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} \, dx \, dy \\
\approx 1.6437.
\]

Consequently, for the hexagonal lattices \( H^i(n, m), H^c(n, m) \) and \( H^f(n, m) \) with toroidal, cylindrical, and free boundary conditions, these lattices have the same asymptotic incidence energy \( \mathcal{E}(H^i(m, n)) = \mathcal{E}(H^c(m, n)) = \mathcal{E}(H^f(m, n)) \approx 3.2875(m+1)(n+1) \).

4.2. The triangular lattice

Our notation for the triangular lattice follows Refs. [4,11]. The triangular lattices with toroidal, cylindrical and free boundary conditions, denoted by \( T^t(n, m), T^c(n, m), \) and \( T^f(n, m) \) are illustrated in Fig. 2, respectively, where \((a_1, a_2^t), (a_2, a_3^t), \ldots, (a_m, a_m^t); (b_1, b_1^t), (b_2, b_2^t), \ldots, (b_n, b_n^t); (b_2, b_3^t), (b_3, b_4^t), \ldots, (b_n, b_{n-1}^t), (b_1, b_n^t) = (a_1, a_m^t); (a_2, a_3^t), (a_3, a_4^t), \ldots, (a_m, a_{m-1}^t)\) are edges in \( T^t(n, m)\). The triangular lattice with cylindrical boundary condition, denoted by \( T^c(n, m), \)

![Image of hexagonal lattices](image-url)

**Fig. 1.** (a) The hexagonal lattice \( H^t(n, m) \) with toroidal boundary condition; (b) the hexagonal lattice \( H^c(n, m) \) with cylindrical boundary condition; (c) the hexagonal lattice \( H^f(n, m) \) with free boundary condition.

![Image of triangular lattices](image-url)

**Fig. 2.** (a) The triangular lattice \( T^t(n, m) \) with toroidal boundary condition; (b) the triangular lattice \( T^c(n, m) \) with cylindrical boundary condition; (c) the triangular lattice \( T^f(n, m) \) with free boundary condition.
Theorem 4.2. For the triangular lattices $T^t(n, m)$, $T^c(n, m)$ and $T^f(n, m)$ with toroidal, cylindrical, and free boundary conditions,

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^t(n, m))}{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^c(n, m))}{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^f(n, m))}{mn}$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{6 + 2 \cos x + 2 \cos y + 2 \cos(x + y)} \, dx \, dy \approx 2.4020,$$

that is, the triangular lattices $T^t(n, m)$, $T^c(n, m)$ and $T^f(n, m)$ with toroidal, cylindrical, and free boundary conditions have the same asymptotic incidence energy $\mathcal{J} \mathcal{E}(T^t(n, m)) = \mathcal{J} \mathcal{E}(T^c(n, m)) = \mathcal{J} \mathcal{E}(T^f(n, m)) \approx 2.4020mn$.

**Proof.** By definitions of $T^t(n, m)$, $T^c(n, m)$ and $T^f(n, m)$, one knows that $T^c(n, m)$ and $T^f(n, m)$ are spanning subgraphs of $T^t(n, m)$. Moreover, almost all vertices of $T^t(n, m)$, $T^c(n, m)$ and $T^f(n, m)$ have degree 6. Hence, by Theorem 3.6,

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^t(n, m))}{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^c(n, m))}{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^f(n, m))}{mn}.$$

It suffices to prove that

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^t(n, m))}{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^c(n, m))}{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^f(n, m))}{mn}.$$

Notice that $T^t(n, m)$ is a 6-regular graph. Let $Q(T^t(n, m))$ be the signless Laplacian matrix of $T^t(n, m)$. Taking into account the techniques in Ref. [4], one can readily obtain that $Q(T^t(n, m))$ is similar to the block diagonal matrix whose diagonal blocks are

$$6 + \omega^i_n + \omega^i_m + \omega^j_n + \omega^j_m + \omega^{-i}_n \omega^{-j}_m,$$

where $\omega_n = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \quad i = 0, 1, \ldots, n - 1; \quad j = 0, 1, \ldots, m - 1$. This implies that the signless Laplacian eigenvalues of $Q(T^t(n, m))$ are

$$6 + 2 \cos \frac{2\pi i}{m} + 2 \cos \frac{2\pi j}{n} + 2 \cos \left(\frac{2\pi i}{m} + \frac{2\pi j}{n}\right), \quad i = 0, 1, \ldots, m - 1; \quad j = 0, 1, \ldots, n - 1.$$

Let $\alpha_i = \frac{2\pi i}{m}, \quad \beta_j = \frac{2\pi j}{n}, \quad i = 0, 1, \ldots, m - 1; \quad j = 0, 1, \ldots, n - 1$. According to the definition of the incidence energy, the incidence energy of $T^t(n, m)$ can be expressed as

$$\mathcal{J} \mathcal{E}(T^t(n, m)) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sqrt{6 + 2 \cos \alpha_i + 2 \cos \beta_j + 2 \cos(\alpha_i + \beta_j)}.$$

Similarly, we can easily arrive to that

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{J} \mathcal{E}(T^t(n, m))}{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sqrt{6 + 2 \cos \alpha_i + 2 \cos \beta_j + 2 \cos(\alpha_i + \beta_j)}$$

$$= \int_0^1 \int_0^1 \sqrt{6 + 2 \cos 2\pi x + 2 \cos 2\pi y + 2 \cos 2\pi (x + y)} \, dx \, dy$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{6 + 2 \cos x + 2 \cos y + 2 \cos(x + y)} \, dx \, dy \approx 2.4020.$$
4.3. The $3^3.4^2$ lattice

Our notation for the $3^3.4^2$ lattice follows Refs. [4, 11]. The $3^3.4^2$ lattice with toroidal boundary condition, denoted by $S^t(n, 2m)$, can be constructed by starting with a $2m \times n$ square lattice and adding a diagonal edge connecting the vertices, that is, the upper left to the lower right corners of each square in every other row as shown in Fig. 3(a), where $a_1 = b_1, a_{2m} = b_1^*, a_1^* = b_n, a_{2m}^* = b_n^*$, and $(a_1, a_1^*), (a_2, a_2^*), \ldots, (a_{2m}, a_{2m}^*)$; $(b_1, b_1^*), (b_2, b_2^*), \ldots, (b_n, b_n^*)$; $(a_1, a_2^*), (a_3, a_4^*), \ldots, (a_{2m-1}, a_{2m}^*)$ are edges in $S^t(n, 2m)$. The $3^3.4^2$ lattice $S^c(n, 2m)$ with cylindrical boundary condition is obtained from $S^t(n, 2m)$ by deleting $(b_1, b_1^*), (b_2, b_2^*), \ldots, (b_n, b_n^*)$ (see Fig. 3(b)). The $3^3.4^2$ lattice $S^f(n, 2m)$ with free boundary condition is obtained from $S^t(n, 2m)$ by deleting $(a_1, a_1^*), (a_2, a_2^*), \ldots, (a_{2m}, a_{2m}^*)$; $(a_1, a_2^*), (a_3, a_4^*), \ldots, (a_{2m-1}, a_{2m}^*)$ (see Fig. 3(c)).

**Theorem 4.3.** For the $3^3.4^2$ lattices $S^t(n, 2m)$, $S^c(n, 2m)$ and $S^f(n, 2m)$ with toroidal, cylindrical, and free boundary conditions,

$$\lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^t(S^t(n, 2m))}{2mn} = \lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^c(S^c(n, 2m))}{2mn} = \lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^f(S^f(n, 2m))}{2mn}$$

that is, the $3^3.4^2$ lattices $S^t(n, 2m)$, $S^c(n, 2m)$ and $S^f(n, 2m)$ with toroidal, cylindrical, and free boundary conditions have the same asymptotic incidence energy $\mathcal{F}^t(S^t(n, 2m)) = \mathcal{F}^c(S^c(n, 2m)) = \mathcal{F}^f(S^f(n, 2m)) \approx 4.2978 mn$.

**Proof.** By definitions of $S^t(n, 2m)$, $S^c(n, 2m)$ and $S^f(n, 2m)$, one knows that $S^t(n, 2m)$ and $S^f(n, 2m)$ are spanning subgraphs of $S^t(n, 2m)$. Furthermore, almost all vertices of $S^t(n, 2m)$, $S^c(n, 2m)$ and $S^f(n, 2m)$ have degree 5. Hence, by Theorem 3.6,

$$\lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^t(S^t(n, 2m))}{2mn} = \lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^c(S^c(n, 2m))}{2mn} = \lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^f(S^f(n, 2m))}{2mn}.$$

It suffices to prove that

$$\lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^t(S^t(n, 2m))}{2mn} = \lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^c(S^c(n, 2m))}{2mn} = \lim _{m \to \infty } \lim _{n \to \infty } \frac{\mathcal{F}^f(S^f(n, 2m))}{2mn} \approx 2.1489.$$

Notice that $S^t(n, 2m)$ is a 5-regular graph. Let $Q(S^t(n, 2m))$ be the signless Laplacian matrix of $S^t(n, 2m)$, then it is not difficult to obtain that $Q(S^t(n, 2m))$ is similar to the block diagonal matrix whose diagonal blocks are

$$Q_{ij} = \begin{pmatrix} 5 + \omega_{i+1} + \omega_{n-j} & 1 + \omega_{i+1} + \omega_{m-j} \\ 1 + \omega_{n-i} + \omega_{m-j} & 5 + \omega_{n-i} + \omega_{n-j} \end{pmatrix}$$

where $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}, i = 0, 1, \ldots, n - 1; j = 0, 1, \ldots, m - 1$. 
Utilizing the techniques in Ref. [4], one can obtain the signless Laplacian eigenvalues of $S^i(n, 2m)$ are
\[ 5 + 2 \cos \frac{2\pi i}{n} \pm \sqrt{3 + 2 \cos \frac{2\pi j}{m} + 2 \cos \frac{2\pi i}{m} + 2 \cos \left( \frac{2\pi i}{m} + \frac{2\pi j}{n} \right)}, \quad i = 0, 1, \ldots, m-1; \quad j = 0, 1, \ldots, n-1. \]

Let $\alpha_i = \frac{2\pi i}{m}$, $\beta_j = \frac{2\pi j}{n}$, $i = 0, 1, \ldots, m-1; \quad j = 0, 1, \ldots, n-1$.

By the definition of the incidence energy, we can easily get that the incidence energy of $M^i(n, 2m)$ can be expressed as
\[
\mathcal{I}_E(S^i(n, 2m)) = \frac{1}{2mn} \left( \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sqrt{5 + 2 \cos \frac{2\pi i}{n} - \sqrt{3 + 2 \cos \frac{2\pi j}{m} + 2 \cos \frac{2\pi i}{m} + 2 \cos \left( \frac{2\pi i}{m} + \frac{2\pi j}{n} \right)}} + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sqrt{5 + 2 \cos \frac{2\pi i}{n} + \sqrt{3 + 2 \cos \frac{2\pi j}{m} + 2 \cos \frac{2\pi i}{m} + 2 \cos \left( \frac{2\pi i}{m} + \frac{2\pi j}{n} \right)}} \right).
\]

Hence, we know that
\[
\lim_{m \to \infty} \lim_{n \to \infty} \frac{\mathcal{I}_E(S^i(n, 2m))}{2mn} = \frac{1}{2} \int_0^1 \int_0^1 \sqrt{5 + 2 \cos 2\pi x - \sqrt{3 + 2 \cos 2\pi x + 2 \cos 2\pi y + 2 \cos 2\pi (x + y)}} \, dx \, dy + \frac{1}{2} \int_0^1 \int_0^1 \sqrt{5 + 2 \cos 2\pi x + \sqrt{3 + 2 \cos 2\pi x + 2 \cos 2\pi y + 2 \cos 2\pi (x + y)}} \, dx \, dy = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{5 + 2 \cos x - \sqrt{3 + 2 \cos x + 2 \cos (x + y)}} \, dx \, dy + \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sqrt{5 + 2 \cos x + \sqrt{3 + 2 \cos x + 2 \cos (x + y)}} \, dx \, dy \approx 2.1489.
\]

The above numerical integration value implies that the $3^3 4^2$ lattices $S^i(n, 2m)$, $S^i(n, 2m)$ and $S^i(n, 2m)$ with toroidal, cylindrical, and free boundary conditions have the same asymptotic incidence energy $\mathcal{I}_E(S^i(n, 2m)) = \mathcal{I}_E(S^i(n, 2m)) = \mathcal{I}_E(S^i(n, 2m)) \approx 4.2978 mn$. □

**Remark 4.4.** We have considered the asymptotic incidence energy of the hexagonal, triangular, and $3^3 4^2$ lattices with toroidal, cylindrical, and free boundary conditions. By similar means one could treat also other two types of boundary conditions, namely Mobius-band and Klein-bottle boundary conditions. Consequently, the hexagonal (resp. triangular and $3^3 4^2$) lattices with these five boundary conditions have the same asymptotic incidence energy. □

5. Concluding remarks

The explicit formulae have been deduced in this paper, which express the incidence energy $\mathcal{I}_E(G)$ of various lattices with toroidal, cylindrical, and free boundary conditions. Moreover, the specific asymptotic values of $\mathcal{I}_E(G)$ in these lattices are obtained via the application of analysis approach with the help of calculational software. At the same time, we showed that for many types of lattices the incidence energy per vertex of the lattices is independent of the boundary conditions.

It is no difficulty to see that the conclusion is true in general. Actually, the approach can be used widely. For instance, dealing with the problem of the asymptotic incidence energy of the hexagonal lattice with the free boundary is not an easy work but we deduced it in a simple approach.

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