Some Properties of D-Concurrence*

WEI Na-Na (wei_nana)† and LI Yuan (liyuan)‡

College of Mathematics and Information Science, Shaanxi Normal University, Xi’an 710062, China

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Abstract In this note, we give a new lower bound for the concurrence, which is related to D-concurrence. For the Werner states, we compare this D-concurrence bound with the other known bound. In addition, some of new upper and lower bounds of D-concurrence are given.

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1 Introduction
Quantum entanglement is very important in quantum information processing and quantum computation. Some measures of entanglement have been defined to quantify the degree of entanglement. The entanglement measure for a bipartite quantum state is zero if and only if the state is separable, and the bigger is the entanglement measure, the more entanglement is the quantum state. The quantum information processing is inevitably exposed to noise in a general quantum system. Therefore, it is necessary to define a good entanglement measure between two mixed quantum states.

One of the most famous measures of entanglement is the concurrence of two-qubit system. The concurrence of a pure two-qubit state $|\psi\rangle$ in the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, which are two Hilbert space $\mathcal{H}_A, \mathcal{H}_B$ for 2 systems $A, B$, is defined by

$$C(|\psi\rangle) = \sqrt{2(1-tr(\rho^2_A)) - \sqrt{2(1-tr(\rho^2_\infty))}},$$

where $\rho_A = tr_B(|\psi\rangle\langle\psi|)$ is the partial trace of $|\psi\rangle\langle\psi|$ over subsystem B, and $\rho_\infty$ has a similar meaning. For a mixed quantum state, the concurrence is defined by the convex roof construction, that is

$$C(\rho) = \min_{\{\rho_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

where $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. It is clear that $C(|\psi\rangle)$ vanishes and is $2(d-1)/d$ for separable and maximally entanglement pure state, respectively. As the entanglement for high dimensional mixed states is difficult to calculate, the bound of concurrence have been paying very attentions by researchers. In reference,[2–4] for arbitrary finite-dimensional bipartite state $\rho$, the authors presented observable lower and upper bounds of the concurrence:

$$\sqrt{2(\text{tr } \rho^2 - \text{tr } \rho^2_\infty)} \leq C(\rho) \leq \sqrt{2(1-\text{tr } \rho^2_\infty)}.$$

In Ref. [5], Chen, Albeverio and Fei also obtained an interesting lower bound of the concurrence.

In recent years, Ma, etc.[6] have introduced a new entanglement measure, which is called D-concurrence. For pure states $\rho$, the D-concurrence is defined as $D(\rho) = \sqrt{\det(I-\rho_A)}$, where det is the determinant function of a matrix. For a mixed quantum state, the D-concurrence is defined by the convex roof construction, that is

$$D(\rho) = \min_{\{\rho_i, |\psi_i\rangle\}} \sum_i p_i D(|\psi_i\rangle\langle\psi_i|),$$

where $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. As the usual concurrence, the D-concurrence has also nice properties, for instance: entanglement monotonicity and the sub-additivity. In particular, it is an interesting conjecture: for bipartite mixed quantum state, the lower bound: $D(\rho) \geq \det(I-\rho_A) - \det(I-\rho)$. However, so far this bound has not been proven in mathematics, although it is verified by the algorithm in Ref. [7]. In Refs. [8–9] Salimi and Mohammadzade have made a further and specific research to the bound of D-concurrence for pair coherent quantum states:

$$\prod_n (1-|C_{nn}|^2) = \left(1+\sum_{n=0}^{N-1} |C_{nn}|^2 \right) \leq D^2(\rho) \leq \prod_n (1-|C_{nn}|^2),$$

where $N = 2, 3, \ldots, \infty$, $\sum_{n=0}^{N-1} |C_{nn}|^2 \leq 1$, and they have researched on a two-qubits resulted from the pair coherent state and expressed the concurrence and D-concurrence of this system in term of the SU(2) coherent states.

In this note, we present a new lower bound for the concurrence, which is related to D-concurrence. For the Werner states, we compare this D-concurrence bound with the other known bound. In addition, some of new upper and lower bounds of D-concurrence are obtained.

2 Main Results
In the following, we always suppose the dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$ are $d$. Let $B(\mathcal{H}_A \otimes \mathcal{H}_B)$ denote the set of all matrices in the compound space $\mathcal{H}_A \otimes \mathcal{H}_B$. The number of
the strictly positive values in the Schmidt decomposition of $\rho$ is called Schmidt rank of $\rho$.

**Proposition 1** Let $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a pure state. Then $D(\rho) \leq (1/2)C(\rho)$, and $D(\rho) = (1/2)C(\rho)$ if and only if $\text{SR}(\rho) \leq 2$, where $\text{SR}(\rho)$ denotes the Schmidt rank of $\rho$.

**Proof** By Schmidt decomposition, we get $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$, where $\lambda_i > 0$ and $\sum_{i=1}^{d} \lambda_i = 1$, so

$$\rho_A = \sum_i \lambda_i |i_A\rangle \langle i_A|, \quad D(|\psi\rangle) = \sqrt{\prod_{i=1}^{d} (1 - \lambda_i)},$$

$$C(|\psi\rangle) = \sqrt{2(1 - \sum_{i=1}^{d} \lambda_i^2)}.$$  

Thus we only need to prove

$$\prod_{i=1}^{d} (1 - \lambda_i) \leq \frac{1}{2} \left(1 - \sum_{i=1}^{d} \lambda_i^2\right) = \sum_{1 \leq i < j \leq d} \lambda_i \lambda_j.$$  

We will use the mathematical induction to get the above inequality. When $d = 2$, it is clear that $(1 - \lambda_1)(1 - \lambda_2) = 1 - \lambda_1 \lambda_2 + \lambda_1 \lambda_2 = 1$, since $\lambda_1 + \lambda_2 = 1$.

Suppose $i = k - 1$, we have

$$\prod_{i=1}^{k-1} (1 - \lambda_i) \leq \sum_{1 \leq i < j \leq k-1} \lambda_i \lambda_j,$$

for $\lambda_i \geq 0$ and $\sum_{i=1}^{k-1} \lambda_i = 1$. When $i = k$, let $\lambda_k' = \lambda_k$ for $i = 1, \ldots, k - 2$ and $\lambda_k' = \lambda_{k-1} + \lambda_k$ for $i = k - 1$. Then

$$\prod_{i=1}^{k} (1 - \lambda_i) = \prod_{i=1}^{k-2} (1 - \lambda_i)(1 - \lambda_{k-1})(1 - \lambda_k) = \prod_{i=1}^{k-2} (1 - \lambda_i)(1 - \lambda_{k-1} + \lambda_k) + \prod_{i=1}^{k-2} (1 - \lambda_i)\lambda_{k-1}\lambda_k \leq \sum_{1 \leq i < j \leq k-1} \lambda_i \lambda_j \lambda_{k-1} \lambda_k + \prod_{i=1}^{k-2} (1 - \lambda_i)\lambda_{k-1}\lambda_k \leq \sum_{1 \leq i < j \leq k} \lambda_i \lambda_j - \lambda_{k-1}\lambda_k + \lambda_{k-1}\lambda_k = \sum_{1 \leq i < j \leq k} \lambda_i \lambda_j,$$

where the fourth inequality is obtained by the hypothesis condition of $i = k - 1$. Thus by the mathematical induction, we have

$$\prod_{i=1}^{d} (1 - \lambda_i) \leq \sum_{1 \leq i < j \leq d} \lambda_i \lambda_j.$$  

If $\text{SR}(\rho) \leq 2$, then in the Schmidt decomposition of $\rho$, there are at most two $\lambda_i > 0$, so it is clear that

$$\prod_{i=1}^{d} (1 - \lambda_i) = \sum_{1 \leq i < j \leq d} \lambda_i \lambda_j.$$  

Conversely, if $D(\rho) = (1/2)C(\rho)$, then

$$\prod_{i=1}^{d} (1 - \lambda_i) = \sum_{1 \leq i < j \leq d} \lambda_i \lambda_j.$$  

We may assume that $\lambda_i$ is a monotone increasing sequence, thus by the proof above, we get

$$\prod_{i=1}^{d-2} (1 - \lambda_i)\lambda_{d-1} \lambda_d = \lambda_{d-1} \lambda_d,$$

which implies $\lambda_1 = \cdots = \lambda_{d-2} = 0$. Hence $\rho$ has one or two nonzero eigenvalues.

The following result shows the connection between the concurrence and $D$-concurrence. We define the Schmidt rank of $\rho$ by

$$\text{SR}(\rho) = \min_{\{p_i, \rho_i\}} \left\{ \text{max SR}(|\psi_i\rangle \langle \psi_i|) \right\},$$

where the minimum is taken over all possible pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, $p_i > 0$.

**Theorem 1** For any state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have $D(\rho) \leq (1/2)C(\rho)$. If $D(\rho) = (1/2)C(\rho)$, then $\text{SR}(\rho) \leq 2$.  

**Proof** Let $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ be the optimal decomposition such that $C(\rho) = \sum_i p_i C(|\psi_i\rangle \langle \psi_i|)$, where $p_i > 0$ for all $i$. Then by proposition 1, we get

$$C(\rho) = \sum_i p_i C(|\psi_i\rangle \langle \psi_i|) \leq 2 \sum_i p_i D(|\psi_i\rangle \langle \psi_i|).$$

Note that inequality (1) implies

$$0 \leq \sum_i p_i C(|\psi_i\rangle \langle \psi_i|) - 2 \sum_i p_i D(|\psi_i\rangle \langle \psi_i|) \leq C(\rho) - 2D(\rho).$$

Thus we get that

$$\sum_i p_i C(|\psi_i\rangle \langle \psi_i|) = 2 \sum_i p_i D(|\psi_i\rangle \langle \psi_i|)$$

follows from $D(\rho) = (1/2)C(\rho)$.

Moreover,

$$\sum_i p_i C(|\psi_i\rangle \langle \psi_i|) = 2 \sum_i p_i D(|\psi_i\rangle \langle \psi_i|)$$

means that

$$\sum_i p_i C(|\psi_i\rangle \langle \psi_i|) - 2D(|\psi_i\rangle \langle \psi_i|) = 0.$$

By proposition 1, we have $C(|\psi_i\rangle \langle \psi_i|) - 2D(|\psi_i\rangle \langle \psi_i|) \geq 0$ for all $i$, so $C(|\psi_i\rangle \langle \psi_i|) = 2D(|\psi_i\rangle \langle \psi_i|)$ for all $i$. Then proposition 1 implies that $\text{SR}(|\psi_i\rangle \langle \psi_i|) \leq 2$ for all $i$, which yields

$$\text{SR}(\rho) = \min_{\{p_i, \rho_i\}} \left\{ \text{max SR}(|\psi_i\rangle \langle \psi_i|) \right\} \leq 2.$$  

In Refs. [2–4], for a bipartite state $\rho$, the authors presented observable lower and upper bounds of the concurrence:

$$\sqrt{2(\text{tr } \rho^2 - \text{tr } \rho_A^2)} \leq C(\rho) \leq \sqrt{2(1 - \text{tr } \rho_A^2)}.$$
In the following, we will give the example to show that $2D(\rho)$ is a better lower bound than $\sqrt{2(\text{tr} \rho^2 - \text{tr} \rho_A^2)}$ for some states. Let us first recall the definition of Werner states.

**Werner State** In Refs. [10–12], the Werner states are a class of mixed state for $d \times d$ system ($d \geq 2$) which are invariant by the transformations $U \otimes U$, for any unitary operator $U$. The density matrix of Werner states are expressed as

$$\rho_f = \frac{1}{d^2 - d}[ (d - f) I + (df - 1) \mathcal{F}],$$

where $\mathcal{F}$ is defined by $\mathcal{F}(\phi \otimes \psi) = \psi \otimes \phi$. In the computational basis $|ij\rangle$, $\mathcal{F}$ is of the form $\mathcal{F} = \sum_{i,j}^{d} |ij\rangle \langle ji|$. Here $f$ is a constant $f = \text{tr}(\mathcal{F} \rho_f)$ satisfying $-1 \leq f \leq 1$. In Ref. [6], $D(\rho_f)$ (from the method of Ref. [12]) are given by

$$D(\rho_f) = \begin{cases} \frac{-f}{2} & \text{for } -1 \leq f < 0, \\ 0 & \text{for } 0 \leq f \leq 1. \end{cases}$$

**Theorem 2** For Werner state $\rho_f$,

$$2D(\rho_f) > \sqrt{2(\text{tr} \rho_f^2 - \text{tr} \rho_{fA}^2)},$$

if and only if one of the following conditions satisfies:

(i) $d = 2$ and $-1 < f < 0$;

(ii) For arbitrary $d \geq 3$ and $-1 \leq f < 0$.

**Proof** First, we take the square to

$$2D(\rho_f) > \sqrt{2(\text{tr} \rho_f^2 - \text{tr} \rho_{fA}^2)},$$

then we have $4D^2(\rho_f) > 2(\text{tr} \rho_f^2 - \text{tr} \rho_{fA}^2)$. In the following, we show this inequality. From Ref. [6], we get that

$$4D^2(\rho_f) = \begin{cases} f^2 & \text{for } -1 \leq f < 0, \\ 0 & \text{for } 0 \leq f \leq 1. \end{cases}$$

It is easy to see that $\mathcal{F}^2 = I$ and $\text{tr} \mathcal{F} = d^2$, so the eigenvalues of $\mathcal{F}$ are only $1$, and the multiplicities of them are $(d^2 + d)/2$ and $(d^2 - d)/2$, respectively, which mean the eigenvalues of $\rho_f$ are only $(d - f + df - 1)/(d^2 - d)$, $(d - f - 1 - df)/(d^2 - d)$ with multiplicities $(d^2 + d)/2$ and $(d^2 - d)/2$, respectively. Clearly, $\rho_{fA} = I_A / d$ (the reduced density matrix) implies that the eigenvalues of $\rho_{fA}$ are $1/d$ with the multiplicities $d$. Thus

$$2(\text{tr} \rho_f^2 - \text{tr} \rho_{fA}^2) = 2\left(\frac{d - f + df - 1}{d^2 - d}\right)^2 \cdot \frac{d^2 + d}{2} + 2\left(\frac{d - f - 1 - df}{d^2 - d}\right)^2 \cdot \frac{d^2 - d}{2} - \frac{2d^2}{d^2}$$

$$= \frac{(1 + f)^2}{d^2 + d} + \frac{(1 - f)^2}{d^2 - d} - \frac{2}{d},$$

which means

$$2(\text{tr} \rho_f^2 - \text{tr} \rho_{fA}^2) - 4D^2(\rho_f) = \frac{(1 + f)^2}{d^2 + d} + \frac{(1 - f)^2}{d^2 - d} - \frac{2}{d} - f^2$$

$$= 3d - d^3 f^2 - \frac{4}{d^3 - d} f + \frac{-2d^2 + 2d + 2}{d^3 - d}.$$
have \( \prod_{j=1}^{d} (1 - t_is_j) \geq 1 - t_i \), which implies
\[
\prod_{i,j=1}^{d} (1 - t_is_j) \geq \prod_{i=1}^{d} (1 - t_i),
\]
so
\[
\sqrt{\prod_{i,j=1}^{d} (1 - t_is_j)} \geq \sqrt{\prod_{i=1}^{d} (1 - t_i)}.
\]

Similarly, we get
\[
\sqrt{\prod_{i,j=1}^{d} (1 - t_is_j)} \geq \sqrt{\prod_{i=1}^{d} (1 - s_j)}.
\]

The following results give the uniform upper bound of \( D(\rho) \).

**Proposition 3** For any state \( \rho \in \mathcal{H}_A \otimes \mathcal{H}_B \), we have \( D(\rho) \leq \sqrt{(1 - 1/d)^d} \).

**Proof** Firstly, we assume that \( \rho \) is a pure state and the eigenvalues of \( \rho \) are \( t_i \), \( i = 1, 2, \ldots, d \), then
\[
D(\rho) = \sqrt{\prod_{i=1}^{d} (1 - t_i)}.
\]

The function \( f(t) = -\prod_{i=1}^{d} (1 - t_i) \) with \( t_1 \geq \cdots \geq t_d \geq 0 \) is Schur-convex, by Lemma 1 in Appendix. It is clear that
\[
\left( \frac{1}{d} \frac{1}{d} \cdots \frac{1}{d} \right) \lesssim t \lesssim (1, 0, \ldots, 0),
\]
so Lemma 2 in Appendix implies
\[
f\left( \frac{1}{d} \frac{1}{d} \cdots \frac{1}{d} \right) \leq f(t) \leq f((1, 0, \ldots, 0)),
\]
which yields
\[
-(1 - \frac{1}{d})^d \leq f(t) \leq 0.
\]

Thus we have
\[
D(\rho) \leq \sqrt{(1 - 1/d)^d}.
\]

If \( \rho \) is a mixed state, then by the definition of \( D \)-concurrence and the proof above, we have \( D(\rho) \leq \sqrt{(1 - 1/d)^d} \).

**Corollary 1** For any state \( \rho, \sigma \), we have \( D(\rho \otimes \sigma) \leq \sqrt{(1 - 1/d^2)^d} \).

3 Conclusion

In this paper, we have got a new lower bound for the concurrence, which is related to \( D \)-concurrence. For the Werner states, we have compared is \( D \)-concurrence bound with the other known bound. Also, some of new upper and lower bounds of \( D \)-concurrence of the compound states are obtained.

Appendix

**Definition 1** (Majorization\[13\]–\[14\]) If \( x \) and \( y \) are \( n \)-dimensional real vector, \( x = (x_1, x_2, x_3, \ldots, x_n) \) and \( x_1 \geq x_2 \geq \cdots \geq x_n \), \( y = (y_1, y_2, y_3, \ldots, y_n) \), and \( x_1 \geq x_2 \geq \cdots \geq x_n \), then we call \( x \) is majorized by \( y \) if
\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n-1, \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i
\]
denoted by \( x \prec y \).

**Lemma 1** The differentiable real-value function \( f : R^n \to R \), then \( f \) is Schur-convex if and only if: for arbitrary \( x \in R^n \), and \( x_1 \geq x_2 \geq \cdots \geq x_n \), if \( i < j \), then \( \partial f / \partial x_i \geq \partial f / \partial x_j \).

**Lemma 2** (Ref. \[14\]) If \( x, y \in R^n \), the differentiable real-value function function \( f : R^n \to R \) is Schur-convex, then \( x \prec y \Rightarrow f(x) \leq f(y) \).

References
