Complete solution of equation $W(L^3(T)) = W(T)$ for the Wiener index of iterated line graphs of trees

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**Abstract**

Let $G$ be a graph. Denote by $L^i(G)$ its $i$-iterated line graph and denote by $W(G)$ its Wiener index. In Knor et al. (in press) we show that there is an infinite class $T$ of trees $T$ satisfying $W(L^3(T)) = W(T)$, which disproves a conjecture of Dobrynin and Entringer. In this paper we prove that except the trees of $T$, there is no non-trivial tree $T$ satisfying $W(L^i(T)) = W(T)$. Consequently, for a tree $T$ and $i \geq 3$, the equation $W(L^i(T)) = W(T)$ holds if and only if $T \in T$ and $i = 3$.

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1. Introduction

Let $G$ be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For any two vertices $u$, $v$ let $d(u, v)$ be the distance from $u$ to $v$. The Wiener index of $G$, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken over unordered pairs of vertices of $G$. The Wiener index was introduced by Wiener in [22]. Since it is related to several properties of chemical molecules (see [12]), it is widely studied by chemists. The interest of mathematicians was attracted in the 1970s, when it was reintroduced as the transmission and the distance of a graph; see [21] and [9], respectively. Recently, several special issues of journals were devoted to mathematical properties of the Wiener index (see [11,10]). For surveys and some up-to-date papers related to the Wiener index of trees and line graphs see [5,6], [8,19,20,24] and [2,3,7,13,23], respectively.

By the definition, if $G$ has a unique vertex, then $W(G) = 0$. In this case, we say that the graph $G$ is trivial. We set $W(G) = 0$ also when the set of vertices of $G$ is empty.

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http://dx.doi.org/10.1016/j.dam.2014.02.007

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Theorem 1.5. The following:

\[
W(G, L(G)) = W(T) - \binom{n}{2}.
\]

Theorem 1.6. Let \( T \) be a tree whcih is not homeomorphic to a path, claw \( K_{1,3} \) or \( H_0 \), and let \( i \geq 3 \). Then \( W(L(T)) = W(T) \). However, there are trees \( T \) satisfying \( W(L(T)) = W(T) \), see e.g. [4]. In [5], the following problem was posed:

Problem 1.2. Is there any tree \( T \) satisfying equality \( W(L(T)) = W(T) \) for some \( i \geq 3 \)?

As observed above, if \( T \) is a trivial tree, then \( W(L(T)) = W(T) \) for every \( i \geq 1 \), although here the graph \( L(T) \) is empty.

The real question is, if there is a nontrivial tree \( T \) and \( i \geq 3 \) such that \( W(L(T)) = W(T) \).

In papers [16,15,18,17] (see [17, Corollary 1.4]) we solved Problem 1.2 for \( i \geq 4 \):

Theorem 1.3. Let \( T \) be a tree and \( i \geq 4 \). Then we have

\[
\begin{align*}
W(L(T)) &= W(T) \quad \text{if } T \text{ is trivial,} \\
W(L(T)) &< W(T) \quad \text{if } T \text{ is a nontrivial path or the claw } K_{1,3}, \\
W(L(T)) &> W(T) \quad \text{otherwise.}
\end{align*}
\]

In this paper we consider Problem 1.2 for the remaining case \( i = 3 \). Let \( H_0 \) be the tree on six vertices, out of which two have degree 3 and four have degree 1. In [15, Corollary 1.6], we proved:

Theorem 1.4. Let \( T \) be a tree which is not homeomorphic to a path, claw \( K_{1,3} \) or \( H_0 \), and let \( i \geq 3 \). Then \( W(L(T)) > W(T) \).

(Recall that two graphs \( G_1 \) and \( G_2 \) are homeomorphic if and only if there is a third graph \( H \), such that both \( G_1 \) and \( G_2 \) can be obtained from \( H \) by means of edge subdivision.)

By Theorem 1.4, to solve Problem 1.2 for \( i = 3 \), it suffices to consider paths and trees homeomorphic to the claw \( K_{1,3} \) and \( H_0 \).

First, let us concentrate to paths. Denote by \( P_n \) a path on \( n \) vertices. If \( n \geq 2 \), then \( W(P_n) = W(P_{n-1}) \), since \( P_{n-1} \) is a subgraph embedded isometrically in \( P_n \). Since \( L(P_n) = P_{n-1} \) if \( n \geq 2 \), while \( L(P_1) \) is an empty graph, we have \( W(L(P_n)) = W(P_n) \) for every \( i \geq 1 \) if \( P_n \) is a nontrivial path.

Similarly, there is no solution of Problem 1.2 among trees homeomorphic to the claw \( K_{1,3} \), namely, in Section 3 we prove the following:

Theorem 1.5. Let \( T \) be a tree homeomorphic to \( K_{1,3} \). Then \( W(L(T)) \neq W(T) \).

However, there is a non-trivial solution of Problem 1.2 among trees homeomorphic to \( H_0 \). Denote by \( H_{a,b,c,d,e} \) a specific tree homeomorphic to \( H_0 \), defined as follows: In \( H_{a,b,c,d,e} \), the two vertices of degree 3 are joined by a path of length \( e+1 \), \( e \geq 0 \). Hence, this path has \( e \) vertices of degree 2. Further, at one vertex of degree 3 there start two pendant paths of lengths \( a \) and \( b \), where \( a, b \geq 1 \), and at the other vertex of degree 3 there start another two pendant paths of lengths \( c \) and \( d \), where \( c, d \geq 1 \). Thus \( H_{a,b,c,d,e} \) has \( a + b + c + d + e + 2 \) vertices (see Fig. 1 for \( H_{3,3,4,2,2} \)). By symmetry, we may assume that \( a \geq b \), \( c \geq d \) and \( b \geq d \). That is, we assume that the shortest pendant path in \( H_{a,b,c,d,e} \) has length \( d \).

In Section 4, we prove the following:

Theorem 1.6. The equation \( W(L(H_{a,b,c,d,e})) = W(H_{a,b,c,d,e}) \) holds if and only if \( d = e = 1 \) and there are \( i, j \in \mathbb{Z}, i \geq j \), such that

\[
\begin{align*}
 a &= 128 + 3i^2 + 3j^2 - 3ij + i \\
b &= 128 + 3i^2 + 3j^2 - 3ij + j \\
c &= 128 + 3i^2 + 3j^2 - 3ij + i + j.
\end{align*}
\]
We remark that the "if" part of Theorem 1.6 was already proved in [14]. The smallest tree satisfying (1) is \( H_{128,128,1,1} \) on 388 vertices obtained when \( i = j = 0 \).

We can summarize our results regarding Problem 1.2 in the following theorem:

**Theorem 1.7.** Let \( T \) be a tree and \( i \geq 3 \). Then we have

(i) \( W(L^1(T)) = W(T) \) if \( T \) is trivial or \( i = 3 \) and \( T \) is \( H_{a,b,c,1,1} \), where \( a, b, c \) satisfy (1);

(ii) \( W(L^1(T)) \neq W(T) \) if \( i = 3 \) and \( T \) is homeomorphic to \( K_{1,3} \) or \( H_0 \) with the exception of trees mentioned in (i);

(iii) \( W(L^2(T)) < W(T) \) if \( T \) is a nontrivial path or the claw \( K_{1,3} \);

(iv) \( W(L^3(T)) > W(T) \) otherwise.

It is obvious that trees mentioned in (ii) either satisfy \( W(L^3(T)) < W(T) \) or \( W(L^3(T)) > W(T) \). If \( T \neq K_{1,3} \), in some cases we prove \( W(L^3(T)) > W(T) \), but in the others using congruences we can only show \( W(L^3(T)) \neq W(T) \), see below.

In the next section we present a lemma, with the help of which we prove Theorems 1.5 and 1.6 in Sections 3 and 4, respectively.

2. Preliminaries

A degree of a vertex, say \( v \), is denoted by \( \deg(v) \), or when convenient, by \( d_v \). Analogously as a vertex of \( L(G) \) corresponds to an edge of \( G \), a vertex of \( L^2(G) \) corresponds to a path of length two in \( G \). For \( x \in V(L^2(G)) \) we denote the corresponding path by \( B_2(x) \). For two subgraphs \( S_1 \) and \( S_2 \) of \( G \), the shortest distance in \( G \) between a vertex of \( S_1 \) and a vertex of \( S_2 \) is denoted by \( d(S_1,S_2) \). If \( S_1 \) and \( S_2 \) share an edge, then we set \( d(S_1,S_2) = -1 \).

Let \( x \) and \( y \) be two vertices of \( L^2(G) \), such that \( u \) is the center of \( B_2(x) \), the vertex \( v \) is the center of \( B_2(y) \) and \( u \neq v \). Then

\[
d_{L^2(G)}(x,y) = d(B_2(x),B_2(y)) + 2.
\]

Let \( u, v \in V(G) \), \( u \neq v \). Let \( \beta_i(u,v) \) denote the number of pairs \( x, y \in V(L^2(G)) \), with \( u \) being the center of \( B_2(x) \) and \( v \) being the center of \( B_2(y) \), such that \( d(B_2(x),B_2(y)) = d(u,v) - 2 + i \). Since \( d(u,v) - 2 \leq d(B_2(x),B_2(y)) \leq d(u,v) \), we have \( \beta_i(u,v) = 0 \) for all \( i \in \{0,1,2\} \). Moreover, \( \sum_{i=0}^{2} \beta_i(u,v) = \left( \begin{array}{c} d_u \\ 2 \end{array} \right) \left( \begin{array}{c} d_v \\ 2 \end{array} \right) \).

Let

\[
h(u,v) = \left( \begin{array}{c} d_u \\ 2 \end{array} \right) \left( \begin{array}{c} d_v \\ 2 \end{array} \right) - 1 \right) d(u,v) + \beta_1(u,v) + 2\beta_2(u,v).
\]

In [14, Lemma 2.2] we have the following statement:

**Lemma 2.1.** Let \( G \) be a connected graph. Then

\[
W(L^2(G)) - W(G) = \sum_{u \neq v} h(u,v) + \sum_u \left[ 3 \left( \begin{array}{c} d_u \\ 3 \end{array} \right) + 6 \left( \begin{array}{c} d_u \\ 4 \end{array} \right) \right],
\]

where the first sum is taken over unordered pairs of vertices \( u, v \in V(G) \), such that either \( d_u \neq 2 \) or \( d_v \neq 2 \), and the second one is taken over \( u \in V(G) \).

Observe that \( W(P_n) = \left( \begin{array}{c} n - 1 \\ 1 \end{array} \right) + \left( \begin{array}{c} n - 2 \\ 1 \end{array} \right) + \cdots + 1 = \left( \begin{array}{c} n+1 \\ 3 \end{array} \right) \), where this fact, one can show that \( W(H_{a,b,c,d,e}) \) is a polynomial of third degree in \( a, b, c, d \) and \( e \) and so is also \( W(L^2(H_{a,b,c,d,e})) \) (the situation with the claw being similar). However, if we calculate \( W(L^2(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e}) \) with the help of Lemma 2.1, we obtain a polynomial the degree of which is at most 2, since the pairs of vertices \( u, v \) with \( d_u = d_v = 2 \) do not contribute to \( W(L^2(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e}) \) (for detailed calculation see the proofs below).

3. Proof of Theorem 1.5

**Proof of Theorem 1.5.** Let \( C_{a,b,c} \) be a tree homeomorphic to the claw \( K_{1,3} \) in which the paths connecting the vertices of degree 1 with the vertex of degree 3 have lengths \( a, b \) and \( c \), where \( a \geq b \geq c \geq 1 \). The tree \( C_{a,b,c} \) has exactly \( a + b + c + 1 \) vertices, see Fig. 2 for \( C_{4,3,2} \).

We prove Theorem 1.5 by counting the distances in \( L(C_{a,b,c}) \) instead of in \( C_{a,b,c} \) and \( L^2(C_{a,b,c}) \). In \( L(C_{a,b,c}) \) we distinguish 6 vertices \( x_1, x_2, x_3, y_1, y_2 \) and \( y_3 \). The vertices \( x_1, x_2 \) and \( x_3 \) correspond to the pendant edges of \( C_{a,b,c} \), while the vertices \( y_1, y_2 \) and \( y_3 \) correspond to the edges incident with the vertex of degree 3 in \( C_{a,b,c} \), see Fig. 2 for \( L(C_{4,3,2}) \). Observe that if \( c = 1 \) (\( b = 1 \) or \( a = 1 \)), then \( x_3 = y_3 \) (\( x_2 = y_2 \) or \( x_1 = y_1 \)), and in such a case, \( \deg(x_3) = 2 \) (\( \deg(x_2) = 2 \) or \( \deg(x_1) = 2 \), respectively).
In what follows, the graph $L(C_{a,b,c})$ is denoted by $LC$. Further, for $i \in \{1, 2, 3\}$, let $V_i$ be the set of vertices of $V(LC)$ of degree $i$. For $x \in V_1$ and $y \in V_3$, define

$$S^1(x) = \sum_u h(u, x) \quad \text{where} \ u \in V(LC) \setminus V_1,$$

$$M^1 = \sum_{u \neq v} h(u, v) \quad \text{where} \ u, v \in V_1,$$

$$S^3(y) = \sum_u h(u, y) \quad \text{where} \ u \in V_2,$$

$$M^3 = \sum_{u \neq v} h(u, v) \quad \text{where} \ u, v \in V_3,$$

$$D = \sum_u \left[ 3 \left( \frac{u}{3} \right) + 6 \left( \frac{u}{4} \right) \right] \quad \text{where} \ u \in V_3.$$

Observe that $\sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3} S^3(y) + M^3$ sums $h(u, v)$ for all pairs $\{u, v\}$ of vertices such that either $\deg(u) \neq 2$ or $\deg(v) \neq 2$.

Denote $P = W(L^2(C_{a,b,c})) - W(C_{a,b,c})$. Since $C_{a,b,c}$ has $a + b + c + 1$ vertices, we have $W(C_{a,b,c}) = W(LC) + \binom{a+b+c+1}{2}$, by Theorem 1.1. Thus, by Lemma 2.1, we have

$$P = W(L^2(LC)) - W(LC) - \binom{a+b+c+1}{2}$$

$$= \sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3} S^3(y) + M^3 + D - \binom{a+b+c+1}{2}. \quad (3)$$

This naturally splits the problem into four cases according to the size of $V_1$. In each of these cases we evaluate $S^1$'s, $M^1$, $S^3$'s, $M^3$ and $D$, and we solve the equation $P = 0$. To avoid fractions, in some cases we solve the equation $2P = 0$ instead of $P = 0$.

Case $1. a, b, c \geq 2$, that is, $|V_1| = 3$.

We start with evaluating $S^1(x)$, where $x \in V_1$. Since $\deg(x) = 1$, we have $\beta_i(u, x) = 0, 0 \leq i \leq 2$. Hence, $h(u, x) = -d(u, x)$, see (2). The sum of distances from $x_1$ to all interior vertices of $x_1 - x_3$ path is $1 + 2 + \cdots + (a + b - 2) = \binom{a+b-1}{2}$ (see Fig. 2). The sum of distances from $x_1$ to all interior vertices of $x_1 - x_3$ path, not included in the previous calculation, is $a + (a+1) + \cdots + (a+c-2) = \binom{a+c-1}{2} - \binom{a}{2}$. In this way we get $S^1(x_1)$ and analogously we calculate $S^1(x_2)$ and $S^1(x_3)$:

$$S^1(x_1) = -\binom{a+b-1}{2} - \binom{a+c-1}{2} + \binom{a}{2},$$

$$S^1(x_2) = -\binom{a+b-1}{2} - \binom{b+c-1}{2} + \binom{b}{2},$$

$$S^1(x_3) = -\binom{a+c-1}{2} - \binom{b+c-1}{2} + \binom{c}{2}. $$

Now $h(x_1, x_2) = -(a+b-1)$. Using the symmetry we obtain

$$M^1 = -(a+b-1) - (a+c-1) - (b+c-1).$$

In $S^3(y)$ we sum $h(u, y)$, where $\deg(u) = 2$ and $\deg(y) = 3$. Hence, $\binom{d_y}{2} \binom{d_y}{3} - 1 = 2$. Since $\beta_0(u, y) = 2, \beta_1(u, y) = 1$ and $\beta_2(u, y) = 0$, we have $h(u, y) = 2d(u, y) + 1$. Thus, the sum of $h(u, y)$'s for interior vertices $u$ of $y_1 - x_1$ path is $2(1 + 2 + \cdots + (a-2)) + (a-2) = 2 \binom{a-1}{2} + (a-2)$ (see Fig. 2). Analogously, the sum of $h(u, y)$'s for interior vertices

Fig. 2. The graphs $C_{a,b,c}$ and $L(C_{a,b,c}) = LC$. 
of $y_2 - x_2$ path is $2(2 + 3 + \cdots + (b - 1)) + (b - 2) = 2 \left( \frac{b}{2} \right) - 2 + (b - 2) = 2 \left( \frac{b}{2} \right) + (b - 4)$. In this way we get

$$S^3(y_1) = 2 \left( \frac{a-1}{2} \right) + (a-2) + 2 \left( \frac{b}{2} \right) + (b-4) + 2 \left( \frac{c}{2} \right) + (c-4),$$

$$S^3(y_2) = 2 \left( \frac{a}{2} \right) + (a-4) + 2 \left( \frac{b-1}{2} \right) + (b-2) + 2 \left( \frac{c}{2} \right) + (c-4),$$

$$S^3(y_3) = 2 \left( \frac{a}{2} \right) + (a-4) + 2 \left( \frac{b}{2} \right) + (b-4) + 2 \left( \frac{c-1}{2} \right) + (c-2).$$

Consider $h(y_1, y_2)$. Here $\left( \frac{dy_1}{2} \right) \left( \frac{dy_2}{2} \right) - 1 = 8$, $\beta_0(y_1, y_2) = 4$, $\beta_1(y_1, y_2) = 5$ and $\beta_2(y_1, y_2) = 0$ (see Fig. 2). This means that $h(y_1, y_2) = 8 + 5 = 13$, and analogously also $h(y_1, y_3) = 13$ and $h(y_2, y_3) = 13$. Hence

$$M^3 = 13 + 13 + 13.$$

Finally, since $LC$ has exactly three vertices of degree 3 and no vertex of higher degree, we have

$$D = \sum_u \left[ 3 \left( \frac{d_u}{3} \right) + 6 \left( \frac{d_u}{4} \right) \right] = 3 \left[ 3 \left( \frac{3}{3} \right) \right] = 9.$$

By (3), expanding the terms (using a computer package, for instance), we get

$$P = (a^2 + b^2 + c^2) - 3(ab + ac + bc) + (a + b + c) + 21 = (a + b + c)^2 - 5(ab + ac + bc) + (a + b + c) + 21.$$

Now substitute $x = (a + b + c)$ and consider the equation $P = 0$ over $\mathbb{Z}_5$. We get

$$x^2 + x + 1 = 0,$$

which has no solution in $\mathbb{Z}_5$. Consequently, $P = 0$ has no integer solution and $W(L^3(C_{a,b,c})) - W(C_{a,b,c}) \neq 0$ in this case.

Case 2. $a, b \geq 2, c = 1$, that is, $|V_1| = 2$. In this case the vertex $x_3 = y_3$ has degree 2, so we do not need to find $S^1(x_3)$ and $S^1(y_3)$, see (3), but we must include the distances to $x_3$ in $S^1(x_1)$, $S^1(x_2)$, $S^3(y_1)$ and $S^3(y_2)$. Analogously as in the previous case we have

$$S^3(y_1) = 2 \left( \frac{a-1}{2} \right) + (a-2) + 2 \left( \frac{b}{2} \right) + (b-4) + 2 + 1,$n\n
$$S^3(y_2) = 2 \left( \frac{a}{2} \right) + (a-4) + 2 \left( \frac{b-1}{2} \right) + (b-2) + 2 + 1,$n$$

$$M^3 = 13,$n\n
$$D = 2 \cdot 3 \left( \frac{3}{3} \right) = 6.$$

By (3), expanding the terms we get

$$2P = (a^2 + b^2) - 6ab - 5(a + b) + 30 = (a + b)^2 - 8ab - 5(a + b) + 30. \quad (4)$$

Now consider the equation $2P = 0$ over $\mathbb{Z}_5$. We get $(a' + b')^2 + 2a'b' = 0$. It is a matter of routine to check that the only solution in $\mathbb{Z}_5$ is $a' = b' = 0$. Hence, in (4) we have $25 \mid (a + b)^2, 25 \mid 8ab$ and $25 \mid 5(a + b)$. Since $25 \mid 30$, (4) has no integer solution. Thus, $P = 0$ has no solution also in this case.

Case 3. $a \geq 2, b = c = 1$, that is, $|V_1| = 1$. The vertices $x_2 = y_2$ and $x_3 = y_3$ have degree 2, so we do not need to find $S^1(x_2), S^1(x_3), S^3(y_2)$ and $S^3(y_3)$. We have

$$S^1(x_1) = - \left( \frac{a}{2} \right) - a - a,$n$$

$$M^1 = 0,$n$$

$$S^1(x_2) = - \left( \frac{a}{2} \right) - b,$n$$

$$S^1(x_3) = - \left( \frac{a}{2} \right) - c,$n$$

$$S^1(x_4) = - \left( \frac{a}{2} \right) - (b + c).$$
$S^3(y_1) = 2 \left( \frac{a - 1}{2} \right) + (a - 2) + 2 + 1 + 2 + 1,$

$M^3 = 0,$

$D = 3 \left( \frac{3}{3} \right) = 3.$

By (3), expanding the terms we get

$$P = -6a + 6 < 0$$

as $a \geq 2.$ Thus, $P = 0$ has no solution in this case.

Case 4. $a = b = c = 1,$ that is, $|V_1| = 0.$

In this case $C_{a,b,c} = K_{1,3}$ and $L'(K_{1,3})$ is a cycle of length 3 for every $i \geq 1.$ Since $W(G) = 3$ if $G$ is a cycle of length 3, while $W(K_{1,3}) = 9,$ we have $W(L'(C_{1,1,1})) - W(C_{1,1,1}) \neq 0$ also in this case. \(\square\)

4. Proof of Theorem 1.6

**Proof of Theorem 1.6.** We proceed analogously as in the proof of Theorem 1.5. That is, we prove Theorem 1.6 by counting the distances in $L(H_{a,b,c,d,e})$ instead of those in $H_{a,b,c,d,e}$ and $L^3(H_{a,b,c,d,e}).$ In $L(H_{a,b,c,d,e})$ we distinguish 10 vertices $x_1, x_2, \ldots, x_4$ and $y_1, y_2, \ldots, y_6.$ The vertices $x_1, \ldots, x_4$ correspond to pendant edges of $H_{a,b,c,d,e},$ while the vertices $y_1, \ldots, y_6$ correspond to edges incident with vertices of degree 3 in $H_{a,b,c,d,e}$ (see Fig. 3). Observe that if $e = 0,$ then $y_5 = y_6$ and $\text{deg}(y_5) = 4.$ If $d = 1$ ($e = 1, b = 1$ or $a = 1$), then $x_4 = y_4$ ($x_3 = y_3, x_2 = y_2$ or $x_1 = y_1$), and in such a case $\text{deg}(x_4) = 2$ ($\text{deg}(x_3) = 2, \text{deg}(x_2) = 2$ or $\text{deg}(x_1) = 2,$ respectively).

In what follows, the graph $L(H_{a,b,c,d,e})$ is denoted by $LH.$ Further, for $i \in \{1, 2, 3, 4\},$ let $V_i$ be the set of vertices of $V(LH)$ of degree $i.$ For $x \in V_1$ and $y \in V_3 \cup V_4,$ define

$$S^1(x) = \sum_{u \in V(LH) \setminus V_1} h(u, x) \quad \text{where} \quad u \in V_1,$$

$$M^1 = \sum_{u \neq v} h(u, v) \quad \text{where} \quad u, v \in V_1,$$

$$S^3(y) = \sum_{u} h(u, y) \quad \text{where} \quad u \in V_2,$$

$$M^3 = \sum_{u \neq v} h(u, v) \quad \text{where} \quad u, v \in V_3 \cup V_4,$$

$$D = \sum_{u} \left[ 3 \left( \frac{u}{3} \right) + 6 \left( \frac{u}{4} \right) \right] \quad \text{where} \quad u \in V_3 \cup V_4.$$

Observe that once again, $\sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3 \cup V_4} S^3(y) + M^3$ sums $h(u, v)$ for all pairs $(u, v)$ of vertices such that either $\text{deg}(u) \neq 2$ or $\text{deg}(v) \neq 2.$

Denote $P = W(L^2(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e}).$ Since $H_{a,b,c,d,e}$ has $a + b + c + d + e + 2$ vertices, we have $W(H_{a,b,c,d,e}) = W(LH) + \binom{a + b + c + d + e + 2}{2},$ by Theorem 1.1. Thus, by Lemma 2.1, we have

$$P = W(L^2(LH)) - W(LH) - \left( \frac{a + b + c + d + e + 2}{2} \right) = \sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3 \cup V_4} S^3(y) + M^3 + D - \left( \frac{a + b + c + d + e + 2}{2} \right).$$

If $e = 0,$ then we have one vertex of degree 4 in $LH,$ while if $e \geq 1,$ then the greatest degree of a vertex in $LH$ is 3. By symmetry, we distinguish eleven cases. In the first five cases we have $e \geq 1$ and in the next five we have $e = 0.$ In each of these first ten cases (the last case will be solved in a different way) we evaluate $S^1's, M^1, S^3's, M^3$ and $D$ and we solve the equation $P = 0.$ To avoid fractions, in some cases we solve the equation $2P = 0.$

Case 1. $a, b, c, d \geq 2, e \geq 1.$

We start with evaluating $S^1(x),$ where $x \in V_1.$ Since $\text{deg}(x) = 1,$ we have $h_j(u, x) = 0, 0 \leq j \leq 2.$ Hence, $h(u, x) = -d(u, x).$ The sum of distances from $x_1$ to all interior vertices of $x_1 - x_2$ path is $1 + 2 + \cdots + (a + b - 2) = \left( \frac{a+b-1}{2} \right)$ (see Fig. 3). The sum of distances from $x_1$ to all interior vertices of $x_1 - x_3$ path, not included in the previous calculation,
is \( \binom{a+c+d}{2} - \binom{a+c}{2} \). Finally, the sum of distances from \( x_1 \) to all interior vertices of \( x_1 - x_4 \) path, not included previously, is \( \binom{a+b+d}{2} - \binom{a+b+e}{2} \). In this way we get \( S_1(x_1) \) and analogously we calculate \( S_1(x_2), S_1(x_3) \) and \( S_1(x_4) \):

\[
S_1(x_1) = -\left( \frac{a+b-1}{2} \right) - \left( \frac{a+e+c}{2} \right) + \left( \frac{a}{2} \right) - \left( \frac{a+e+d}{2} \right) + \left( \frac{a+e+1}{2} \right),
\]

\[
S_1(x_2) = -\left( \frac{a+b-1}{2} \right) - \left( \frac{b+e+c}{2} \right) + \left( \frac{b}{2} \right) - \left( \frac{b+e+d}{2} \right) + \left( \frac{b+e+1}{2} \right),
\]

\[
S_1(x_3) = -\left( \frac{a+e+c}{2} \right) - \left( \frac{b+e+c}{2} \right) + \left( \frac{e+c+1}{2} \right) - \left( \frac{c+d-1}{2} \right) + \left( \frac{c}{2} \right),
\]

\[
S_1(x_4) = -\left( \frac{a+e+d}{2} \right) - \left( \frac{b+e+d}{2} \right) + \left( \frac{e+d+1}{2} \right) - \left( \frac{c+d-1}{2} \right) + \left( \frac{d}{2} \right).
\]

Now \( h(x_1, x_2) = -(a+b-1) \) and \( h(x_1, x_3) = -(a+e+c) \). Using the symmetry we obtain

\[
M_1 = -(a+b-1) - (a+e+c) - (a+e+d) - (b+e+d) - (c+d-1).
\]

In \( S_3(y) \) we sum \( h(u, y) \), where \( \deg(u) = 2 \) and \( \deg(y) = 3 \). Hence, \( \left( \frac{d_y}{2} \right) \left( \frac{d_y}{2} \right) - 1 = 2 \). Since \( \beta_5(u, y) = 2, \beta_1(u, y) = 1 \) and \( \beta_2(u, y) = 0 \), we have \( h(u, y) = 2d(u, y) + 1 \). Thus, the sum of \( h(u, y) \) for interior vertices \( u \) of \( y_1 - x_1 \) path is \( 2(1 + 2 + \cdots + (a-2)) + (a-2) = 2 \left( \binom{a-1}{2} \right) + (a-2) \) (see Fig. 3). Analogously, the sum of \( h(u, y) \) for interior vertices of \( y_2 - x_2 \) path is \( 2(2 + 3 + \cdots + (b-1)) + (b-2) = 2 \left( \binom{b}{2} \right) + (b-4) \); the sum of \( h(u, y) \) for interior vertices of \( y_3 - x_3 \) path is \( 2((e+3) + (e+4) + \cdots + (e+c)) + (c-2) = 2 \left( \binom{e+c+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (c-2) \). In this way we get

\[
S_3(y_1) = 2 \left( \binom{a-1}{2} \right) + (a-2) + 2 \left( \binom{b}{2} \right) + (b-4) + 2 \left( \binom{e+1}{2} \right) + (e-3)
\]

\[
+ 2 \left( \binom{e+c+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (c-2) + 2 \left( \binom{e+d+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (d-2),
\]

\[
S_3(y_2) = 2 \left( \binom{a}{2} \right) + (a-4) + 2 \left( \binom{b-1}{2} \right) + (b-2) + 2 \left( \binom{e+1}{2} \right) + (e-3)
\]

\[
+ 2 \left( \binom{e+c+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (c-2) + 2 \left( \binom{e+d+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (d-2),
\]

\[
S_3(y_3) = 2 \left( \binom{a+e+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (a-2) + 2 \left( \binom{b+e+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (b-2)
\]

\[
+ 2 \left( \binom{e+1}{2} \right) + (e-3) + 2 \left( \binom{c-1}{2} \right) + (c-2) + 2 \left( \binom{d}{2} \right) + (d-4),
\]

\[
S_3(y_4) = 2 \left( \binom{a+e+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (a-2) + 2 \left( \binom{b+e+1}{2} \right) - 2 \left( \binom{e+3}{2} \right) + (b-2)
\]

\[
+ 2 \left( \binom{e+1}{2} \right) + (e-3) + 2 \left( \binom{c}{2} \right) + (c-4) + 2 \left( \binom{d-1}{2} \right) + (d-2),
\]

\[
S_3(y_5) = 2 \left( \binom{a}{2} \right) + (a-4) + 2 \left( \binom{b}{2} \right) + (b-4) + 2 \left( \binom{e}{2} \right) + (e-1)
\]

\[
+ 2 \left( \binom{e+c}{2} \right) - 2 \left( \binom{e+2}{2} \right) + (c-2) + 2 \left( \binom{e+d}{2} \right) - 2 \left( \binom{e+2}{2} \right) + (d-2),
\]

Fig. 3. The graph \( L(H_{b,c,d,e}) \) for \( e \geq 1 \) and \( e = 0 \).
\[ S^3(y_6) = 2 \left( \frac{a + e}{2} \right) - 2 \left( \frac{e + 2}{2} \right) + (a - 2) + 2 \left( \frac{b + e}{2} \right) - 2 \left( \frac{e + 2}{2} \right) + (b - 2) \\
+ 2 \left( \frac{e}{2} \right) + (e - 1) + 2 \left( \frac{c}{2} \right) + (c - 4) + 2 \left( \frac{d}{2} \right) + (d - 4). \]

Consider \( h(y_i, y_j) \), where \( 1 \leq i < j \leq 6 \). Here \( \left( \frac{d_{y_i}}{2} \right) \left( \frac{d_{y_j}}{2} \right) - 1 = 8 \) and \( \beta_0(y_i, y_j) = 4 \). If \( y_i \) and \( y_j \) lie in a common triangle, then \( \beta_1(y_i, y_j) = 5 \) and \( \beta_2(y_i, y_j) = 0 \), while if \( y_i \) and \( y_j \) do not lie in a common triangle, then \( \beta_1(y_i, y_j) = 4 \) and \( \beta_2(y_i, y_j) = 1 \). This means that \( h(y_1, y_2) = 13, h(y_1, y_3) = 8(e + 2) + 6 = 8e + 22, h(y_1, y_6) = 8(e + 1) + 6 = 8e + 14 \) and \( h(y_5, y_6) = 8e + 6. \) Hence

\[
M^3 = \left( 13 + (8e + 22) + (8e + 22) + 13 + (8e + 14) \right) + \left( (8e + 22) + (8e + 22) \right) + \left( 8e + 14 \right) + \left( (8e + 14) + 13 \right) + \left( (8e + 6) \right).
\]

Finally,

\[
D = \sum_u \left[ 3 \left( \frac{d_{u}}{3} \right) + 6 \left( \frac{d_{u}}{4} \right) \right] = 6 \left( \frac{3}{3} \right) = 18.
\]

By (5), expanding the terms (using a computer package, for instance), we get

\[
2P = 7(a^2 + b^2 + c^2 + d^2 + e^2) - 6(ab + ac + ad + bc + bd + cd) + 4(\bar{a}e + be + ce + de) \\
+ 5(a + b + c + d) + 65e + 234 \\
= 7(a + b + c + d + e)^2 - 20(ab + ac + ad + bc + bd + cd) - 10(\bar{a}e + be + ce + de) \\
+ 5(a + b + c + d) + 65e + 234.
\]

Now substitute \( x = (a + b + c + d + e) \) and consider the equation \( 2P = 0 \) over \( \mathbb{Z}_5 \). We get

\[
2x^2 + 4 = 0.
\]

which has no solution in \( \mathbb{Z}_5 \). Consequently, \( P = 0 \) has no integer solution and \( W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e}) \neq 0 \) in this case.

Case 2. \( a, b, c \geq 2, d = 1, e \geq 1. \)

In this case the vertex \( x_4 = y_4 \) has degree 2, so we do not need to find \( S^1(x_4) \) and \( S^3(y_4) \), but we must include the distances to \( x_4 \) in \( S^1(x_1), S^1(x_2), S^1(x_3), S^3(y_1), S^3(y_2), S^3(y_3), S^3(y_5) \) and \( S^3(y_6) \). Analogously as in the previous case we have

\[
S^1(x_1) = -\left( \frac{a + b - 1}{2} \right) - \left( \frac{a + e + c}{2} \right) + \left( \frac{a}{2} \right) - (a + e + 1),
\]

\[
S^1(x_2) = -\left( \frac{a + b - 1}{2} \right) - \left( \frac{b + e + c}{2} \right) + \left( \frac{b}{2} \right) - (b + e + 1),
\]

\[
S^1(x_3) = -\left( \frac{a + e + c}{2} \right) - \left( \frac{b + e + c}{2} \right) + \left( \frac{e + c + 1}{2} \right) - c,
\]

\[
M^1 = -(a + b - 1) - (a + e + c) - (b + e + c).
\]

\[
S^3(y_1) = 2 \left( \frac{a - 1}{2} \right) + (a - 2) + 2 \left( \frac{b}{2} \right) + (b - 4) + 2 \left( \frac{e + 1}{2} \right) + (e - 3) + 2 \left( \frac{e + c + 1}{2} \right) \\
- 2 \left( \frac{e + 3}{2} \right) + (c - 2) + 2(e + 2) + 1,
\]

\[
S^3(y_2) = 2 \left( \frac{a}{2} \right) + (a - 4) + 2 \left( \frac{b - 1}{2} \right) + (b - 2) + 2 \left( \frac{e + 1}{2} \right) + (e - 3) + 2 \left( \frac{e + c + 1}{2} \right) \\
- 2 \left( \frac{e + 3}{2} \right) + (c - 2) + 2(e + 2) + 1,
\]

\[
S^3(y_3) = 2 \left( \frac{a + e + 1}{2} \right) - 2 \left( \frac{e + 3}{2} \right) + (a - 2) + 2 \left( \frac{b + e + 1}{2} \right) - 2 \left( \frac{e + 3}{2} \right) + (b - 2) \\
+ 2 \left( \frac{e + 1}{2} \right) + (e - 3) + 2 \left( \frac{c - 1}{2} \right) + (c - 2) + 2 + 1,
\]

\[
S^3(y_5) = \text{and } S^3(y_6).
\]
\[ S^3(y_5) = 2 \left( \frac{a}{2} \right) + (a - 4) + 2 \left( \frac{b}{2} \right) + (b - 4) + 2 \left( \frac{e}{2} \right) + (e - 1) + 2 \left( \frac{e + c}{2} \right) - 2 \left( \frac{e + 2}{2} \right) + (c - 2) + 2(e + 1) + 1, \]
\[ S^3(y_6) = 2 \left( \frac{a + e}{2} \right) - 2 \left( \frac{e + 2}{2} \right) + (a - 2) + 2 \left( \frac{b + e}{2} \right) - 2 \left( \frac{e + 2}{2} \right) + (b - 2) + 2 \left( \frac{e + c}{2} \right) + (e - 1) + 2 \left( \frac{c}{2} \right) + (c - 4) + 2 + 1, \]
\[ M^3 = \left( 13 + (8e + 22) + 13 + (8e + 14) \right) + ( (8e + 22) + 13 + (8e + 14) ) + ( (8e + 14) + 13 ) + ( (8e + 6) ), \]
\[ D = 5 \cdot 3 \left( \frac{3}{3} \right) = 15. \]
By (5), expanding the terms we get
\[ P = 3(a^2 + b^2 + c^2 + e^2) - 3(ab + ac + bc) + (ae + be) + 2ce - 2(a + b) - c + 28e + 97. \]
Since \((a - b)^2 + (b - c)^2 + (c - a)^2 = 2(a^2 + b^2 + c^2) - 2(ab + ac + bc) \geq 0\), we have \(3(a^2 + b^2 + c^2) - 3(ab + ac + bc) \geq 0\). Hence, if \(e \geq 2\), then
\[ P \geq 3e^2 + (e - 2)(a + b) + c(2e - 1) + 28e + 97 > 0. \]
This means that if \(P = 0\) then \(e = 1\). For \(e = 1\) we obtain
\[ P = 3(a^2 + b^2 + c^2) - 3(ab + ac + bc) - a - b + c + 128. \]
Substituting \(a = 128 + x, b = 128 + y\) and \(c = 128 + z\) we get
\[ P = 3(x^2 + y^2 + z^2) - 3(xy + xz + yz) - x - y - z. \]
Now we solve the equation \(P = 0\). This gives
\[ 3(x^2 + y^2 + z^2) - 3(xy + xz + yz) = x + y - z = 3t \]
or equivalently
\[ \frac{3}{2} (x - y)^2 + (y - z)^2 + (z - x)^2 = x + y - z = 3t. \]
where \(t\) is a nonnegative integer. Since \(x, y\) and \(z\) were defined using \(a, b\) and \(c\), the differences \((z - y)\) and \((z - x)\) are integer numbers. Set \(l = (z - y)\) and \(j = (z - x)\). Then \((x - y) = (z - y) - (z - x) = l - j\), so that
\[ 2t = (x - y)^2 + (y - z)^2 + (z - x)^2 = (i - j)^2 + (j - i)^2 = 2i^2 + 2j^2 - 2ij \]
and consequently \(3t = 3i^2 + 3j^2 - 3ij = x + y - z\). This gives
\[ x = 3t + (z - y) = 3i^2 + 3j^2 - 3ij + i, \]
\[ y = 3t + (z - x) = 3i^2 + 3j^2 - 3ij + j, \]
\[ z = x + y - 3t = 3i^2 + 3j^2 - 3ij + i + j, \]
which is equivalent to (1).
In [14] we proved that for every triple \(a, b, c\) satisfying (1) and \(e = 1\) it holds \(P = 0\) (that is, \(W(L^3(H_{a,b,c,1,1})) = W(H_{a,b,c,1,1})\)). Thus, \(P = 0\) in this case if and only if \(e = 1\) and \(a, b, c\) satisfy (1).
Case 3. \(a, c \geq 2, b = d = 1, e \geq 1.\)
The vertices \(x_2 = y_2\) and \(x_4 = y_4\) have degree 2, so we do not need to find \(S^1(x_2), S^1(x_4), S^3(y_2)\) and \(S^3(y_4)\). We have
\[ S^1(x_1) = -a - \left( \frac{a + e + c}{2} \right) - (a + e + 1), \]
\[ S^1(x_3) = - \left( \frac{a + e + c}{2} \right) - (e + c + 1) - c, \]
\[ M^1 = -(a + e + c), \]
\[ S^1(y_1) = 2 \left( \frac{a - 1}{2} \right) + (a - 2) + 2 + 2 \left( \frac{e + 1}{2} \right) + (e - 3) + 2 \left( \frac{e + c + 1}{2} \right) - 2 \left( \frac{e + 3}{2} \right) + (c - 2) + 2(e + 2) + 1, \]
The vertices \(D \geq 2\) and \(2 \geq a \geq c = d = 1, e \geq 1\). The vertices \(x_3 = y_3\) and \(x_4 = y_4\) have degree 2, so we do not need to find \(S^1(x_3), S^1(x_4), S^3(y_3)\) and \(S^3(y_4)\). We have

\[
S^1(x_1) = -\left(\frac{a + b - 1}{2}\right) - \left(\frac{a + e + 2}{2}\right) + \left(\frac{a}{2}\right) - (a + e + 1),
\]

\[
S^1(x_2) = -\left(\frac{a + b - 1}{2}\right) - \left(\frac{b + e + 2}{2}\right) + \left(\frac{b}{2}\right) - (b + e + 1),
\]

\[
M^1 = -(a + b - 1),
\]

\[
S^3(y_1) = 2\left(\frac{a - 1}{2}\right) + (a - 2) + 2\left(\frac{b}{2}\right) + (b - 4) + 2\left(\frac{e + 1}{2}\right) + (e - 3) + 2(e + 2) + 1 + 2(e + 2) + 1,
\]

\[
S^3(y_2) = 2\left(\frac{a}{2}\right) + (a - 4) + 2\left(\frac{b - 1}{2}\right) + (b - 2) + 2\left(\frac{e + 1}{2}\right) + (e - 3) + 2(e + 2) + 1 + 2(e + 2) + 1,
\]

\[
S^3(y_3) = 2\left(\frac{a}{2}\right) + (a - 4) + 2\left(\frac{b}{2}\right) + (b - 4) + 2\left(\frac{e}{2}\right) + (e - 1) + 2(e + 1) + 1 + 2(e + 1) + 1,
\]

\[
S^3(y_6) = 2\left(\frac{a + e}{2}\right) - 2\left(\frac{e + 2}{2}\right) + (a - 2) + 2\left(\frac{b + e}{2}\right) - 2\left(\frac{e + 2}{2}\right) + (b - 2) + 2\left(\frac{e}{2}\right) + (e - 1) + 2 + 1 + 2 + 1,
\]

\[
M^3 = \left(13 + 13 + (8e + 14)\right) + \left(13 + (8e + 14)\right) + (8e + 6),
\]

\[
D = 4 \cdot 3 \left(\frac{3}{3}\right) = 12.
\]

By (5), expanding the terms we get

\[
2P = 5(a^2 + b^2 + e^2) - 6ab - 13(a + b) + 47e + 148.
\]

Since \(4(a - b)^2 = 4a^2 + 4b^2 - 8ab \geq 0\) and \((a + b - 7)^2 = a^2 + b^2 + 2ab - 14(a + b) + 49 \geq 0\), we get

\[
2P \geq a^2 + b^2 + 5e^2 + 2ab - 13(a + b) + 47e + 148
\]

\[
\geq 5e^2 + (a + b) + 47e + 99 > 0.
\]

Thus, the equation \(P = 0\) has no solution in this case.
Case 5. $a \geq 2, b = c = d = 1, e \geq 1$.

The vertices $x_2 = y_2, x_3 = y_3$ and $x_4 = y_4$ have degree 2, so we have

$$S^1(x_i) = -a - \left(\frac{a + e + 2}{2}\right) - (a + e + 1),$$

$$M^1 = 0,$$

$$S^3(y_1) = 2\left(\frac{a - 1}{2}\right) + (a - 2) + 2 + 2\left(\frac{e + 1}{2}\right) + (e - 3) + 2(e + 2) + 1 + 2(e + 2) + 1,$$

$$S^3(y_5) = 2\left(\frac{a - 4}{2}\right) + (a - 4) + 2 + 2\left(\frac{e}{2}\right) + (e - 1) + 2(e + 1) + 1 + 2(e + 1) + 1,$$

$$S^3(y_6) = 2\left(\frac{a + e}{2}\right) - 2\left(\frac{e + 2}{2}\right) + (a - 2) + 2(e + 1) + 1 + 2\left(\frac{e}{2}\right) + (e - 1) + 2 + 1 + 2 + 1,$$

$$M^3 = \left(13 + (8e + 14)\right) + (8e + 6),$$

$$D = 3 \cdot 3 \cdot 3 = 9.$$ 

By (5), expanding the terms we get

$$P = 2a^2 + 2e^2 - 10a + 17e + 48.$$ 

Since $(a - 5)^2 = a^2 - 10a + 25 \geq 0$, we get

$$P \geq a^2 + 2e^2 + 17e + 23 > 0.$$ 

Thus, the equation $P = 0$ has no solution in this case.

Case 6. $a, b, c, d \geq 2, e = 0$.

In this case, and also in the next four, we have $y_5 = y_6$ and the degree of $y_5$ is 4 (see Fig. 3). This does not affect $S^1(x_i),\ M^1$ and $S^3(y_j)$, where $1 \leq i, j \leq 4$. Hence, analogously as above we get

$$S^1(x_1) = -\left(\frac{a + b - 1}{2}\right) - \left(\frac{a + c}{2}\right) + \left(\frac{a}{2}\right) - \left(\frac{a + d}{2}\right) + \left(\frac{a + 1}{2}\right),$$

$$S^1(x_2) = -\left(\frac{a + b - 1}{2}\right) - \left(\frac{b + c}{2}\right) + \left(\frac{b}{2}\right) - \left(\frac{b + d}{2}\right) + \left(\frac{b + 1}{2}\right),$$

$$S^1(x_3) = -\left(\frac{a + c}{2}\right) - \left(\frac{b + c}{2}\right) + \left(\frac{c + 1}{2}\right) - \left(\frac{c + d - 1}{2}\right) + \left(\frac{c}{2}\right),$$

$$S^1(x_4) = -\left(\frac{a + d}{2}\right) - \left(\frac{b + d}{2}\right) + \left(\frac{d + 1}{2}\right) - \left(\frac{c + d - 1}{2}\right) + \left(\frac{d}{2}\right).$$

$$M^1 = -(a + b - 1) - (a + c) - (a + d) - (b + c) - (b + d) - (c + d - 1),$$

$$S^3(y_1) = 2\left(\frac{a - 1}{2}\right) + (a - 2) + 2\left(\frac{b}{2}\right) + (b - 4) + 2\left(\frac{c + 1}{2}\right) + (c - 8) + 2\left(\frac{d + 1}{2}\right) + (d - 8),$$

$$S^3(y_2) = 2\left(\frac{a}{2}\right) + (a - 4) + 2\left(\frac{b - 1}{2}\right) + (b - 2) + 2\left(\frac{c + 1}{2}\right) + (c - 8) + 2\left(\frac{d + 1}{2}\right) + (d - 8),$$

$$S^3(y_3) = 2\left(\frac{a + 1}{2}\right) + (a - 8) + 2\left(\frac{b + 1}{2}\right) + (b - 8) + 2\left(\frac{c - 1}{2}\right) + (c - 2) + 2\left(\frac{d}{2}\right) + (d - 4),$$

$$S^3(y_4) = 2\left(\frac{a + 1}{2}\right) + (a - 8) + 2\left(\frac{b + 1}{2}\right) + (b - 8) + 2\left(\frac{c}{2}\right) + (c - 4) + 2\left(\frac{d - 1}{2}\right) + (d - 2),$$

where we simplified expressions as $2\left(\frac{a + 0 + 1}{2}\right) - 2\left(\frac{a + 0}{2}\right) - (a - 2)$ to $2\left(\frac{a + 1}{2}\right) + (a - 8)$. 

Now we discuss the terms containing $h(u, y_5)$. In $S^3(y_5)$ we sum $h(u, y_5)$, where deg$(u) = 2$ and deg$(y_5) = 4$. Hence $\frac{d}{2} \left(\frac{\beta_3(u, y_5)}{2}\right) - 1 = 5$. Since $\beta_3(u, y_5) = 3, \beta_1(u, y_5) = 3$ and $\beta_2(u, y_5) = 0$, we have $h(u, y_5) = 5d(u, y_1) + 3$. Thus, the sum of $h(u, y_5)$ for interior vertices $u$ of $y_1 - x_1$ path is $5(2 + 3 + \cdots + (a - 1)) + 3(a - 2) = 5\left(\frac{a}{2}\right) - 5 + 3(a - 2)$ (see Fig. 3).

In this way we get

$$S^3(y_5) = 5\left(\frac{a}{2}\right) - 5 + 3(a - 2) + 5\left(\frac{b}{2}\right) - 5 + 3(b - 2) + 5\left(\frac{c}{2}\right) - 5 + 3(c - 2) + 5\left(\frac{d}{2}\right) - 5 + 3(d - 2).$$
Now consider \( h(y_1, y_3), \ 1 \leq i \leq 4 \). Here \( \left( \frac{d_i}{2} \right) \left( \frac{d_{n-i}}{2} \right) - 1 = 17 \) and \( \beta_0(y_1, y_3) = 2 \cdot 3 = 6 \). Since \( y_1 \) and \( y_3 \) always lie in a common triangle, we have \( \beta_1(y_1, y_3) = 11 \) and \( \beta_2(y_1, y_3) = 1 \) (see Fig. 3). Thus, \( h(y_1, y_3) = 17 \cdot 1 + 11 + 2 \cdot 1 = 30 \). As regards \( h(y_i, y_j) \), where \( 1 \leq i < j \leq 4 \), analogously as above we get \( h(y_1, y_2) = 13 \) and \( h(y_1, y_3) = 8e + 22 = 22 \). Hence

\[
M^3 = (13 + 22 + 22 + 30) + (22 + 22 + 30) + (13 + 30) + 30.
\]

Finally,

\[
D = \sum_{u} \left[ 3 \left( \frac{d_u}{3} \right) + 6 \left( \frac{d_u}{4} \right) \right] = 4 \left[ 3 \left( \frac{3}{3} \right) + 3 \left( \frac{4}{3} \right) + 6 \left( \frac{4}{4} \right) \right] = 12 + 18.
\]

By (5), expanding the terms we get

\[
P = 4(a^2 + b^2 + c^2 + d^2) - 3(ab + ac + ad + bc + bd + cd) + 3(a + b + c + d) + 137
\]

\[
= 4(a + b + c + d)^2 - 11(ab + ac + ad + bc + bd + cd) + 3(a + b + c + d) + 137.
\]

Substitute \( x = (a + b + c + d) \) and consider the equation \( P = 0 \) over \( \mathbb{Z}_{11} \). We get

\[
4x^2 + 3x + 5 = 0,
\]

which has no solution in \( \mathbb{Z}_{11} \). Consequently, \( P = 0 \) has no integer solution and \( W(L^3(H_{a,b,c,d,0})) - W(H_{a,b,c,d,0}) \neq 0 \) in this case.

Case 7. \( a, b, c \geq 2, d = 1, e = 0 \).

In this case the vertex \( x_4 = y_4 \) has degree 2, so we do not need to find \( S^1(x_4) \) and \( S^3(x_4) \). Analogously as in the previous case we have

\[
S^1(x_1) = -\left( \frac{a + b - 1}{2} \right) - \left( \frac{a + c}{2} \right) + \left( \frac{a}{2} \right) - (a + 1),
\]

\[
S^1(x_2) = -\left( \frac{a + b - 1}{2} \right) - \left( \frac{b + c}{2} \right) + \left( \frac{b}{2} \right) - (b + 1),
\]

\[
S^1(x_3) = -\left( \frac{a + c}{2} \right) - \left( \frac{b + c}{2} \right) + \left( \frac{c + 1}{2} \right) - c,
\]

\[
M^1 = -(a + b - 1) - (a + c) - (b + c),
\]

\[
S^3(y_1) = 2 \left( \frac{a - 1}{2} \right) + (a - 2) + 2 \left( \frac{b}{2} \right) + (b - 4) + 2 \left( \frac{c + 1}{2} \right) + (c - 8) + 4 + 1,
\]

\[
S^3(y_2) = 2 \left( \frac{a}{2} \right) + (a - 4) + 2 \left( \frac{b - 1}{2} \right) + (b - 2) + 2 \left( \frac{c + 1}{2} \right) + (c - 8) + 4 + 1,
\]

\[
S^3(y_3) = 2 \left( \frac{a + 1}{2} \right) + (a - 8) + 2 \left( \frac{b + 1}{2} \right) + (b - 8) + 2 \left( \frac{c - 1}{2} \right) + (c - 2) + 2 + 1,
\]

\[
S^3(y_3) = 5 \left( \frac{a}{2} \right) - 5 + 3(a - 2) + 5 \left( \frac{b}{2} \right) - 5 + 3(b - 2) + 5 \left( \frac{c}{2} \right) - 5 + 3(c - 2) + 5 + 3.
\]

\[
M^3 = (13 + 22 + 30) + (22 + 30) + 30,
\]

\[
D = 3 \left( \frac{3}{3} \right) + 3 \left( \frac{4}{3} \right) + 6 \left( \frac{4}{4} \right) = 9 + 18.
\]

By (5), expanding the terms we get

\[
2P = 7(a^2 + b^2 + c^2) - 6(ab + ac + bc) - 3(a + b) - c + 232.
\]

Since \( 3(a - b)^2 + 3(b - c)^2 + 3(c - a)^2 = 6(a^2 + b^2 + c^2) - 6(ab + ac + bc) \geq 0 \) and also \( (a - 2)^2 + (b - 2)^2 + (c - 1)^2 = (a^2 + b^2 + c^2) - 4(a + b) - 2c + 9 \geq 0 \), we get

\[
2P \geq (a^2 + b^2 + c^2) - 3(a + b) - c + 232
\]

\[
\geq a + b + c + 223 > 0.
\]

Thus, the equation \( P = 0 \) has no solution in this case.
Case 8. \(a, c \geq 2, b = d = 1, e = 0\).
The vertices \(x_2 = y_2\) and \(x_4 = y_4\) have degree 2, so we have

\[
S^1(x_1) = -a - \left(\frac{a + c}{2}\right) - (a + 1),
\]
\[
S^1(x_3) = -\left(\frac{a + c}{2}\right) - (c + 1) - c,
\]
\[
M^1 = -(a + c),
\]
\[
S^3(y_1) = 2 \left(\frac{a - 1}{2}\right) + (a - 2) + 2 + 1 + 2 \left(\frac{c + 1}{2}\right) + (c - 8) + 4 + 1,
\]
\[
S^3(y_3) = 2 \left(\frac{a + 1}{2}\right) + (a - 8) + 4 + 1 + 2 \left(\frac{c - 1}{2}\right) + (c - 2) + 2 + 1,
\]
\[
S^3(y_5) = 5 \left(\frac{a}{2}\right) - 5 + 3(a - 2) + 5 + 3 + 5 \left(\frac{c}{2}\right) - 5 + 3(c - 2) + 5 + 3,
\]
\[
M^3 = (22 + 30) + 30.
\]
\[
D = 2 \left(3 \left(\frac{3}{3}\right)\right) + \left(3 \left(\frac{4}{3}\right) + 6 \left(\frac{4}{4}\right)\right) = 6 + 18.
\]

By (5), expanding the terms we get

\[
P = 3(a^2 + c^2) - 3ac - 5(a + c) + 92.
\]

Since \(2(a - c)^2 = 2(a^2 + c^2) - 4ac \geq 0\) and \((a - 3)^2 + (c - 3)^2 = (a^2 + c^2) - 6(a + c) + 18 \geq 0\), we get

\[
P \geq (a^2 + c^2) + ac - 5(a + c) + 92
\]
\[
\geq ac + (a + c) + 74 > 0.
\]

Thus, the equation \(P = 0\) has no solution in this case.

Case 9. \(a, b \geq 2, c = d = 1, e = 0\).
The vertices \(x_3 = y_3\) and \(x_4 = y_4\) have degree 2, so we have

\[
S^1(x_1) = -\left(\frac{a + b - 1}{2}\right) - (a + 1) - (a + 1) - a,
\]
\[
S^1(x_2) = -\left(\frac{a + b - 1}{2}\right) - (b + 1) - (b + 1) - b,
\]
\[
M^1 = -(a + b - 1),
\]
\[
S^3(y_1) = 2 \left(\frac{a - 1}{2}\right) + (a - 2) + 2 \left(\frac{b}{2}\right) + (b - 4) + 4 + 1 + 4 + 1,
\]
\[
S^3(y_2) = 2 \left(\frac{a}{2}\right) + (a - 4) + 2 \left(\frac{b - 1}{2}\right) + (b - 2) + 4 + 1 + 4 + 1,
\]
\[
S^3(y_5) = 5 \left(\frac{a}{2}\right) - 5 + 3(a - 2) + 5 \left(\frac{b}{2}\right) - 5 + 3(b - 2) + 5 + 3 + 5 + 3,
\]
\[
M^3 = (13 + 30) + 30.
\]
\[
D = 2 \left(3 \left(\frac{3}{3}\right)\right) + \left(3 \left(\frac{4}{3}\right) + 6 \left(\frac{4}{4}\right)\right) = 6 + 18.
\]

By (5), expanding the terms we get

\[
P = 3(a^2 + b^2) - 3ab - 6(a + b) + 92.
\]

Since \(2(a - b)^2 = 2(a^2 + b^2) - 4ab \geq 0\) and \((a - 3)^2 + (b - 3)^2 = (a^2 + b^2) - 6(a + b) + 18 \geq 0\), we get

\[
P \geq (a^2 + b^2) + ab - 6(a + b) + 92
\]
\[
\geq ab + 74 > 0.
\]

Thus, the equation \(P = 0\) has no solution in this case.
Case 10. \( a \geq 2, \ b = c = d = 1, \ e = 0 \).
The vertices \( x_2 = y_2, x_3 = y_3 \) and \( x_4 = y_4 \) have degree 2, so we have

\[
S^1(x_1) = -\binom{a+1}{2} - (a + 1) - (a + 1) - a,
\]

\[
M^1 = 0,
\]

\[
S^3(y_1) = 2\binom{a-1}{2} + (a - 2) + 2 + 4 + 1 + 4 + 1,
\]

\[
S^4(y_2) = 5\binom{a}{2} - 5 + 3(a - 2) + 5 + 3 + 5 + 3 + 5 + 3,
\]

\[
M^3 = 30,
\]

\[
D = 3\binom{3}{3} + 3\binom{4}{3} + 6\binom{4}{4} = 3 + 18.
\]

By (5), expanding the terms we get

\[
2P = 5a^2 - 19a + 130.
\]

Since \( 5(a - 2)^2 = 5a^2 - 20a + 20 \), we get

\[
2P \geq a + 110.
\]

Thus, the equation \( P = 0 \) has no solution in this case.

Case 11. \( a = b = c = d = 1, \ e \geq 0 \).

In [16, Theorem 1.5] we proved that \( W(L^i(T)) > W(T) \) for every \( i \geq 3 \) and for every tree \( T \) which is different from a path and the claw \( K_{1,3} \) and in which no leaf is adjacent to a vertex of degree 2. By this statement, for \( H = H_{1,1,1,1,\ldots} \) we have \( W(L^3(H)) > W(H) \), which completes the proof. \( \square \)

Acknowledgments

The first author acknowledges partial support by Slovak research grant VEGA 1/0065/13. The first two authors acknowledge partial support by Slovak-Slovenian grant. The first and fourth authors acknowledge partial support by Slovenian research agency ARRS, program no. P1-00383, project no. L1-4292, and Creative Core FISNM 3330-13-500033.

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