On the sum of all distances in bipartite graphs

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The transmission of a connected graph $G$ is the sum of all distances between all pairs of vertices in $G$, it is also called the Wiener index of $G$. In this paper, sharp bounds on the transmission are determined for several classes of connected bipartite graphs. For example, in the class of all connected $n$-vertex bipartite graphs with a given matching number $q$, the minimum transmission is realized only by the graph $K_{q,n-q}$; in the class of all connected $n$-vertex bipartite graphs of diameter $d$, the extremal graphs with the minimal transmission are characterized. Moreover, all the extremal graphs having the minimal transmission in the class of all connected $n$-vertex bipartite graphs with a given vertex connectivity (resp. edge-connectivity) are also identified.

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1. Introduction

In this paper, we only consider connected, simple and undirected graphs. Let $G = (V_G, E_G)$ be a graph with $u, v \in V_G$. Then $G - v, G - uv$ denote the graph obtained from $G$ by deleting vertex $v \in V_G$, or edge $uv \in E_G$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is obtained from $G$ by adding an edge $uv \notin E_G$. For $v \in V_G$, let $N_G(v)$ (or $N(v)$ for short) denote the set of all the adjacent vertices of $v$ in $G$ and $d_G(v) = |N_G(v)|$, the degree of $v$ in $G$. In particular, let $\Delta(G) = \max\{d_G(x) | x \in V_G\}$ and $\delta(G) = \min\{d_G(x) | x \in V_G\}$. For convenience, let $N_G[u] = N_G(u) \cup \{u\}$. The distance $d_G(u, v)$ between vertices $u$ and $v$ in $G$ is defined as the length of a shortest path between them. The diameter of $G$ is the maximal distance between any two vertices of $G$. $D_G(u)$ denotes the sum of all distances from $u$ in $G$.

Recall that $G$ is called $k$-connected if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V_G$ with $|X| < k$. The greatest integer $k$ such that $G$ is $k$-connected is the connectivity $\kappa(G)$ of $G$. Thus, $\kappa(G) = 0$ if and only if $G$ is disconnected or $K_1$, and $\kappa(K_n) = n - 1$ for all $n \geq 1$.

Analogously, if $|G| > 1$ and $G - E'$ is connected for every set $E' \subseteq E_G$ of fewer than $l$ edges, then $G$ is called $l$-edge-connected. The greatest integer $l$ such that $G$ is $l$-connected is the edge-connectivity $\kappa'(G)$ of $G$. In particular, $\kappa'(G) = 0$ if $G$ is disconnected.

A bipartite graph $G$ is a simple graph, whose vertex set $V_G$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that every edge of $G$ joins a vertex of $V_1$ with a vertex of $V_2$. A bipartite graph in which every two vertices from different partition classes are adjacent is called complete, which is denoted by $K_{m,n}$, where $m = |V_1|, n = |V_2|$. 

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A vertex (edge) independent set of a graph $G$ is a set of vertices (edges) such that any two distinct vertices (edges) of the set are not adjacent (incident on a common vertex). The vertex (edge) independence number of $G$, denoted by $\alpha(G)$ ($\alpha'(G)$), is the maximum of the cardinalities of all vertex (edge) independent sets. A vertex (edge) cover of a graph $G$ is a set of vertices (edges) such that each edge (vertex) of $G$ is incident with at least one vertex (edge) of the set. The vertex (edge) cover number of $G$, denoted by $\beta(G)$ ($\beta'(G)$), is the minimum of the cardinalities of all vertex (edge) covers. For a connected graph $G$ of order $n$, its matching number $\alpha'(G)$ satisfies $1 \leq \alpha'(G) \leq \lfloor \frac{n}{2} \rfloor$. When we consider an edge cover of a graph, we always assume that the graph contains no isolated vertex. It is known that for a graph $G$ of order $n$, $\alpha(G) + \beta(G) = n$; and if in addition $G$ has no isolated vertex, then $\alpha'(G) + \beta'(G) = n$. For a bipartite graph $G$, one has $\alpha'(G) = \beta(G)$, and $\alpha(G) = \beta'(G)$.

Let $\mathcal{A}^d_n$ be the class of all bipartite graphs of order $n$ with matching number $k$; $\mathcal{R}^d_n$ be the class of all bipartite graphs of order $n$ with diameter $d$; $\mathcal{C}^d_n$ (resp. $\mathcal{R}^d_n$) be the class of all $n$-vertex bipartite graphs with connectivity $s$ (resp. edge-connectivity $t$).

The transmission of $G$ is the sum of distances between all pairs of vertices of $G$, which is denoted by

$$W(G) = \sum_{u,v \in V_G} d_G(u,v) = \frac{1}{2} \sum_{v \in V_G} \sum_{w \in V_G} d_G(v,w).$$

This quantity was introduced by Wiener in [11] and has been extensively studied in the monograph [1] and was named ‘gross status’ [13], ‘total status’ [1], ‘graph distance’ [8] and ‘transmission’ [19, 20]. In the chemical literature $W(G)$ is nowadays known exclusively under the name ‘Wiener index’. For a mathematical work mentioning the Wiener index see [17]. It is related to several properties of chemical molecules; see [12]. For this reason Wiener index is widely studied by chemists, although it has interesting applications also in computer networks (see [7]). Recently, several special issues of journals were devoted to (mathematical properties of) Wiener index [10, 9, 5]. For surveys and some up-to-date papers related to Wiener index of trees and line graphs, see [4, 17, 15, 16, 18, 22] and [2, 3, 6, 14, 21], respectively.

In this paper we study the quantity $W$ in the case of $n$-vertex bipartite graphs, which is an important class of graphs in graph theory. Based on the structure of bipartite graphs, sharp bounds on $W$ among $\mathcal{A}^d_n$ (resp. $\mathcal{R}^d_n$, $\mathcal{C}^d_n$, $\mathcal{R}^d_n$) are determined. The corresponding extremal graphs are identified, respectively.

Further on we need the following lemma, which is the direct consequence of the definition of $W$.

**Lemma 1.** Let $G$ be a connected graph of order $n$ and not isomorphic to $K_n$. Then for each edge $e \in \overline{G}$, $W(G) > W(G + e)$.

**2. The graph with minimum transmission among $\mathcal{A}^d_n$**

In this section, we determine the sharp lower bound on the transmission of all $n$-vertex bipartite graphs with matching number $q$. The unique corresponding extremal graph is identified.

**Theorem 2.1.** Let $G$ be in $\mathcal{A}^d_n$. Then $W(G) \geq n^2 + q^2 - qn - n$ with equality if and only if $G \cong K_{q,n-q}$.

**Proof.** It is routine to check that

$$W(K_{q,n-q}) = n^2 + q^2 - qn - n.$$

So in what follows, we show that $K_{q,n-q}$ is the unique graph in $\mathcal{A}^d_n$ with the minimum transmission.

Choose $G$ in $\mathcal{A}^d_n$ such that its transmission is as small as possible. If $q = \lfloor \frac{n}{2} \rfloor$, by **Lemma 1.1** the extremal graph is just $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, as desired. So in what follows, we consider $q < \lfloor \frac{n}{2} \rfloor$.

Let $(U, W)$ be the bipartition of the vertex set of $G$ such that $|W| \geq |U| \geq q$, and let $M$ be a maximal matching of $G$. By **Lemma 1.1**, the sum of all distances of a graph decreases with addition of edges, so if $|U| = q$, then the extremal graph is $G = K_{q,n-q}$. So we assume that $|U| > q$ in what follows.

Let $U_M$, $W_M$ be the sets of vertices of $U$, $W$ which are incident to the edges of $M$, respectively. Therefore, $|U_M| = |W_M| = q$. Note that $G$ contains no edges between the vertices of $U \setminus U_M$ and the vertices of $W \setminus W_M$, otherwise any such edge may be united with $M$ to produce a matching of cardinality greater than that of $M$, violating the maximality of $M$.

Adding all possible edges between the vertices of $U_M$ and $W_M$, $U_M$ and $W \setminus W_M$, $U \setminus U_M$ and $W_M$, we get a graph $G'$ with $W(G') < W(G)$. Note that the matching number of $G'$ is at least $k + 1$. Hence, $G' \notin \mathcal{A}^d_n$ and $G \cong G'$. Based on $G'$, we construct a new graph, say $G''$, which is obtained from $G'$ by deleting all the edges between $U \setminus U_M$ and $W_M$, and adding all the edges between $U \setminus U_M$ and $U_M$, $G''$ is depicted in **Fig. 1**. It is routine to check that $G'' \cong K_{n-k}$.

Let $|U \setminus U_M| = n_1$, $|W \setminus W_M| = n_2$. Suppose $n_2 \geq n_1$. We partition $V_G = V_G'$ into $U_M \cup W_M \cup (U \setminus U_M) \cup (W \setminus W_M)$ as shown in **Fig. 1**. By direct calculation, for all $x \in W \setminus W_M$ (resp. $y \in U_M$, $z \in W_M$, $w \in U \setminus U_M$), one has

$$D_G(x) = 3q + 3n_1 + 2n_2 - 2, \quad D_G'(x) = 3q + 2n_1 + 2n_2 - 2, \quad D_G(y) = 3q + 2n_1 + n_2 - 2, \quad D_G'(y) = 3q + 2n_1 + n_2 - 2,$$

$$D_G(z) = 3q + 2n_2 + 2n_2 - 2, \quad D_G'(z) = 3q + 2n_2 + n_1 - 2, \quad D_G'(w) = 3q + 2n_2 + n_1 - 2.$$
This gives
\[
W(G') - W(G'') = \frac{1}{2} \left( \sum_{x \in V'} D_G(x) - \sum_{x \in V''} D_G(x) \right)
= \frac{1}{2} \left( \sum_{x \in W, w_m} D_G(x) - \sum_{x \in W, w_m} D_G(x) + \sum_{y \in U} D_G(y) - \sum_{y \in U} D_G(y) \right)
+ \frac{1}{2} \left( \sum_{z \in W, w_m} D_G(z) - \sum_{z \in W, w_m} D_G(z) \right)
\]
\[
\frac{1}{2} \left[ n_z (3q + 3n_1 + 2n_2 - 2) - n_z (3q + 2n_1 + 2n_2 - 2) + qn_1 - qn_1
+ n_l (3q + 3n_2 + 2n_1 - 2) - n_l (3q + 2n_2 + 2n_1 - 2) \right]
\]
\[
= n_1 n_2 > 0.
\]
This completes the proof. \(\square\)

According to the relationship among the parameters such as \(\alpha(G), \alpha'(G), \beta(G), \beta'(G)\) of a connected bipartite graph \(G\), the following is a direct consequence of Theorem 2.1.

**Corollary 2.2.** The graph \(K_{n,n-\sigma}\) is the unique graph that minimizes the transmission among all connected bipartite graphs of order \(n\) with vertex cover number or vertex independence number or edge cover number \(\sigma\).

### 3. The graph with minimum transmission among \(\mathcal{B}_n^d\)

Let \(G\) be a graph in \(\mathcal{B}_n^d\). Clearly there exists a partition \(V_0, V_1, \ldots, V_d\) of \(V_G\) such that \(|V_0| = 1\) and \(d(u, v) = i\) for each vertex \(v \in V_i\) and \(u \in V_0\) \((i = 0, 1, \ldots, d)\). We call \(V_i\) a block of \(V_G\). Two blocks \(V_i, V_j\) of \(V_G\) are adjacent if \(|i - j| = 1\). For convenience, let \(|V_i| = i\) throughout this section.

**Lemma 3.1** ([18]). For any graph \(G \in \mathcal{B}_n^d\) with the above partition of \(V_G\), \(G[V_i]\) induces an empty graph (i.e. containing no edge) for each \(i \in \{0, 1, \ldots, d\}\).

Given a complete bipartite graph \(K_{\frac{n-d+2}{2}, \frac{n-d+2}{2}}\) with bipartition \((X, Y)\) satisfying \(|Y| = \lceil \frac{n-d+2}{2} \rceil\) and \(|X| = \lfloor \frac{n-d+2}{2} \rfloor \geq 2\), choose a vertex \(x\) (resp. \(y\)) in \(X\) (resp. \(Y\)) and let
\[
G' = K_{\frac{n-d+2}{2}, \frac{n-d+2}{2}} - xy,
\]
where \(G'\) is depicted in Fig. 2. Let \(G^*\) be the graph obtained from \(G'\) by attaching paths \(P_{\frac{n-d+2}{2}}\) and \(P_{\frac{n-d+2}{2}}\) at \(x\) and \(y\), respectively. It is routine to check that \(G^* \in \mathcal{B}_n^d\) for odd \(d\).

Given a complete bipartite graph \(K_{p,q}\) with bipartition \((X, Y)\) satisfying \(|X| = p \geq 3\), \(|Y| = q \geq 2\), and \(p + q = n - d + 4\), choose two different vertices, say \(x, y\) in \(X\) and let
\[
G'' = K_{p,q} - \{xw : w \in V' \subseteq Y\} - \{yw' : w' \in Y \setminus V'\},
\]
where \(G''\) is depicted in Fig. 2. Let \(\hat{G}[p, q]\) be the graph obtained from \(G''\) by attaching paths \(P_{\frac{p+q}{2}}\) and \(P_{\frac{p+q}{2}}\) at \(x\) and \(y\), respectively. It is routine to check that \(\hat{G}[p, q] \in \mathcal{B}_n^d\) for even \(d\). Set:
\[
\mathcal{B} = \{\hat{G}[p, q] : p + q = n - d + 4, |(p-2) - q| \leq 1\}. 
\]
Theorem 3.2. Let $G$ be in $\mathcal{G}_d$ with the minimum transmission.

(i) If $d = 2$, then $G \cong K_{n, \frac{n}{2}}$.

(ii) If $d \geq 3$, then $G \cong G^*$ for odd $d$ and $G \in \mathcal{B}$ for even $d$, where $G^*$ and $\mathcal{B}$ are defined as above.

Proof. Choose $G \in \mathcal{G}_d$ with bipartition $(X, Y)$ such that its transmission is as small as possible.

(i) If $d = 2$, then by Lemma 1.1, $G \cong K_{n-1, d}$, where $t = n - t \geq 2$. Let $|X| = n - t$, $|Y| = t$. Then it is routine to check that, for all $x$ (resp. $y$) in $X$ (resp. $Y$), one has

$$D_G(x) = 2n - t - 2, \quad D_G(y) = n + t - 2.$$

This gives

$$W(K_{n-1, t}) = \frac{1}{2} \left( \sum_{x \in X} D_G(x) + \sum_{y \in Y} D_G(y) \right) = \frac{1}{2} (n - t)(2n - t - 2) + \frac{1}{2} t(n + t - 2) = t^2 - nt + n^2 - n.$$

If $n$ is odd, then $W(K_{n-1, t}) \geq \frac{3}{2} n^2 - \frac{3}{2} n + \frac{1}{2}$ with equality if and only if $t = \frac{n-1}{2}$, or $t = \frac{n+1}{2}$, i.e., $G \cong K_{\frac{n+1}{2}, \frac{n-1}{2}}$; and if $n$ is even, then $W(K_{n-1, t}) \geq \frac{3}{2} n^2 - \frac{1}{2} n$ with equality if and only if $t = \frac{n}{2}$, i.e., $G \cong K_{\frac{n}{2}, \frac{n}{2}}$, as desired.

(ii) First we show the following facts.

Fact 1. $G[V_{t-1} \cup V_t]$ induces a complete bipartite subgraph for each $i \in \{1, 2, \ldots, d\}$, and $|V_d| = 1$ for $d \geq 3$.

Proof of Fact 1. The first part follows directly from Lemmas 1.1 and 3.1. So in what follows, we prove the second part.

Let $d \geq 3$, $x \in V_t$ and $w \in V_{d-3}$. If $|V_d| \geq 2$, then $G + zw \in \mathcal{G}_d$ and $V_0 \cup V_1 \cup \cdots \cup V_{d-3} \cup (V_{d-2} \cup \{w\}) \cup V_{d-1} \cup (V_d \setminus \{w\})$ is a partition of $V_{d+2w}$. By Lemma 1.1 $W(G + zw) < W(G)$, a contradiction.

This completes the proof of Fact 1. $\Box$

Fact 2. Consider the vertex partition $V_G = V_0 \cup V_1 \cup \cdots \cup V_d$ of $G$.

(i) If $d$ is odd, then

$$|V_0| = |V_1| = \cdots = |V_{\frac{d+1}{2}-1}| = |V_{\frac{d+1}{2}+2}| = \cdots = |V_{d-1}| = |V_d| = 1, \quad |V_{\frac{d+1}{2}}| - |V_{\frac{d+1}{2}+1}| \leq 1. \quad (3.1)$$

(ii) If $d$ is even, then

$$|V_0| = |V_1| = \cdots = |V_{\frac{d}{2}-2}| = |V_{\frac{d}{2}+2}| = \cdots = |V_{d-1}| = |V_d| = 1, \quad |V_{\frac{d}{2}}| - \left( |V_{\frac{d}{2}-1}| + |V_{\frac{d}{2}+1}| \right) \leq 1. \quad (3.2)$$

Proof of Fact 2. (i) Note that $|V_0| = |V_d| = 1$, here we only show that $|V_t| = 1$ holds. Similarly, we can also show $|V_2| = \cdots = |V_{\frac{d+1}{2}-1}| = |V_{\frac{d+1}{2}+2}| = \cdots = |V_{d-1}| = 1$, we omit the procedure here.

In fact, if $d = 3$, our result is trivial. So we consider that $d \geq 5$. If $|V_t| \geq 2$, then choose $u \in V_t$ and let $G' = G - u_0 v + \{ux : x \in V_4\}$. In fact, the vertex partition of $G'$ is $V_0 \cup (V_1 \setminus \{u\}) \cup V_2 \cup (V_3 \cup \{u\}) \cup V_4 \cup \cdots \cup V_d$; in view of Fact 1 and the choice of $G$, any two of adjacent blocks of $V_{d}$ induce a complete bipartite subgraph and $|V_d| = 1$ for $d \geq 5$.

Note that $\sum_{i=4}^d l_i \geq d - 3 \geq d - \left\lceil \frac{d}{2} \right\rceil = \left\lceil \frac{d}{2} \right\rceil > l_0 = 1$. $D_G(u) = D_G(u) + 2 \sum_{i=4}^d l_i$ and $D_G(v) = D_G(v) - 2$ for all $v \in V_0$; $D_G(v) = D_G(v)$ for all $v \in (V_1 \setminus \{u\}) \cup V_2 \cup V_3$; $D_G(v) = D_G(v) + 2$ for all $v \in V_4 \cup V_5 \cup \cdots \cup V_d$. This gives

$$W(G) - W(G') = \frac{1}{2} \left( \sum_{v \in G} D_G(v) - \sum_{v \in G'} D_G(v) \right)$$

$$= \frac{1}{2} \left( \sum_{v \in V_0} (D_G(v) - D_G(v)) + \sum_{j=4}^d \sum_{v \in V_j} (D_G(v) - D_G(v)) + D_G(u) - D_G(u) \right)$$

$$= 2 \left( \sum_{j=4}^d l_j - 1 \right) > 0,$$

i.e. $W(G') < W(G)$, a contradiction to the choice of $G$. Hence, $|V_t| = 1$. 

Fig. 2. Graphs $G'$ and $G''$. 

Next we show that if \( d \) is odd, then \(|V\frac{d}{2}+1| - |V\frac{d}{2}| \leq 1\). Without loss of generality, we assume that \(|V\frac{d}{2}+1| \geq |V\frac{d}{2}|\). Then it suffices to show that \(|V\frac{d}{2}+1| - |V\frac{d}{2}| \leq 1\). If this is not true, then \(|V\frac{d}{2}+1| - |V\frac{d}{2}| \geq 2\). Choose \( w \in V\frac{d}{2}+1 \), let

\[ G' = G - \{ wx : x \in V\frac{d}{2} \cup V\frac{d}{2}+1 \} + \{ wy : y \in V\frac{d}{2} \cup V\frac{d}{2}+1 \} . \]

Then the vertex partition of \( G' \) is \( V_0 \cup V_1 \cup \cdots \cup V_{\frac{d}{2}} \cup (V_{\frac{d}{2}+1} \setminus \{ w \}) \cup (V_{\frac{d}{2}+1} \cup \{ w \}) \cup V_{\frac{d}{2}+2} \cup \cdots \cup V_d \) and each of the two adjacent blocks of \( V' \) induces a complete bipartite graph. By direct calculation, we have

\[ W(G') - W(G) = \left[ \left| V\frac{d}{2} \right| - 1 \right] + 2 \left| V\frac{d}{2}+1 \right| - \left[ \left| V\frac{d}{2} \right| - 1 \right] + \left| V\frac{d}{2}+1 \right| \]

\[ = - \left| V\frac{d}{2} \right| - \left( \left| V\frac{d}{2}+1 \right| + 1 \right) \leq -1 < 0 , \]

a contradiction to the choice of \( G \).

(ii) By the same discussion as the proof of the first part of (i) as above, we can show that \(|V_0| = |V_1| = \cdots = |V_{\frac{d}{2}-1}| = |V_{\frac{d}{2}+1}| = \cdots = |V_{d-1}| = |V_d| = 1\), we omit the procedure here.

Now we show that if \( d \) is even, then \(|V\frac{d}{2}| - (|V\frac{d}{2}-1| + |V\frac{d}{2}+1|) \leq 1\). Without loss of generality, we assume that \(|V\frac{d}{2}| < |V\frac{d}{2}-1| + |V\frac{d}{2}+1|\). Then it suffices to show that \(|V\frac{d}{2}-1| + |V\frac{d}{2}+1| - |V\frac{d}{2}| \leq 1\). If this is not true, then \(|V\frac{d}{2}-1| + |V\frac{d}{2}+1| - |V\frac{d}{2}| \geq 2\).

It is routine to check that at least one of \( V\frac{d}{2}-1 \) and \( V\frac{d}{2}+1 \) contains at least two vertices. Hence, we assume without loss of generality that \(|V\frac{d}{2}-1| \geq 2\). Choose \( w \in V\frac{d}{2}-1 \) and let

\[ G' = G - \{ wx : x \in V\frac{d}{2}-2 \cup V\frac{d}{2} \} + \{ wy : y \in V\frac{d}{2}-1 \cup V\frac{d}{2}+1 \} . \]

Then the vertex partition of \( G' \) is \( V_0 \cup V_1 \cup \cdots \cup (V\frac{d}{2}-1 \setminus \{ w \}) \cup (V\frac{d}{2}-1 \cup \{ w \}) \cup V\frac{d}{2}+1 \cup \cdots \cup V_d \) and each of the two adjacent blocks of \( V' \) induces a complete bipartite graph. By direct calculation, we have

\[ W(G') - W(G) = \left[ \left| V\frac{d}{2}+1 \right| + |V\frac{d}{2}-1| - 1 \right] + 2 \left| V\frac{d}{2} \right| - \left[ \left| V\frac{d}{2}+1 \right| + |V\frac{d}{2}-1| - 1 \right] + \left| V\frac{d}{2} \right| \]

\[ = - \left| V\frac{d}{2}+1 \right| + |V\frac{d}{2}-1| - \left( \left| V\frac{d}{2} \right| + 1 \right) \leq -1 < 0 , \]

a contradiction to the choice of \( G \).

This completes the proof of Fact 2. \( \Box \)

Now we come back to show the second part of Theorem 3.2. In view of Fact 2(i), if \( d \) is odd, note that \(|V\frac{d}{2}+1| + |V\frac{d}{2}+1| = n - d + 1\), together with \(|V\frac{d}{2}+1| - |V\frac{d}{2}+1| \leq 1\), we obtain that \( G \cong G' \), as desired.

In view of Fact 2(ii), if \( d \) is even, note that \(|V\frac{d}{2}| + |V\frac{d}{2}-1| + |V\frac{d}{2}+1| = n - d + 2\), together with \(|V\frac{d}{2}| - (|V\frac{d}{2}-1| + |V\frac{d}{2}+1|) \leq 1\), we obtain that \( G \in \mathcal{B} \). Furthermore, \( \mathcal{B} = \{ \hat{G}[p, q] : p + q = n - d + 4, p = \frac{n-d-6}{2} \} \) for even \( n \) and \( \mathcal{B} = \{ \hat{G}[p, q] : p + q = n - d + 4, p = \frac{n-d-2}{2} \} \) for odd \( n \).

This completes the proof. \( \Box \)

4. The graph with minimal transmission among \( \varphi_n^a \) (resp. \( \varphi_n^b \))

In this section, we determine sharp lower bounds on the sum of all distances of graphs among \( \varphi_n^a \) and \( \varphi_n^b \), respectively.

In \( K\alpha\beta \), we assume that \( p \geq q \) and by \( K_{0,0}, p \geq 1 \), we mean \( pK_1 \). We define two graphs \( O_n \cup_1 (K_{n_1,n_2} \cup K_{m_1,m_2}) \) and \( O_n \cup_2 (K_{n_1,n_2} \cup K_{m_1,m_2}) \), where \( \cup \) is the union of two graphs, \( O_n (s \geq 1) \) is an empty graph of order \( s \) and \( \cup_1 \) is a graph operation that joins all the vertices in \( O_n \) to the vertices belonging to the partitions of cardinality \( n_1 \) in \( K_{n_1,n_2} \) and \( m_1 \) in \( K_{m_1,m_2} \) respectively; whereas \( \cup_2 \) is a graph operation that joins all the vertices in \( O_n \) to the vertices belonging to the partitions of cardinality \( n_2 \) in \( K_{n_1,n_2} \) and \( m_2 \) in \( K_{m_1,m_2} \) respectively. Note that \( \cup_2 \) is defined only when \( n_2 \geq 1 \) and \( m_2 \geq 1 \).

Lemma 4.1. If \( s + q > p \) and \( p + s \geq 1 \), then \( W(O_n \cup_1 (K_1 \cup K_{p,q})) > W(O_n \cup_1 (K_1 \cup K_{p+1,q-1})) \).

Proof. Let us denote \( O_n \cup_1 (K_1 \cup K_{p,q}) \) by \( G \) and \( O_n \cup_1 (K_1 \cup K_{p+1,q-1}) \) by \( G' \). Here \( G \) and \( G' \) are depicted in Fig. 3. We partition \( V_G = V_C \cup C \cup A \cup B \cup [b_q] \), where \( C = \{ c_1, c_2, \ldots, c_s \} \), \( A = \{ a_1, a_2, \ldots, a_p \} \) and \( B = \{ b_1, b_2, \ldots, b_{q-1} \} \).

Note that:

\[ D_C(a) = D_{C'}(a) - 1 \quad \text{for any } a \in A; \quad D_C(b) = D_{C'}(b) + 1 \quad \text{for any } b \in B; \]

\[ D_C(c) = D_{C'}(c) + 1 \quad \text{for any } c \in C; \quad D_C(v) = D_{C'}(v) + 1; \quad D_C(b_q) = D_{C'}(b_q) + s + q - p. \]
Hence, this gives

\[
W(G) - W(G') = \frac{1}{2} \left( \sum_{v \in V_G} D_G(v) - \sum_{v \in V_{G'}} D_{G'}(v) \right)
\]

\[
= \frac{1}{2} \left( \sum_{a \in A} (D_G(a) - D_{G'}(a)) + \sum_{b \in B} (D_G(b) - D_{G'}(b)) + \sum_{c \in C} (D_G(c) - D_{G'}(c)) \right)
\]

\[
+ \frac{1}{2} (D_G(v) - D_{G'}(v) + D_G(u) - D_{G'}(u))
\]

\[
= \frac{1}{2} \left( -p + (q - 1) + s + 1 + s + q - p \right)
\]

\[
= s + q - p > 0.
\]

This completes the proof. \(\square\)

The following result is the direct consequence of the above lemma.

**Corollary 4.2.** If \(q \geq 1\), then \(W(O_q \cup (K_1 \cup K_{p,q})) \geq W(O_q \cup (K_1 \cup K_{p,q})).\) The equality holds only when \(p = q.\)

**Lemma 4.3.** If \(s + q + 2 < p\), then \(W(O_q \cup (K_1 \cup K_{p,q})) > W(O_q \cup (K_1 \cup K_{p-1,q+1})).\)

**Proof.** Let us denote \(O_q \cup (K_1 \cup K_{p,q})\) by \(G\) and \(O_q \cup (K_1 \cup K_{p-1,q+1})\) by \(G'.\) We partition \(V_G = V_{G'}\) into \(\{v\} \cup A \cup B \cup C \cup \{u\},\) where \(A = \{a_1, a_2, \ldots, a_{p-1}\},\) \(B = \{b_1, b_2, \ldots, b_q\}\) and \(C = \{c_1, c_2, \ldots, c_l\}\) (see Fig. 4).

Note that

\[
D_G(a) = D_{G'}(a) + 1 \quad \text{for any } a \in A; \quad D_G(b) = D_{G'}(b) - 1 \quad \text{for any } b \in B;
\]

\[
D_G(c) = D_{G'}(c) - 1 \quad \text{for any } c \in C; \quad D_G(v) = D_{G'}(v) - 1, \quad D_G(u) = D_{G'}(u) + p - s - q - 2.
\]

Hence, this gives

\[
W(G) - W(G') = \frac{1}{2} \left( \sum_{v \in V_G} D_G(v) - \sum_{v \in V_{G'}} D_{G'}(v) \right)
\]

\[
= \frac{1}{2} \left( \sum_{a \in A} (D_G(a) - D_{G'}(a)) + \sum_{b \in B} (D_G(b) - D_{G'}(b)) + \sum_{c \in C} (D_G(c) - D_{G'}(c)) \right)
\]

\[
+ \frac{1}{2} (D_G(v) - D_{G'}(v) + D_G(u) - D_{G'}(u))
\]
lemma 4.5. If $G$ is a graph with the minimal transmission in $\mathcal{C}_n^s$, then we may assume that $\Delta(G) = s$.\newline\indent This completes the proof. □

Similar to the above lemma we have the following lemma.

**Corollary 4.4.** If $1 \leq s \leq \lfloor \frac{n-2}{3} \rfloor$, then $W(K_{s,n-s}) \geq W(O_v \cup (K_1 \cup K_{n-s-2,1}))$. The equality holds if and only if $n = 2s + 3$.

**Lemma 4.5.** If $G \in \mathcal{C}_n^s$ and $U$ is a vertex-cut set of order $s$ in $G$ such that $G - U$ has two nontrivial components, then $G$ cannot be the graph with the minimal transmission in $\mathcal{C}_n^s$.

**Proof.** Assume that $G_1$ and $G_2$ are two nontrivial components of $G - U$ with bipartitions $(A, B)$ and $(C, D)$, respectively. Let $U_1 \cup U_2$ be the bipartition of $U$ induced by the bipartition of $G$. Now joining all possible edges between the vertices of $A$ and $B$, and $C$ and $D$, we get a graph $G$ in $\mathcal{C}_n^s$ such that $W(G) \geq W(C_n^s)$. Therefore we suppose that $G = G_2$; see Fig. 5.

If there exists some vertex $u$ in $G - U$ such that $d_G(u) = s$, then forming a complete bipartite graph within the vertices of $G \setminus \{u\}$ we would get a graph in $\mathcal{C}_n^s$ with smaller transmission. Thus we may assume that each vertex in $G - U$ has degree greater than $s$. Let $|A| = m_1$, $|B| = m_2$, $|C| = n_1$, $|D| = n_2$, $|U_1| = t$, $|U_2| = k$. We choose a vertex $u_0$ from $C$ and observe that $d_G(u_0) = t + |D| > s$, where $t(0 \leq t \leq s)$ is the total number of edges joining $u_0$ and the vertices of $U_1$. Note that $U_1 \cup U_2$ is the vertex-cut set of order $s$, hence $m_1, n_1 > t$, $m_2, n_2 > k$. Without loss of generality we may assume that $m_1 = \max\{m_1, m_2, n_1, n_2\}$ and note that $s \geq 1$, hence $m_1 \geq 2$. We now choose a subset $D_2$ of $D$ such that $|D_2| = |D| - k > 0$. Let

$$G^* = G - \{u_0X : x \in D_2\} + \{bc : b \in B, c \in C \setminus \{u_0\} \} + \{pq : p \in D, c \in A\}.$$ 

It is routine to check that $G^* \in \mathcal{C}_n^s$ with bipartition $(X, Y)$, where $X = B \cup D_2 \cup U_1 \cup D_1$, $Y = A \cup B \cup C' \cup \{u_0\}$ with $|U_1| = t$, $|U_2| = k$, $|A| = m_1$, $|B| = m_2$, $|C'| = n_1 - 1$, $|D_1| = k$, and $|D_2| = n_2 - k$. Here, $G^*$ is depicted in Fig. 5.

It is routine to check that $D_{G^*}(u_0) = D_{G^*}(u_0) + 2k - 2n_2$. Note that, for any $a \in A$ (resp. $b \in B, c \in C$, $d \in D_1, d' \in D_2$).

Then one has

$$D_{G^*}(a) = D_{G^*}(a) + 2m_2 - 2; \quad D_{G^*}(b) = D_{G^*}(b) + 2n_1 - 2; \quad D_{G^*}(c) = D_{G^*}(c) + 2m_2; \quad D_{G^*}(d) = D_{G^*}(d) + 2m_1 - 2; \quad D_{G^*}(d') = D_{G^*}(d') + 2m_1 - 2.$$ 

This gives

$$W(G) - W(G^*) = \frac{1}{2} \left( \sum_{v \in V_C} D_{G^*}(v) - \sum_{v \in V_{C'}} D_{G^*}(v) \right)$$

$$= \frac{1}{2} \left( \sum_{a \in A} (D_{G^*}(a) - D_{G^*}(a)) + \sum_{b \in B} (D_{G^*}(b) - D_{G^*}(b)) + \sum_{c \in C} (D_{G^*}(c) - D_{G^*}(c)) \right)$$

$$= \frac{1}{2} \left( D_{G^*}(u_0) - D_{G^*}(u_0) + \sum_{d \in D_1} (D_{G^*}(d) - D_{G^*}(d)) + \sum_{d' \in D_2} (D_{G^*}(d') - D_{G^*}(d')) \right)$$

$$= \frac{1}{2} \left[ m_1(2m_2 - 2) + m_2(2n_1 - 2) + 2(n_1 - 1)m_2 + 2k - 2m_2 + 2km_1 + (n_2 - k)(2m_1 - 2) \right]$$

$$= (m_2 - 1)m_1 + 2(n_1 - 1)m_2 + m_1k + (n_2 - k)(m_2 - 2)$$

$$> (n_2 - k)(m_2 - 2) \geq 0.$$ 

This completes the proof. □
Let $X$ and $Y$ be sets of vertices (not necessarily disjoint) of a graph $G$. We denote by $E_G[X, Y]$ the set of edges of $G$ with one end in $X$ and the other end in $Y$.

**Lemma 4.6.** If $G \in \mathcal{P}_1^n$ and $E_t$ is an edge cut-set of order $t$ in $G$ such that $G - E_t$ has two nontrivial components, then $G$ cannot be the graph with the minimal transmission in $\mathcal{P}_1^n$.

**Proof.** Let $G_1$ and $G_2$ be the nontrivial components of $G - E_t$ with bipartitions $(A, B)$ and $(C, D)$, respectively. Now joining all possible edges between the vertices of $A$ and $B$, $C$ and $D$ yields a graph, say $G$, in $\mathcal{P}_1^n$ such that $W(G) \geq W(G_t)$. Therefore we suppose that $G = G_t$; see Fig. 6.

Note that for any vertex $v \in G$ we have $d_G(v) \geq t$. If there exists some vertex $w$ in $G$ such that $d_G(w) = t$. Without loss of generality, adding all possible edges within the subgraph of $G$ induced by the vertices of $V_G \setminus \{w\}$, we will arrive at a bipartite graph $G'$. If $G \neq G'$, then $W(G) > W(G')$ by Lemma 3.1. Thus we may assume that each vertex in $G$ has degree greater than $t$.

Let $|A| = m_1$, $|B| = m_2$, $|C| = n_1$, $|D| = n_2$ and the number of edges between $A$ and $C$ (resp. $B$ and $D$), in $G$, is $a$ (resp. $b$). It is routine to check that $m_1 + m_2 + n_1 + n_2 = n$ and $a + b = t$.

Without loss of generality, we choose a vertex $c_0$ from $C$ and observe that $d_G(c_0) = h + |D| > t$, where $h(0 \leq h \leq t)$ is the total number of edges joining $c_0$ and the vertices of $A$. Note that $E_3[A, C] \cup E_3[B, D]$ is an edge-cut set of size $a + b = t$ (see Fig. 6), hence $m_1, n_1 > a$, $m_2, n_2 > b$. We now pick a subset $D_2$ of $D$ satisfying $|D_2| = |D| - (d_G(c_0) - h) > 0$. Let

$$G^* = G - \{u_0 : x \in D_2\} + \{ac : a \in A, c \in C \setminus \{u_0\}\} + \{pq : p \in B, q \in D\}.$$  

It is routine to check that $G^* \in \mathcal{P}_1^n$; see Fig. 6.

Denote the sets of the end-vertices of the edges of $E_t$ in $A, B, C,$ and $D$ by $S_1, S_2, S_3,$ and $S_4$, respectively. Let $|A - S_1| = a_1$, $|B - S_2| = a_2$, $|C - S_3| = a_3$, and $|D - S_4| = a_4$.

By direct calculation, $G$ contains

$$m_1m_2 + n_1n_2 + t = |E_G|$$

pairs of vertices at distance 1, $m_1n_1 + m_2n_2 - t$ pairs of vertices at distance 3, and $a_1a_4 + a_2a_3$ pairs of vertices at distance 4. All other vertex pairs, namely

$$\binom{n}{2} - |E_G| - (m_1n_1 + m_2n_2 - t) - (a_1a_4 + a_2a_3)$$

are at distance 2. Consequently,

$$W(G) = |E_G| + 3[m_1n_1 + m_2n_2 - t] + 4[a_1a_4 + a_2a_3]$$

$$+ 2\left[\binom{n}{2} - |E_G| - (m_1n_1 + m_2n_2 - t) - (a_1a_4 + a_2a_3)\right].$$  \hspace{1cm} (4.1)

Similarly, $G^*$ contains

$$m_1m_2 + (n_1 - 1)n_2 + m_2n_2 + (n_1 - 1)m_1 + t = |E_{G^*}|$$

pairs of vertices at distance 1, $m_1 + n_2 - t$ pairs of vertices at distance 3. All other vertex pairs, namely

$$\binom{n}{2} - |E_{G^*}| - (m_1 + n_2 - t)$$

are at distance 2. Consequently,

$$W(G^*) = |E_{G^*}| + 2\left[\binom{n}{2} - |E_{G^*}| - (m_1 + n_2 - t)\right] + 3(m_1 + n_2 - t).$$  \hspace{1cm} (4.2)
In view of (4.1) and (4.2), we have that
\[
W(G) - W(G^s) = 2(a_1a_4 + a_2a_3) + 2(m_2n_2 - n_2 + n_1m_1 - m_1) \\
= 2(a_1a_4 + a_2a_3) + 2n_2(m_2 - 1) + 2m_1(n_1 - 1) \\
> 2(a_1a_4 + a_2a_3) \geq 0.
\]
This completes the proof. \(\square\)

**Theorem 4.7.** Let \(G\) be the graph in \(\mathcal{G}_n^s\) with the minimal transmission with \(1 \leq s \leq \lfloor \frac{n-1}{2} \rfloor \). Then \(G \in \{G_1^s, G_2^s\}\) if \(n\) is odd and \(G \cong G_2^s\) otherwise. Graphs \(G_1^s\), \(G_2^s\) and \(G_3^s\) are depicted in Fig. 7.

**Proof.** Let \(G\) be a graph with the minimal transmission in \(\mathcal{G}_n^s\). Let \(U\) be a vertex cut of \(G\) containing \(s\) vertices, whose deletion yields the components \(G_1, G_2, \ldots, G_i\) of \(G - U\), where \(t \geq 2\). If some component \(G_i\) of \(G - U\) has at least two vertices, then it must be complete bipartite. Again if some component \(G_i\) of \(G - U\) is a singleton, say \(G_i = \{u\}\), then \(u\) is adjacent to all the vertices of \(U\); otherwise \(\varepsilon(G) < s\); hence the subgraph \(G[U]\) induced by \(U\) contains no edges, and belongs to the same partition of \(G\). We now have the following cases.

Case 1. All the components of \(G - U\) are singletons. In this case, we have \(G = K_{s,n-s}\), for \(s = \lfloor \frac{n-1}{2} \rfloor \) or \(\lfloor \frac{n-3}{2} \rfloor \). It is routine to check that \(K_{s,n-s} \cong G_1^s\), if \(n\) is odd and \(K_{s,n-s} \cong G_2^s\), if \(n\) is even, as desired.

Let us assume that \(1 \leq s \leq \lfloor \frac{n-3}{2} \rfloor \). Then by Corollary 4.4, \(W(K_{s,n-s}) \geq W(O_s \cup (K_1 \cup K_{n-s-2,1}))\), which contradicts the minimality of \(G\). Therefore not all the components of \(G - U\) can be singletons.

Case 2. One component of \(G - U\), say \(G_1\), contains at least two vertices. In this case, \(G - U\) contains exactly two components; otherwise, forming a complete bipartite graph within the vertices of \(G_1 \cup G_2 \cup \cdots \cup G_{i-1}\) we obtain a new graph \(G^*\) from \(G\) with smaller transmission such that \(G^* \in \mathcal{G}_n^s\), a contradiction. Let \(G_1, G_2\) be the components of \(G - U\). By Lemma 4.5, either \(G_1 = K_1\) or \(G_2 = K_1\). Without loss of generality assume that \(G_2 = K_1 = \{u\}\). Then \(u\) joins all vertices of \(U\), and each vertex of \(U\) joins every vertex of \(G_1\) which are in the same partition as \(u\). Note that \(G\) is a graph with the minimal transmission, hence by Corollary 4.2, \(G = O_s \cup (K_1 \cup K_{p,q})\) for some \(p\) and \(q\). We note that \(p \geq s\), otherwise \(s\) cannot be the vertex connectivity of \(G\). If \(q + s \leq p \leq q + s + 2\), then the result follows. Again if \(q + s > p\), then by repeated application of Lemma 4.1, \(G = G_1^s\), if \(n\) is odd and \(G = G_2^s\), if \(n\) is even. Finally if \(p \geq q + s + 3\), then by using Lemma 4.3 repeatedly, we have \(G\) is either \(G_2^s\) or \(G_3^s\) according to \(n\) is odd or even.

This completes the proof. \(\square\)

By a similar argument as in the proof of Theorem 4.7, we can show the following result, we omit its procedure here.

**Theorem 4.8.** Let \(G\) be a graph in \(\mathcal{G}_n^s\) with minimal transmission with \(1 \leq s \leq \lfloor \frac{n-1}{2} \rfloor \). Then \(G \in \{G_1^s, G_2^s\}\) if \(n\) is odd and \(G \cong G_2^s\) otherwise. Graphs \(G_1^s\), \(G_2^s\) and \(G_3^s\) are depicted in Fig. 7.

References