Wiener index of iterated line graphs of trees homeomorphic to H

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\textbf{A B S T R A C T}

This is fourth paper out of five in which we completely solve a problem of Dobrynin, Entringer and Gutman. Let $G$ be a graph. Denote by $L_i(G)$ its $i$-iterated line graph and denote by $W(G)$ its Wiener index. Moreover, denote by $H$ a tree on six vertices, out of which two have degree 3 and four have degree 1. Let $j \geq 3$. In previous papers we proved that for every tree $T$, which is not homeomorphic to a path, claw $K_{1,3}$ and $H$, it holds $W(L_i(T)) > W(T)$. Here we prove that $W(L_4(T)) > W(T)$ for every tree $T$ homeomorphic to $H$. As a consequence, we obtain that with the exception of paths and the claw $K_{1,3}$, for every tree $T$ it holds $W(L_i(T)) > W(T)$ whenever $i \geq 4$.

\begin{abstract}

This is fourth paper out of five in which we completely solve a problem of Dobrynin, Entringer and Gutman. Let $G$ be a graph. Denote by $L_i(G)$ its $i$-iterated line graph and denote by $W(G)$ its Wiener index. Moreover, denote by $H$ a tree on six vertices, out of which two have degree 3 and four have degree 1. Let $j \geq 3$. In previous papers we proved that for every tree $T$, which is not homeomorphic to a path, claw $K_{1,3}$ and $H$, it holds $W(L_i(T)) > W(T)$. Here we prove that $W(L_4(T)) > W(T)$ for every tree $T$ homeomorphic to $H$. As a consequence, we obtain that with the exception of paths and the claw $K_{1,3}$, for every tree $T$ it holds $W(L_i(T)) > W(T)$ whenever $i \geq 4$.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph. For any two of its vertices, say $u$ and $v$, denote by $d_G(u, v)$ (or by $d(u, v)$ if no confusion is likely) the distance from $u$ to $v$ in $G$. The \textit{Wiener index} (\textit{Wiener number}) of $G$, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of $G$. Wiener index was introduced by Wiener in [22]. It is related to several properties of chemical molecules; see [14]. For this reason Wiener index is widely studied by chemists, although it has interesting applications also in computer networks; see e.g. [9]. The interest of mathematicians was attracted in 1970s, when it was reintroduced as the \textit{transmission} and the \textit{distance of a graph}; see [21] and [10], respectively. Recently, several special issues of journals were devoted to (mathematical properties) of Wiener index (see [12,13]). For surveys and some up-to-date papers related to Wiener index of trees and line graphs, see [5,6,18–20,24] and [2,3,7,11,23], respectively.

By the definition, if $G$ has a unique vertex, then $W(G) = 0$. In this case, we say that the graph $G$ is \textit{trivial}. We set $W(G) = 0$ also when the set of vertices (and hence also the set of edges) of $G$ is empty.

The line graph of $G$, $L(G)$, has vertex set identical with the set of edges of $G$. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges share an endpoint in $G$. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0 \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In [1] we have the following statement.

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Theorem 1.1. Let $T$ be a tree on $n$ vertices. Then $W(L(T)) = W(T) - \binom{n}{3}$.

Since $\binom{n}{3} > 0$ if $n \geq 2$, there is no non-trivial tree for which $W(L(T)) = W(T)$. However, there are trees $T$ satisfying $W(L^2(T)) = W(T)$; see e.g. [4]. In [5], the following problem was posed:

Problem 1.2. Is there a tree $T$ satisfying the equality $W(L^i(T)) = W(T)$ for some $i \geq 3$?

As observed above, if $T$ is a trivial tree, then $W(L^i(T)) = W(T)$ for every $i \geq 1$, although here the graph $L^i(T)$ is empty. The real question is if there is some other tree $T$ and $i \geq 3$ such that $W(L^i(T)) = W(T)$. In this paper we solve Problem 1.2 for $i \geq 4$; see Corollary 1.4 below.

The essential part of this paper deals with trees homeomorphic to the graph $H$ on 6 vertices, depicted in Fig. 1, that is, the trees that have precisely two vertices of degree 3, four vertices of degree 1 and all other vertices of degree 2. (Recall that graphs $G_1$ and $G_2$ are homeomorphic if and only if the graphs obtained from $G_1$ and $G_2$, respectively, by repeatedly substituting the vertices of degree two together with the two incident edges with a single edge, are isomorphic.) In particular, we prove the following result:

Theorem 1.3. Let $T$ be a tree homeomorphic to $H$. Then $W(L^i(T)) > W(T)$ for every $i \geq 4$.

In [16] we proved an analogue of Theorem 1.3 for trees different from a path which are not homeomorphic to neither $K_{1,3}$ nor $H$, and in [17] for trees homeomorphic (but not isomorphic) to $K_{1,3}$. By observing that $W(L^i(T)) < W(T)$ for $T \cong K_{1,3}$ or $P_n$, we can summarize our results for $i \geq 4$ as follows:

Corollary 1.4. Let $T$ be a tree and $i \geq 4$. Then the following holds:

- $W(L^i(T)) < W(T)$ if $T$ is the claw $K_{1,3}$ or a path $P_n$ with $n \geq 2$;
- $W(L^i(T)) = W(T)$ if $T$ is the trivial graph $P_1$;
- $W(L^i(T)) > W(T)$ otherwise.

We remark that the case $i = 3$, which resolves Problem 1.2, is dealt with in a forthcoming paper.

2. Preliminaries

In paper [15], it was shown that for every connected graph $G$, the function $i \mapsto W(L^i(G))$ is convex. This implies that in order to prove Theorem 1.3, it suffices to prove

$$W(L^4(T)) > W(T)$$

for every tree $T$ homeomorphic to $H$.

By convexity of function $i \mapsto W(L^i(T))$, it follows that $W(L^4(T)) + W(L^2(T)) > 2W(L^3(T))$ and therefore $W(L^4(T)) = W(T) > 2W(L^3(T)) - W(L^2(T)) - W(T)$. Define the quantity $D(T)$ as

$$D(T) = 2W(L^3(T)) - W(L^2(T)) - W(T).$$

By the analysis above, (1) holds if the following lemma is true.

Lemma 2.1. Let $T$ be a tree homeomorphic to $H$. Then $D(T) > 0$.

A ray (a pendant path) in $T$ is a (directed) path, the first vertex of which has degree at least 3, its last vertex has degree 1, and all of its internal vertices (if any exist) have degree 2 in $G$. Let $T$ be a tree homeomorphic to $H$ such that a longest ray of $T$ has length at least 2. By pred($T$) we denote a tree obtained from $T$ by deleting the pendant vertex (and its incident edge) of a longest ray. Obviously, pred($T$), standing for a predecessor of $T$, induces a partial ordering on trees homeomorphic to $T$. We prove Lemma 2.1 by induction applied to this partial ordering.

We start with the base of the induction, that is with trees, all rays of which have length 1. In [15] we proved that if $T$ is a tree distinct from a path and the claw $K_{1,3}$ and all rays of $T$ have length 1, then $W(L^3(T)) > W(T)$. By convexity of function $i \mapsto W(L^i(T))$ we get $W(L^3(T)) > W(T) = D(T) > 0$.

Proposition 2.2. Let $T$ be a tree, all rays of which have length 1, distinct from a path and the claw $K_{1,3}$. Then $2W(L^1(T)) - W(L^2(T)) - W(T) = D(T) > 0$. 
Now we describe the induction step. Let $T$ be a tree with a longest ray of length at least 2, and let $R$ be a longest ray in $T$. Remove the last vertex of $R$ (i.e., the vertex of degree 1) and the edge incident with this vertex, and denote the resulting graph by $T^-$. Obviously, $T^-$ is pred($T$). By the induction hypothesis we assume that $D(T^-) > 0$ and our aim is to prove $D(T) > 0$. Define

$$
\Delta T = D(T) - D(T^-).
$$

(3)

To establish the induction step it suffices to prove:

**Lemma 2.3.** Let $T$ be a tree homeomorphic to $H$ with a longest ray of length at least 2 and let $T^-$ and $\Delta T$ be as defined above. Then $\Delta T \geq 0$.

Now we introduce some necessary notation used in the following sections. If $z$ is a vertex of a graph $G$, then its degree is denoted by $d_z$. By the definition, every vertex $w \in V(L(G))$ corresponds to an edge of $G$. Let us denote by $B_1(w)$ this edge of $G$. For two subgraphs $S_1$ and $S_2$ of $G$, by $d(S_1, S_2)$ we denote the shortest distance in $G$ between a vertex of $S_1$ and a vertex of $S_2$. If $S_1$ and $S_2$ share an edge then we set $d(S_1, S_2) = -1$. Let $w$ and $z$ be two vertices of $L(G)$. Then $d_{L(G)}(w, z) = d(B_1(w), B_1(z)) + 1$ (see also [15]).

Let $u, v \in V(G), u \neq v$. Denote by $\alpha_i(u, v)$ the number of pairs $w, z \in V(L(G))$, with $u \in V(B_1(w))$ and $v \in V(B_2(z))$, such that $d(B_1(w), B_1(z)) = d(u, v) - 1 + i$. Since $d(u, v) - 2 \leq d(B_1(w), B_1(z)) \leq d(u, v)$, we have $\alpha_i(u, v) = 0$ for all $i \not\in \{-1, 0, 1\}$. In [15] we have the following statement:

**Proposition 2.4.** Let $G$ be a connected graph. Then

$$
W(L(G)) = \frac{1}{4} \sum_{u \neq v} d_u d_v (d(u, v) - \alpha_{-1}(u, v) + \alpha_1(u, v)) + \frac{1}{4} \sum_u \left( \frac{d_u}{2} \right),
$$

where the first sum is taken over unordered pairs $u, v \in V(G)$ and the second one over $u \in V(G)$.

In the next section we find a formula for $\Delta T$ and in the last one we prove Lemma 2.3.

3. Formula for $\Delta T$

From now on, we will work with line graphs of trees. To simplify the notation, define $LG = L(G)$ for an arbitrary graph $G$. Let us assume the structure of line graphs of trees. For any tree $F$, the graph $LF$ consists of cliques. Denote by $\mathcal{C}(LF)$ the set of maximal cliques of $LF$. Then every vertex of $LF$ belongs to at most two cliques from $\mathcal{C}(LF)$; each pair of cliques from $\mathcal{C}(LF)$ intersect in at most one vertex; and the cliques of $\mathcal{C}(LF)$ have a “tree structure”, i.e., there are no distinct cliques $C_0, C_1, \ldots, C_{t-1}, t \geq 3$, such that $C_i$ and $C_{i+1}$ have nonempty intersection, $0 \leq i \leq t - 1$, the addition being modulo $t$.

Consequently, for every pair of vertices $u, v$ of $LF$, there is a unique shortest path starting at $u$ and terminating at $v$.

Before we state an exact formula for $\Delta T$, we specify some vertices of $T$ and $LT$ related to $T^-$ and $LT^-$ via $R^-$. Denote by $b'$ the last vertex of $R^-$ and denote by $a'$ its neighbor. Then $V(T^-) = V(T) \setminus \{b'\}$. Further, denote by $b$ the edge $a'b'$ and denote by $a$ the other edge of $T$ incident with $a'$. Then $ab$ is an edge of $LT$ and $V(LT^-) = V(LT) \setminus \{b\}$; see Fig. 2. In the next formulas, all the degrees and distances are considered in $LT^-$ (rather than in $LT$). For $u \in V(LT^-) \setminus \{a\}$ define

$$
\begin{aligned}
\phi(u, a) &= \begin{cases} 
(d_u - 1)(d_a - 2) & \text{if } d(u, a) = 1 \\
0 & \text{otherwise}.
\end{cases}
\end{aligned}
$$

(4)

where

$$
\begin{aligned}
h(u) &= \left( d_u \left( (d_u - 1)d_a - \frac{1}{2} \right) - 1 \right) d(u, a) + (d_u - 1) \left( 2d_u d_a - 2d_a - d_u - \frac{1}{2} \right) - 2 - 2\phi(u, a),
\end{aligned}
$$

Proposition 3.1. Let $T$ be a tree with a longest ray of length at least 2. Further, let $T^-$ and $\Delta T$ be as in Section 2 and let $a$ be as defined above. Then we have

$$
\Delta T = \sum_u h(u) + d_a \left( (d_a - 1)(2d_a - 1) - \frac{1}{2} \right) - 3,
$$

where the sum is taken over the vertices $u$ of $V(LT^-) \setminus \{a\}$ and the degrees and distances are considered in $LT^-$. 
Proof. For \( i \in \{0, 2, 3\} \) denote
\[
\Delta WL^i = W(L(T^-)) - W(L(T^+)).
\]
Then
\[
\Delta T = 2\Delta WL^3 - \Delta WL^2 - \Delta WL^0. \tag{5}
\]
In (3) and (4) of [16] we derived
\[
\Delta WL^3 = \sum_u \left[ \left( \frac{d_u}{2} \right) d_a d(u, a) + (d_u - 1) \left( d_u d_a - \frac{1}{2} d_a \right) - \phi(u, a) \right] + \frac{1}{2} d_a (d_a - 1) (2d_a - 1), \tag{6}
\]
and in (7) of [16] we have
\[
\Delta WL^0 = \sum_u \left[ d(u, a) + 2 \right] + 3. \tag{7}
\]
In both (6) and (7) the sum is taken over the verteces \( u \in V(L(T^-)) \setminus \{a\} \) and the degrees and distances are considered in \( LT^- \). (We remark that in [16], instead of \( T \) and \( T^- \) we use \( T^* \) and \( T \), respectively.) Hence, it remains to find \( \Delta WL^2 \). Observe that
\[
\Delta WL^2 = W(L(T^2)) - W(L(T)) = W(L(T)) - W(L(LT^-)).
\]
Let \( F \in \{T, T^-\} \). By Proposition 2.4, we have
\[
W(L(LF)) = \frac{1}{4} \sum_{u \neq v} \left[ d_u d_v d(u, v) - \alpha_{-1}(u, v) + \alpha_1(u, v) \right] + \frac{1}{4} \sum_u \left( \frac{d_u}{2} \right), \tag{8}
\]
where the degrees and distances are considered in \( LF \), the first sum is taken over unordered pairs \( u, v \in V(LF) \) and the second one over \( u \in V(LF) \). We find \( \alpha_{-1}(u, v) \) and \( \alpha_1(u, v) \).

Let \( u \) and \( v \) be distinct verteces of \( LF \). Recall that \( \alpha_2(u, v) \) is the number of pairs \( w, z \in V(L(LF)) \), with \( u \in V(B_1(w)) \) and \( v \in V(B_1(z)) \), such that \( d(B_1(w), B_1(z)) = d(u, v) - 1 + i \). As mentioned above, there is a unique shortest \( u \rightarrow v \) path in \( LF \). Denote this path by \( u = x_0, x_1, \ldots, x_t = v \). If \( d(B_1(w), B_1(z)) = d(u, v) - 2 \) then \( w \) is the edge \( ux_1 \) and \( z \) is the edge \( x_{t-1}v \). Hence, \( \alpha_{-1}(u, v) = 1 \).

To find \( \alpha_0(u, v) \) we distinguish two cases:

Case 1. \( d(u, v) \geq 2 \). In this case \( u \) and \( v \) do not belong to a common clique from \( C(LF) \). Let \( d(B_1(w), B_1(z)) = d(u, v) - 1 \), it holds either \( x_1 \in V(B_1(w)) \) or \( x_{t-1} \in V(B_1(z)) \), but not both. In the first case we obtain \( (d_u - 1) \) pairs \( w, z \) and in the second \( (d_v - 1) \) pairs \( w, z \). Thus, \( \alpha_0(u, v) = d_u + d_v - 2 \).

Case 2. \( d(u, v) = 1 \). In this case \( u \) and \( v \) belong to a common clique. All pairs \( w, z \) mentioned in the previous case contribute to \( \alpha_0(u, v) \), but we have to add pairs \( w, z \) such that \( v \notin V(B_1(w)) \) and \( u \notin V(B_1(z)) \). If \( d(B_1(w), B_1(z)) = d(u, v) = 1 \) then \( w \) is the edge \( B_1(w) \) and \( z \) is the edge \( B_1(z) \). Hence, \( \alpha_1(u, v) = 1 \).

Now we evaluate \( \Delta WL^2 = W(L(LT^-)) - W(L(LT^+)) \). The graph \( LT \) has one more vertece than \( LT^- \), namely the vertece \( b \) of degree 1, and the degree of \( a \) is \( d_a + 1 \) in \( LT \) (observe that by \( d_u \) we denoted the degree of \( a \) in \( LT^- \)). Therefore, all the terms of (8) for pairs \( u, v \) which do not contain either \( a \) nor \( b \), cancel out in \( W(L^2(LT^-)) - W(L^2(LT^+)) \). But we need to subtract the terms for pairs \( u, a \) in \( LT^- \), to add the terms for pairs \( u, a \) and \( u, b \) in \( LT \), for each \( u \in V(LT^-) \setminus \{a\} \), and finally to add the term for pair \( a, b \). As regards the second sum in (8), we have to subtract the term corresponding to \( a \) in \( LT^- \) and add the terms corresponding to \( a \) and \( b \) in \( LT \), the latter one being 0 as the degree of \( b \) is 1 in \( LT \). If \( d(u, v) = 1 \) then the order of the clique from \( C \) containing both \( u \) and \( v \) is denoted by \( c(u, v) \). If \( d(u, v) \geq 2 \) we set \( c(u, v) = 2 \). Observe that for every \( u \in V(LT^-) \) we have \( c(u, b) = 2 \). We obtain
\[
\Delta WL^2 = -\frac{1}{4} \sum_u \left[ d_u d_a d(u, a) - 1 + (d_u - 1)(d_a - 1) - (c(u, a) - 2) \right] + \frac{1}{4} \sum_u \left[ d_u (d_u + 1) d(u, a) - 1 + (d_u - 1)d_a - (c(u, a) - 2) \right] + \frac{1}{4} \sum_u \left[ d_u 1(d(u, a) + 1) - 1 + 0 - (c(u, b) - 2) \right].
\]
\[
+ \frac{1}{4} \left[ \left(1(d_a + 1) 1 - 1 + 0 - (c(a, b) - 2) \right) - \frac{1}{4} \left( \frac{d_a}{2} \right) + \frac{1}{4} \left( \frac{d_a + 1}{2} \right) \right] = \frac{1}{2} \sum_u \left[ d_u d(u, a) + d_u - 1 \right] + \frac{1}{2} d_a, \tag{9}
\]

where \( u \in V(\Delta T^-) \setminus \{a\} \). Now substituting (6), (9) and (7) into (5) we obtain the result. \( \square \)

4. Proof of Lemma 2.3

With the notation as in the previous sections, denote by \( l + 2 \) the length of \( \Delta T^- \). Since we assume that the longest ray of \( T \) has length at least 2 (see the definition of \( \Delta T^- \)), we have \( l \geq 0 \). Further, all the rays of \( T^- \) have length at most \( l + 2 \), and the ray terminating at \( a' \) has length \( l + 1 \). Consequently, all rays of \( \Delta T^- \) have length at most \( l + 1 \) and the ray terminating at \( a \) has length \( l \); see Fig. 2.

We prove Lemma 2.3 in two steps. First we prove it in the case \( l = 0 \) (Lemma 4.1) and then in the case \( l \geq 1 \) (Lemma 4.2). If \( l = 0 \) then \( a' \) is adjacent to a vertex of degree at least 3 in \( T^- \), so that in \( \Delta T^- \) we have \( d_a \geq 2 \). On the other hand, if \( l \geq 1 \) then \( d_a = 1 \).

Let \( v \) be an endvertex of a ray, say \( R \), in \( LT \). Then \( d_v = 1 \). By \( \overline{v} \) we denote the first vertex of \( R \), i.e., a closest vertex to \( v \) whose degree is at least 3. Observe that if \( u \) and \( v \) are distinct vertices of degree 1 in \( LT \), then \( \overline{v} \neq \overline{u} \).

In the next lemma we do not need to restrict ourselves to trees homeomorphic to \( H \). However, we require that \( \Delta T^- \) has length 2, and as this length equals \( l + 2 \), we require \( l = 0 \).

**Lemma 4.1.** Let \( T \) be a tree different from a path, in which the longest ray has length exactly 2. Let \( \Delta T \) be as in Section 2. Then \( \Delta T^- \geq 0 \).

**Proof.** With the notation as in the beginning of Section 3, we find a lower bound for \( \sum_u h(u) \), where \( u \in V(\Delta T) \setminus \{a\} \).

Consider three cases:

Case 1. \( d_a = 1 \). Then \( d(u, a) > 1 \), so that \( h(u) = -\frac{3}{2} d(u, a) - 2 \) by (4).

Case 2. \( d_a = 2 \). Since \((d_a - 1) (d_a - 2) = 0\), we have \( \phi(u, a) = 0 \) also in this case. By (4) we have \( h(u) = (2d_a - 2) d(u, a) + (2d_a - \frac{5}{2}) - 2 \geq 0 \), as \( d_a \geq 2 \).

Case 3. \( d_a \geq 3 \). By (4) we have
\[
h(u) \geq \left( d_u \left( (d_u - 1) d_u - \frac{1}{2} \right) - 1 \right) d(u, a) + (d_u - 1) \left( 2d_u d_a - 2d_a - (d_u - \frac{1}{2}) \right) - 2 - 2(d_a - 1) d_u - 2 \geq \frac{19}{2} d(u, a) + (d_u - 1) \left( 2d_u d_a - 2d_a - (d_u - \frac{1}{2}) - 2 \right) - 2 \geq \frac{19}{2} d(u, a) + (d_u - 1) \left( 2d_u \right) - 2 \geq \frac{19}{2} d(u, a) + 3
\]

as \( d_u \geq 3 \) and \( d_a \geq 2 \).

Hence,
\[
h(u) \geq \begin{cases} -\frac{3}{2} d(u, a) - 2 & \text{if } d_u = 1, \\ 0 & \text{if } d_u = 2, \\ \frac{19}{2} d(u, a) + 3 & \text{if } d_u \geq 3. \end{cases}
\]

Since \( l = 0 \), all rays of \( T^- \) have length at most 2, and consequently all rays of \( \Delta T^- \) have length at most 1. Hence, if \( d_u = 1 \) then \( d(u, \overline{u}) = 1 \) in \( \Delta T^- \). Thus,
\[
h(u) + h(\overline{u}) \geq \frac{3}{2} d(u, a) - \frac{7}{2} + \frac{19}{2} d(u, a) + 3 \geq 8d(\overline{u}, a) - \frac{1}{2} \geq 0.
\]

(10)
Denote by \( V_1 \) the set of vertices of degree 1 in \( V(T^-) \). Notice that \( a \not\in V_1 \). Since \( \overline{u} \neq \overline{v} \) whenever \( u, v \in V_1 \), \( u \neq v \), by (10) we have

\[
\sum_u h(u) \geq \sum_{u \in V_1} (h(u) + h(\overline{u})) \geq 0.
\]

Finally, since \( d_a [(d_a - 1)(2d_a - 1) - \frac{1}{2}] - 3 \geq 0 \) if \( d_a \geq 2 \), we have

\[
\Delta T = \sum_u h(u) + d_a \left( (d_a - 1)(2d_a - 1) - \frac{1}{2} \right) - 3 \geq 0,
\]

by Proposition 3.1. \( \Box \)

If \( l \geq 1 \), that is if \( R^- \) has length at least 3, then \( h(u) < 0 \) even if \( d_a = 2 \). Thus, our estimations need to be more tight in this case. For this reason we concentrate only to trees homeomorphic to \( H \).

**Lemma 4.2.** Let \( l \geq 1 \). Further, let \( T \) be a tree homeomorphic to \( H \), in which the longest ray has length \( l + 2 \). Let \( \Delta T \) be as in Section 2. Then \( \Delta T \geq 0 \).

**Proof.** We use the notation as in Section 3. Since \( d_a = 1 \), by Proposition 3.1 we have

\[
\Delta T = \sum_u h(u) - \frac{7}{2},
\]

where

\[
h(u) = \begin{cases} 
-\frac{3}{2}d(u, a) - 2 & \text{if } d_a = 1, \\
-\frac{5}{2} & \text{if } d_a = 2, \\
\frac{7}{2}d(u, a) - 1 & \text{if } d_a = 3, \\
\frac{9}{2}d(u, a) + \frac{5}{2} & \text{if } d_a = 4.
\end{cases}
\]

see (4). Hence,

\[
h(u) = \begin{cases} 
-\frac{3}{2}d(u, a) - 2 & \text{if } d_a = 1, \\
-\frac{5}{2} & \text{if } d_a = 2, \\
\frac{7}{2}d(u, a) - 1 & \text{if } d_a = 3, \\
\frac{9}{2}d(u, a) + \frac{5}{2} & \text{if } d_a = 4.
\end{cases}
\]

Observe that \( LT^- \) has no vertex of degree greater than 4.

Suppose that there is a ray \( R' \) in \( T^- \) starting at \( c' \). Denote by \( c \) the vertex of \( LT^- \) corresponding to the first edge of \( R' \), and denote by \( R(c) \) the set of vertices of \( LR' \). We find \( \sum_u h(u) \) where \( u \in R(c) \), \( u \neq a \). We distinguish three cases:

**Case 1.** \( c = \overline{a} \). Then \( d(c, a) = 1 \). As \( l \geq 1 \) and \( T^- \) is a tree homeomorphic to \( H \), \( R(c) \) has one vertex of degree 3, namely \( c \), and \( l - 1 \) vertices of degree 2. By (12), we have

\[
\sum_{u \in R(c) \setminus \{a\}} h(u) = \frac{7}{2}d(c, a) - 1 - (l - 1) \left( \frac{5}{2} \right) = \frac{7}{2}l - \frac{7}{2} - \frac{5}{2}l + \frac{5}{2} = l + \frac{3}{2}.
\]

**Case 2.** \( c \neq \overline{a} \) and \( |R(c)| = 1 \). Since \( R(c) \) has a unique vertex, namely \( c \), the length of \( R' \) is 1. Since \( T^- \) is homeomorphic to \( H \), the edge of \( T^- \) corresponding to \( c \) has one endvertex of degree 3 and the other endvertex of degree 1. Thus \( d_3 = 2 \), so that by (12) we have

\[
\sum_{u \in R(c)} h(u) = -\frac{5}{2}.
\]

**Case 3.** \( c \neq \overline{a} \) and \( |R(c)| \geq 2 \). Since the length of \( R' \) is at most \( l + 2 \), the length of ray \( LR' \) is at most \( l + 1 \). Since \( T^- \) is a tree homeomorphic to \( H \), \( R(c) \) has one vertex of degree 3, at most \( l \) vertices of degree 2, and one vertex of degree 1. By (12), we have

\[
\sum_{u \in R(c)} h(u) \geq \frac{7}{2}d(c, a) - 1 - \frac{5}{2} - \frac{3}{2} \left( d(c, a) + l + 1 \right) - 2 = 2d(c, a) - 4l - \frac{9}{2}.
\]
Denote $S(c) = \sum_{u \in R(c) \setminus \{a\}} h(u)$. By the previous analysis we have

\[
S(c) \geq \begin{cases} 
  l + \frac{3}{2} & \text{if } c = \bar{a}, \\
  -\frac{5}{2} & \text{if } c \neq \bar{a} \text{ and } |R(c)| = 1, \\
  2d(c, a) - 4l - \frac{9}{2} & \text{if } c \neq \bar{a} \text{ and } |R(c)| \geq 2.
\end{cases}
\tag{13}
\]

In Fig. 2 we have $T$, that is, a tree homeomorphic to $H$, and also its line graph. Denote by $P'$ the path connecting the two vertices of degree 3 in $T$ and denote by $P$ the line graph of $P'$. Further, denote by $c_1, c_2, \ldots, c_6$ vertices of $LT$ corresponding to edges incident with vertices of degree 3 in $T$; see Fig. 2. We assume that $c_1 = \bar{a}$, $c_1 c_2 \in E(LT)$ and $c_5, c_6 \in V(P)$. Denote by $t$ the length of $P$. Then $t \geq 0$, and in the case $t = 0$ we have $c_5 = c_6$. Denote $SP = \sum_{u \in V(P)} h(u)$. Then

\[
\sum_{u \in V(LT) \setminus \{a\}} h(u) = \sum_{i=1}^{4} S(c_i) + SP.
\tag{14}
\]

We distinguish three cases:

Case 1. $t = 0$. Then $c_5 = c_6$ and the degree of $c_5$ is 4. Since $l \geq 1$, $d(c_2, a) = d(c_5, a) = l + 1$ and $d(c_3, a) = d(c_4, a) = l + 2$, by (13) and (12) we have

\[
S(c_1) \geq l + \frac{3}{2},
\]

\[
S(c_2) \geq \min \left\{ -\frac{5}{2}, -2l - \frac{5}{2} \right\} = -2l - \frac{5}{2},
\]

\[
S(c_3) \geq \min \left\{ -\frac{5}{2}, -2l - \frac{1}{2} \right\} = -2l - \frac{1}{2},
\]

\[
S(c_4) \geq \min \left\{ -\frac{5}{2}, -2l - \frac{1}{2} \right\} = -2l - \frac{1}{2},
\]

\[
SP = 9(l + 1) + \frac{5}{2} = 9l + \frac{23}{2}.
\]

By (14) we have $\sum_{u \in V(LT) \setminus \{a\}} h(u) = 4l + \frac{19}{2}$, and by (11) we conclude $\Delta T \geq 4l + \frac{19}{2} - \frac{7}{2} \geq 0$.

Case 2. $t \geq 1$ and $t \leq l - 1$. Then both $c_5$ and $c_6$ have degree 3 and $P$ has $t - 1$ vertices of degree 2. Since $l \geq 1$, $d(c_2, a) = d(c_5, a) = l + 1$, $d(c_6, a) = l + t + 1$ and $d(c_3, a) = d(c_4, a) = l + t + 2$, by (13) and (12) we have

\[
S(c_1) \geq l + \frac{3}{2},
\]

\[
S(c_2) \geq \min \left\{ -\frac{5}{2}, -2l - \frac{5}{2} \right\} = -2l - \frac{5}{2},
\]

\[
S(c_3) \geq \min \left\{ -\frac{5}{2}, 2t - 2l - \frac{1}{2} \right\} \geq \min \left\{ -\frac{5}{2}, 2(l - 1) - 2l - \frac{1}{2} \right\} = \frac{-5}{2},
\]

\[
S(c_4) \geq \min \left\{ -\frac{5}{2}, 2t - 2l - \frac{1}{2} \right\} \geq \min \left\{ -\frac{5}{2}, 2(l - 1) - 2l - \frac{1}{2} \right\} = \frac{-5}{2},
\]

\[
SP = \frac{7}{2}(l + 1) - 1 - \frac{5}{2}(t - 1) + \frac{7}{2}(l + t + 1) - 1 = 7l + t + \frac{15}{2}.
\]

By (14) we have $\sum_{u \in V(LT) \setminus \{a\}} h(u) = 6l + t + \frac{3}{2}$, and by (11) we conclude $\Delta T \geq 6l + t + \frac{3}{2} - \frac{7}{2} \geq 0$.

Case 3. $t \geq 1$ and $t \leq l - 2$. Then again, both $c_5$ and $c_6$ have degree 3 and $P$ has $t - 1$ vertices of degree 2, $d(c_2, a) = d(c_5, a) = l + 1$, $d(c_6, a) = l + t + 1$ and $d(c_3, a) = d(c_4, a) = l + t + 2$. Since $l \geq 1$, by (13) and (12) we have

\[
S(c_1) \geq l + \frac{3}{2},
\]

\[
S(c_2) \geq \min \left\{ -\frac{5}{2}, -2l - \frac{5}{2} \right\} = -2l - \frac{5}{2},
\]

\[
S(c_3) \geq \min \left\{ -\frac{5}{2}, 2t - 2l - \frac{1}{2} \right\} = 2t - 2l - \frac{1}{2}.
\]
\[ S(c_4) \geq \min \left\{ \frac{5}{2}, 2t - 2l - \frac{1}{2} \right\} = 2t - 2l - \frac{1}{2} \]

\[ SP = 7l + t + \frac{15}{2}. \]

By (14) we have \( \sum_{u \in V(LT^a)} h(u) = 2l + 5t + \frac{11}{2} \), and by (11) we conclude \( \Delta T \geq 2l + 5t + \frac{11}{2} - \frac{7}{2} \geq 0. \)

Hence, in every case \( \Delta T \geq 0 \) as required. \qed

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References