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Finite-time generalized function matrix projective lag synchronization of coupled dynamical networks with different dimensions via the double power function nonlinear feedback control method

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Abstract
This paper investigates the problem of finite-time generalized function matrix projective lag synchronization between two different coupled dynamical networks with different dimensions of network nodes. The double power function nonlinear feedback control method is proposed in this paper to guarantee that the state trajectories of the response network converge to the state trajectories of the drive network according to a function matrix in a given finite time. Furthermore, in comparison with the traditional nonlinear feedback control method, the new method improves the synchronization efficiency, and shortens the finite synchronization time. Numerical simulation results are presented to illustrate the effectiveness of this method.

Keywords: coupled dynamical network, finite-time synchronization, double power function, nonlinear feedback control

1. Introduction

Networks can be abstracted to a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. Recently, since the discovery of ‘small-world networks’ [1] and ‘scale-free networks’ [2], network science has gained a lot of attention in various fields of science, engineering and humanity worldwide, and has been deeply investigated in recent years in areas such as the Internet, the World Wide Web, biological networks, metabolic networks, linguistic networks, social networks, transportation networks, phone networks, communication networks, aviation networks, interpersonal relationship networks, electricity distribution networks and so forth.

Among various dynamical behaviors, synchronization is a significant and interesting phenomenon. Synchronization processes are ubiquitous in our lives and play a very important role in many different contexts, such as in synchronous communication, signal synchronization (for example, synchronization between video and audio signals), firefly bioluminescence synchronization in biology, geostationary satellites, synchronous motors, database synchronization etc. The synchronization of coupled dynamical networks has been a focus in various fields of science and engineering, especially in the field of control. In the past decade, the control and synchronization of coupled dynamical networks has attracted much attention, and some relevant theoretical results have been established [3–13]. In real-world applications, however, it is often desired that synchronization of coupled dynamical networks should be achieved in finite time as quickly as possible, particularly in engineering fields. Finite-time synchronization is widely studied as a kind of synchronization,
and therefore more and more people are beginning to realize the importance of finite-time synchronization. In [14–20], the authors studied finite-time control or synchronization of chaotic (hyperchaotic) systems. Finite-time synchronization of coupled dynamical networks was studied in [21]. The authors studied the finite-time synchronization of coupled dynamical networks with time delay in [22] and [23]. Finite-time stochastic synchronization was investigated in [24]. In [25], the authors studied finite-time stochastic outer synchronization between two dynamical networks with different topologies. Finite-time mixed outer synchronization of dynamical networks with coupling time-varying delay was investigated in [26]. In [27], the authors studied finite-time generalized outer synchronization between two different coupled dynamical networks. At present, there are not many published articles regarding finite-time synchronization between two coupled dynamical networks [21–27].

To the best of our knowledge, most research has primarily been concerned with asymptotical or finite-time synchronization between two coupled dynamical networks with identical or non-identical nodes, and the dynamical equations of these network nodes have the same dimension. However, in many real physics systems, the synchronization is carried out through oscillators with different dimensions, especially systems in biological science and social science. And thus far, there are few published papers considering finite-time generalized synchronization of coupled dynamical networks with different dimensions [27–29]. Therefore, based on actual demands, it is essential to study the finite-time synchronization of coupled dynamical networks with different dimensions. One example is the synchronization of different dimension chaotic systems (or coupled dynamical networks) which have been applied in secure communications, and which can improve the degree of security in these communications. Moreover, we usually hope two systems or networks achieve synchronization in a finite time, so that the useful signal can be restored as soon as possible. And thus an estimation of the synchronization time can be obtained. Besides, synchronization has been widely observed in neuronal networks, multi-agent networks and cortical networks etc [30–32]. In [33], we studied adaptive generalized matrix projective lag synchronization between two different coupled dynamical networks with different dimensions. Using the matrix as a bridge, we achieve the asymptotical synchronization between two different coupled dynamical networks with different dimensions. Motivated by the above discussion, in this paper we will extend our study to the finite-time synchronization of coupled dynamical networks with different dimensions. Therefore, the finite-time generalized function matrix projective lag synchronization (GFMLPS) between two different dimensional coupled dynamical networks is investigated in this paper. GFMLPS includes projective synchronization (PS) [34], lag synchronization (LS) [35], function projective synchronization (FPS) [36], matrix projective synchronization (MPS) [33] and generalized function projective synchronization (GFFPS) [37], and it is a more general form of generalized synchronization. Based on the traditional finite-time control theory, in order to further improve the synchronization efficiency, and shorten the finite synchronization time, the double power function nonlinear feedback controller is proposed in this paper. The new method can effectively improve the convergence speed of the synchronization error system, and shorten the finite synchronization time $T$; numerical simulation results are presented to illustrate the effectiveness of this method.

The rest of the present paper is organized as follows. In section 2 the network models and some useful preliminaries are given. In section 3, by using the double power function nonlinear feedback control method, some sufficient finite-time synchronization criteria are derived to guarantee the finite-time GFMLPS between two completely different coupled dynamical networks with different dimensions. In section 4 the simulation results are used to validate the effectiveness of the proposed approach. Finally, in section 5, some concluding remarks are drawn and our future work is outlined.

2. Network models and preliminaries

2.1. Network models

Some necessary mathematical notations that will be utilized throughout this paper are first introduced. Let $A$ (or $x$) be the transpose of the matrix $A$ (or vector $x$). $\|x\|$ means the 2-norm of the vector $x$, and $\otimes$ denotes the Kronecker product of two matrices. $\lambda_{\text{max}}(A)$ represents the maximum eigenvalue of a square matrix $A$.

Consider a general dynamical network consisting of $N$ nodes with linear couplings, which is described by

$$
\dot{x}_i(t) = A_i x_i(t) + f_i(x_i(t)) + \sum_{j=1}^{N} c_{ij} P x_j(t),
$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in})^T \in R^n$ is the state vector of the $i$th node; $A_i \in R^{n\times n}$ is a constant matrix and $f_i : R^n \rightarrow R^n$ is a smooth nonlinear vector field; the nonlinear function $\phi_i(x_i(t)) = A_i x_i(t) + f_i(x_i(t))$ describes the dynamics of node $i$ in the absence of interactions with other nodes; $P \in R^{n\times n}$ is an inner coupling matrix, which means two coupled nodes are linked through their $i$th state variables. $C = (c_{ij})_{N \times N} \in R^{N \times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the network. The matrix $C$ is defined as follows: if there exists a connection from node $j$ to node $i$ ($i \neq j$), then $c_{ij} \neq 0$; otherwise $c_{ij} = 0$. The diagonal elements of matrix $C$ are defined by $c_{ii} = -\sum_{j=1, j \neq i}^{N} c_{ij}, \quad i = 1, 2, ..., N$.

We take the network given by equation (1) as the drive network, and the response network with a nonlinear control
scheme is given by
\[ \dot{y}(t) = B \chi(t) + g(\chi(t)) + \sum_{j=1}^{N} d_{ij} \phi_{y}(t) + u_{i}(t), \]
\[ i = 1, 2, ..., N, \tag{2} \]
where \( \chi = (\chi_{1}, \chi_{2}, ..., \chi_{m})^{T} \in R^{m} \) is the state vector of the \( i \)th node; \( B \in R^{m \times m} \) is the constant matrix and \( g: R^{m} \to R^{m} \) is a smooth nonlinear vector field; \( u_{i}(t) \in R^{m} \) is a nonlinear feedback controller; \( Q \in R^{m \times m} \) is also an inner coupling matrix and \( D = (d_{ij})_{N \times N} \in R^{m \times m} \) is the coupling configuration matrix, which has the same meaning as that of matrix \( C \).

2.2. Preliminaries

The synchronization error signal for GFMPLS is defined as follows:
\[ e_{i}(t) = \chi_{i}(t) - M(t)x_{i}(t - \tau(t)) \]
\[ i = 1, 2, ..., N, \tag{3} \]
where \( \tau(t) > 0 \) is the time-varying delay, \( M(t) = (m_{ij}(t)) \in R^{m \times m} \) is the time-varying scaling matrix, and the element in each row cannot be equal to zero at the same time.

If the element in each row is equal to zero at the same time, then the network synchronization is meaningless.

From (3), the time derivative of \( e_{i}(t) \) will be
\[ \dot{e}_{i}(t) = \dot{\chi}_{i}(t) - \dot{M}(t)x_{i}(t - \tau(t)) \]
\[ - M(t)x_{i}(t - \tau(t))(1 - \tau(t)), \]
\[ i = 1, 2, ..., N. \tag{5} \]

By substituting equations (1) and (2) into equation (5), the synchronization error dynamical system is obtained as follows:
\[ \dot{e}_{i}(t) = B e_{i}(t) + \left( B M(t) - \dot{M}(t)A_{j} \right)x_{i}(t - \tau(t)) + g(\chi(t)) - \dot{M}(t)x_{i}(t - \tau(t)) \]
\[ + \sum_{j=1}^{N} d_{ij} Q e_{j}(t) - \sum_{j=1}^{N} \left( c_{ij} \dot{M}(t) P - d_{ij} Q M(t) \right)x_{j} \]
\[ \times (t - \tau(t)) + u_{i}(t), \]
\[ i = 1, 2, ..., N, \tag{6} \]
where \( \dot{M}(t) = (1 - \tau(t))M(t) \).

Definition 1. Let \( x_{i}(t - \tau(t)) \) be the time delay state of the drive network (1), and \( \chi(t) \) be the current state of the response network (2). Given the time-varying delay \( \tau(t) > 0 \), if there exists the time-varying scaling function matrix \( M(t) = (m_{ij}(t)) \in R^{m \times m} \), and the fact that the element in each row cannot be equal to zero at the same time, then if there exists a constant \( T > 0 \) such that
\[ \lim_{t \to T} \|e_{i}(t)\| = 0, \quad i = 1, 2, ..., N, \tag{7} \]
and \( \|e_{i}(t)\| \equiv 0, \quad \forall t \geq T \), then the GFMPLS between coupled dynamical networks (1) and (2) is achieved in a finite time \( T \).

It is clear that the finite-time synchronization problem can be transformed to the equivalent problem of the finite-time stabilization of the error dynamical system (6). The objective of this paper is to design a suitable nonlinear feedback controller \( u_{i}(t) \) such that for any given drive network (1) and the response network (2), the finite-time stability of the resulting error dynamical system (6) can be achieved in the sense of definition 1.

Remark 1. If \( m = n \) and the scaling functions matrix
\[ M(t) = \text{diag}(k_{1}(t), k_{2}(t), ..., k_{m}(t)), \]
where \( k_{i}(t) \) is the function of the time \( t \), then the drive and response networks would realize generalized function projective lag synchronization (GFLPS). If \( k_{i}(t) \) is the nonzero constant, then the GFLPS degenerates into generalized projective lag synchronization (GPLS). In particular, when the nonzero constant is chosen as 1 , then the GPLS degenerates into the common lag synchronization (LS). In short, GFMPLS is a more general form that includes many kinds of synchronization as its special cases.

Assumption 1. The time-varying delay \( \tau(t) > 0 \) is a continuous monotone increasing/decreasing function, or \( \tau(t) = \tau > 0 \) is a constant. At the same time, the time-varying delay \( \tau(t) \) is a bounded function.

Assumption 2. The time-varying scaling functions matrix
\[ M(t) = (m_{ij}(t)) \in R^{m \times m} \]
is a bounded matrix, and \( m_{ij}(t) \) is a continuous bounded function or a constant.

Lemma 1. [38] Assume that a continuous, positive-definite function \( V(t) \) satisfies the following differential inequality:
\[ V(t) \leq -\alpha V(t), \quad \forall t \geq t_{0}, \quad V(t_{0}) \geq 0, \tag{8} \]
where \( \alpha > 0, \quad 0 < \eta < 1 \) are two constants. Then, for any given \( t_{0}, V(t) \) satisfies the following differential inequality:
\[ V^{1-\eta}(t) \leq V^{1-\eta}(t_{0}) - \alpha(1 - \eta)(t - t_{0}), \tag{9} \]
and
\[ V(t) \equiv 0, \quad \forall t \geq t_{1}, \tag{10} \]
with \( t_{1} \) given by
\[ t_{1} \leq T_{1} = t_{0} + \frac{V^{1-\eta}(t_{0})}{\alpha(1 - \eta)}. \tag{11} \]
Lemma 2. [38] For $x_1, x_2, \cdots, x_n \in \mathbb{R}$, the following inequality holds:
\[
|x_1| + |x_2| + \cdots + |x_n| \geq \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}.
\]  

(12)

3. Synchronization criteria

In this section, we will study the finite-time GFMLS between two different coupled dynamical networks with different dimensions of network nodes via the double power function nonlinear feedback control method.

3.1. The synchronization theory and proof

Theorem 1. Given the time-varying scaling matrix $M(t) = \{m_{ij}(t)\} \in \mathbb{R}^{m \times m}$, and the fact that the element in each row cannot be equal to zero at the same time, the finite-time GFMLS between two different coupled dynamical networks (1) and (2) with different dimensions can be achieved by using the following double power function nonlinear feedback controller:

\[
u_i(t) = \eta_i(t) - \left( BM(t) - \hat{M}(t)A \right)x_i(t - \tau(t)) - g_e\left(x_i(t) + \hat{M}(t)f_i(x_i(t - \tau(t)))\right) + \hat{M}(t)x_i(t - \tau(t)) + \sum_{j=1}^{N} \left( e_j(t) - \hat{d}_jQ(t)\right) x_i(t - \tau(t))
\]

where $\eta_i(t) = -ke_i(t) - \mu \left( \|e(t)\|^{a} + \|e(t)\|^{b}\right) \text{sign}(e_i(t))$ is the double power function feedback control part; $\mu$ and $k$ are positive control parameters;

\[a > 1.0 < b < 1\]

and

\[e(t) = (e_1(t), e_2(t), \cdots, e_N(t))^T \in \mathbb{R}^{mN}.\]

Then under the set of controller (13), the coupled dynamical networks are synchronized in a finite time

\[
T = t_o + \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{1-\theta_1}(t_o) + b} \right) + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{a + c}{c} \right), \quad V(t_o) \in (1, +\infty).
\]

\[
T = t_o + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{aV^{1-\theta_2}(t_o) + c}{c} \right), \quad V(t_o) \in (0.1]
\]

where $a = 2(k - \lambda^*)$, $b = \mu 2^b$, $c = \mu 2^b$, $\theta_1 = \frac{1 + \mu}{2}$, $\theta_2 = \frac{1 + \mu}{2}$.

\[V(t_o) = \frac{1}{2}e(t_0)^T e(t_0) > 0, \quad e(t_0) \text{ is the initial condition of } e(t), \quad \text{and } V(t) \equiv 0, \forall t \geq T.
\]

Proof. Firstly, by putting the double power function nonlinear feedback controller (13) into equation (6), the synchronization error dynamical system can be expressed by

\[
\dot{e}_i(t) = B e_i(t) + \sum_{j=1}^{N} d_jQ e_j(t) - ke_i(t)
\]

\[-\mu \left( \|e(t)\|^{a} + \|e(t)\|^{b}\right) \text{sign}(e_i(t))
\]

\[i = 1, 2, \cdots, N.\]

(15)

Because $e(t) = (e_1(t), e_2(t), \cdots, e_N(t))^T \in \mathbb{R}^{mN}$, then the network synchronization error dynamical system can be expressed by

\[\dot{e}(t) = \hat{B} e(t) + \hat{D} e(t) - ke(t)
\]

\[-\mu \left( \|e(t)\|^{a} + \|e(t)\|^{b}\right) \text{sign}(e(t)),
\]

where

\[
\hat{B} = \text{diag}(B_1, B_2, \cdots, B_N) \in \mathbb{R}^{mN \times mN},
\]

and

\[
\hat{D} = D \otimes Q \in \mathbb{R}^{mN \times mN}.
\]

Then, based on the double power function nonlinear feedback controller (13) and the network synchronization error dynamical system (16), we construct the Lyapunov candidate function as follows:

\[V(t) = \frac{1}{2}e(t)^T e(t).\]

(17)

Obviously, $V(t) \geq 0$. Taking the time derivative of $V(t)$ along the trajectories of the network synchronization error dynamical system (16), we have

\[V(t) = e(t)^T \dot{e}(t)
\]

\[= e(t)^T (\hat{B} e(t) + e(t)^T \hat{D} e(t) - ke(t))
\]

\[= e(t)^T \hat{B} e(t) + e(t)^T \hat{D} e(t) - k e(t)
\]

\[-\mu \left( \|e(t)\|^{a} + \|e(t)\|^{b}\right) \text{sign}(e(t)),
\]

(18)

Because of $e(t)^T \hat{B} e(t) + e(t)^T \hat{D} e(t) \leq \lambda^* e(t)^T e(t)$, $\lambda^* = \lambda_{\max} \left( \frac{b + \beta}{2} \right) + \lambda_{\max} \left( \frac{\beta + b}{2} \right)$. And according to lemma 2, $e(t)^T \text{sign}(e(t)) = \sum_{i=1}^{N} \sum_{j=1}^{N} |e_j(t)| \geq \left( \sum_{i=1}^{N} |e_j(t)| \right)^{\frac{1}{2}} = \|e(t)\|$. 


If we choose \( k > \lambda^* \), then,
\[
\dot{V}(t) \leq - (k - \lambda^*) e(t)^{T} e(t)
\]
\[
- \mu \left( \| e(t) \|^2 + \| e(t) \|^{2^\theta} \right)
\]
\[
= - 2(k - \lambda^*) V(t) - \mu \left( \| e(t) \|^2 \right) \left( \frac{1}{2} \| e(t) \|^{2^\theta} \right)
\]
\[
= - \mu \left( \frac{1}{2} \| e(t) \|^2 \right) \left( \frac{1}{2} \| e(t) \|^{2^\theta} \right)
\]
\[
= - 2(k - \lambda^*) V(t) - \mu \left( \frac{1}{2} \| e(t) \|^2 \right) \left( \frac{1}{2} \| e(t) \|^{2^\theta} \right)
\]
\[
= - a V(t) - b (V(t))^{\theta_1} - c (V(t))^{\theta_2} < 0. \quad (19)
\]
where
\[
a = 2(k - \lambda^*) \in (0, + \infty),
\]
\[
b = \mu 2^{\theta_1} \in (0, + \infty),
\]
\[
c = \mu 2^{\theta_2} \in (0, + \infty),
\]
\[
\theta_1 = \frac{1 + \theta}{2} \in (1, + \infty),
\]
\[
\theta_2 = \frac{1 + \theta}{2} \in \left( \frac{1}{2}, 1 \right).
\]

Up until now, GFMLPS between two different coupled dynamical networks (1) and (2) with network nodes of different dimensions could be achieved by using the controller (13). Next, we will prove that synchronization can be accomplished in finite time.

1. If \( V(t_0) \in (1, + \infty) \), we will calculate the finite time \( T \) divided into two parts.
   a) From initial state \( V(t_0) \in (1, + \infty) \) arrived to \( V(\hat{t}_1) = 1 \):

   In this phase, because of \( \theta_1 \in (1, + \infty) \), \( \theta_2 \in \left( \frac{1}{2}, 1 \right) \) and \( V(t) > 1 \) \( 0 < t < \hat{t}_1 \), the change rate of \( V(t) \) mainly depends on the first two items \( -a V(t) - b (V(t))^{\theta_1} \) of the expression (19). Thus we can ignore the effect of the last item \( -c (V(t))^{\theta_2} \) of the expression (19). Then,
\[
\dot{V}(t) \leq - a V(t) - b (V(t))^{\theta_1}. \quad (20)
\]
In order to write this more conveniently, we will replace \( V(t) \) with \( V \).
\[
\dot{V}(t) \leq - a V(t) - b (V(t))^{\theta_1}
\]
\[
\Leftrightarrow \frac{dV}{dt} \leq - a V - b V^{\theta_1}
\]
\[
\Rightarrow \text{dr} \leq \frac{dV}{-a V - b V^{\theta_1}},
\]
\[
= \frac{V^{-\theta_1} dV}{-a V^{\theta_1} - b}
\]
\[
= \frac{1}{\theta_1 - 1} \left( \frac{dV^{\theta_1}}{aV^{\theta_1} + b} \right) \quad (21)
\]

Taking integral of both sides of equation (21) from \( t_0 \) to \( \hat{t}_1 \), we have
\[
\hat{t}_1 - t_0 \leq \frac{1}{\theta_1 - 1} \int_{t^{\theta_1}(t_0)}^{t^{\theta_1}(\hat{t}_1)} \frac{dV^{\theta_1}}{aV^{\theta_1} + b}
\]
\[
\Leftrightarrow \hat{t}_1 - t_0 \leq \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right)
\]
\[
= t_0 + \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right). \quad (22)
\]
a) From \( V(\hat{t}_1) = 1 \) arrived to \( V(\hat{t}_2) = 0 \):

In the second phase, because of \( \theta_1 \in (1, + \infty) \), \( \theta_2 \in \left( \frac{1}{2}, 1 \right) \) and \( V(t) \in \left( 0, 1 \right) \) \( \hat{t}_1 < t < \hat{t}_2 \), the change rate of \( V(t) \) mainly depends on the item \( -a V(t) - c (V(t))^{\theta_2} \) of the expression (19). Thus we can ignore the effect of the second item \( -b (V(t))^{\theta_1} \) of the expression (19). Then,
\[
\dot{V}(t) \leq - a V(t) - c (V(t))^{\theta_2}. \quad (23)
\]
Based on same argument, we have
\[
\hat{t}_2 \leq \hat{t}_1 + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{a + c}{c} \right)
\]
\[
\hat{t}_2 \leq T = t_0 + \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{a + c}{c} \right). \quad (24)
\]
Therefore,
\[
T = t_0 + \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{a + c}{c} \right). \quad (25)
\]
In summary, the GFMLPS between two different coupled dynamical networks (1) and (2) with network nodes of different dimensions can be achieved using the double power function nonlinear feedback controller (13) in the finite time \( T \) with \( V(t_0) > 1 \), and \( V(t) \equiv 0, \forall t \geq T \). In other words, \( \lim_{t \to T} \| e(t) \| = 0 \), and \( e(t) \equiv 0, \forall t \geq T \).
(1) If $V(t_0) \in (0, 1]$, we will calculate the finite time $T$ according to the equation (23).

Taking the integral of both sides of equation (23) from $t_0$ to $t$, and letting $V(t) = 0$, we have

$$\hat{t}_2 \leq T = t_0 + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{aV^{-\theta_2}(t_0) + c}{c} \right).$$

(26)

To sum up, the complex networks are synchronized in a finite time

$$\begin{cases}
T = t_0 + \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{1-\theta_1}(t_0) + b} \right) + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{a + c}{c} \right), \\
V(t_0) \in (1, + \infty)
\end{cases}$$

$$\begin{cases}
T = t_0 + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{aV^{-\theta_2}(t_0) + c}{c} \right), \\
V(t_0) \in (0, 1]
\end{cases}$$

(27)

Hence, the proof is completed.

**Remark 2.** We ignore the minor factor in calculating the finite time $T$, so the actual synchronization time is less than the finite time $T$.

**Remark 3.** In our work, the coupling configuration matrices $C$ and $D$ need not be symmetric or irreducible. In addition, there is no constraint imposed on the inner coupling matrices $P$ and $Q$. Furthermore, each node of the networks may have different local dynamics. The proposed approach is applicable to all kinds of coupled dynamical networks.

**Remark 4.** The parameters $k$, $\mu$, $\alpha$ and $\beta$ can be chosen appropriately beforehand to adjust the synchronization rate. However, the inequality $k \geq \lambda_{\text{max}} \left( \frac{\hat{b} + \hat{d}'}{2} \right) + \lambda_{\text{max}} \left( \frac{\hat{b} + \hat{d}'}{2} \right)$ is just a sufficient condition but not a necessary one.

### 3.2. Finite-time comparison

In comparison with the double power function nonlinear feedback method, if we choose the traditional nonlinear feedback control method to study the finite-time GFMPLS, $\eta(t) = -ke(t) - \mu\|e(t)\|^\theta \text{sign}(e(t))$ is chosen in the nonlinear feedback controller (13).

Then, the network synchronization error dynamical system can be expressed by

$$\dot{e}(t) = \hat{B}e(t) + \hat{D}e(t) - ke(t) - \mu\|e(t)\|^\theta \text{sign}(e(t)).$$

(28)

The Lyapunov candidate function is constructed in the same way as expression (17). Then, we have

$$\begin{align*}
V(t) &= \dot{e}(t)^T \dot{e}(t) \\
&= \dot{e}(t)^T \left( \hat{B}e(t) + \hat{D}e(t) - ke(t) - \mu\|e(t)\|^\theta \text{sign}(e(t)) \right) \\
&= \mu\|e(t)\|^\theta \text{sign}(e(t)) \dot{e}(t)^T \left( - \hat{B}e(t) - \hat{D}e(t) + ke(t) \right) - \mu\|e(t)\|^\theta \text{sign}(e(t))^T \dot{e}(t) \\
&= \mu\|e(t)\|^\theta \text{sign}(e(t)) \dot{e}(t)^T \left( - \hat{B}e(t) - \hat{D}e(t) + ke(t) \right) - \mu\|e(t)\|^\theta \text{sign}(e(t))^T \dot{e}(t) \\
&= - \mu\|e(t)\|^\theta \text{sign}(e(t))^T \dot{e}(t) \\
&< 0,
\end{align*}$$

(29)

If we choose $k \geq \lambda_{\text{max}} \left( \frac{\hat{b} + \hat{d}'}{2} \right) + \lambda_{\text{max}} \left( \frac{\hat{b} + \hat{d}'}{2} \right)$, then we can obtain

$$V(t) \leq - c(V(t))^{\theta_1} < 0,$$

where $c > 0$, $\theta_1 \in [\frac{1}{2}, 1]$. According to Lemma 1,

$$T_i = t_0 + \frac{V^{1-\theta_1}(t_0)}{c(1 - \theta_2)}.$$

(30)

The $\Delta T$ is defined as follows:

$$\Delta T = T_i - T.$$

(31)

If $\Delta T > 0$, then $T_i > T$, which means the double power function nonlinear feedback method is better than the traditional feedback control method. Otherwise $T_i \leq T$, meaning the traditional feedback control method is better.

**Theorem 2.** For $a \in (0, + \infty)$, $b \in (0, + \infty)$, $c \in (0, + \infty)$, $\theta_1 \in (1, + \infty)$ and $\theta_2 \in [\frac{1}{2}, 1]$, the following inequality holds:

$$\begin{cases}
T < t_0 + \frac{1 - V^{1-\theta_1}(t_0)}{b(\theta_1 - 1)} + \frac{1}{c(1 - \theta_2)} V(t_0) \in (1, + \infty) \\
T < t_0 + \frac{V^{1-\theta_1}(t_0)}{c(1 - \theta_2)} V(t_0) \in (0, 1]
\end{cases}$$

(32)

In order to prove $\Delta T > 0$, the inequality of finite time $T$ (27) is reasonably enlarged in theorem 2. The proof is given in appendix A.

**Theorem 3.** For $a \in (0, + \infty)$, $b \in (0, + \infty)$, $c \in (0, + \infty)$, $\theta_1 \in (1, + \infty)$ and $\theta_2 \in [\frac{1}{2}, 1]$, the following inequality holds:

$$\Delta T = T_i - T > 0.$$

(33)

The proof is given in appendix B.

In comparison with the traditional feedback control method, the double power function nonlinear feedback method is better than the traditional method.
4. Illustrative example

In this section, the simulations are performed to verify the effectiveness of the proposed synchronization scheme in the previous section. The total finite-time synchronization error is defined as: \( E(t) = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{n} e_{ij}^2(t)} \).

The total finite-time synchronization error \( E(t) \) is used to measure the quality of the finite-time synchronization process. It is obvious that when \( \lim_{t \to T} ||E(t)|| = 0 \), and \( E(t) \equiv 0 \), \( \forall t \geq T \), the drive and response networks achieve the desired synchronization globally in the finite time \( T \).

In the simulations, the node equations of the drive network (1) are described by the following ten three-dimensional chaotic systems [39]:

\[
\begin{bmatrix}
\dot{x}_{1i}(t) \\
\dot{x}_{2i}(t) \\
\dot{x}_{3i}(t)
\end{bmatrix}
= \begin{bmatrix}
-35 & 35 & 0 \\
0 & -28 & 0 \\
0 & 0 & -8
\end{bmatrix}
\begin{bmatrix}
x_{1i}(t) \\
x_{2i}(t) \\
x_{3i}(t)
\end{bmatrix}
+ \begin{bmatrix}
-x_{1i}(t)x_{3i}(t) \\
(x_{1i}(t) - x_{1i}(t))^4 - 10x_{1i}(t)x_{3i}(t) \\
x_{1i}^2(t) + x_{2i}^2(t)
\end{bmatrix},
\]

\( i = 1, 2, \ldots, 10. \) (34)

For the response network (2), 10 nodes are described by the following four-dimensional hyperchaotic Chen systems [40]:

\[
\begin{bmatrix}
\dot{y}_{1i}(t) \\
\dot{y}_{2i}(t) \\
\dot{y}_{3i}(t) \\
\dot{y}_{4i}(t)
\end{bmatrix}
= \begin{bmatrix}
-35 & 35 & 0 & 0 \\
4 & -21 & 0 & 4 \\
0 & 0 & -3 & 0 \\
-2 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_{1i}(t) \\
y_{2i}(t) \\
y_{3i}(t) \\
y_{4i}(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-10y_{1i}(t)y_{3i}(t) \\
y_{3i}^2(t) \\
0
\end{bmatrix},
\]

\( i = 1, 2, \ldots, 10. \) (35)

Two different directed networks are drawn in figures 1 and 2. They are used as the drive and response networks, respectively. The coupling configuration matrices are given respectively as follows:

\[
C = \begin{bmatrix}
-4 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -3 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -3 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -4
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
-2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & -4 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & -3 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -3 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & -4 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & -5
\end{bmatrix}
\]

The inner coupling matrices are taken arbitrarily as \( P = \text{diag}(0,1,0.2,0.3), Q = \text{diag}(0.1,1.0,1.2,0.2) \). It is easy to obtain \( \hat{\lambda}_{\text{max}} \left( \frac{\hat{b} + \hat{\beta}^2}{2} \right) \geq 27.2163 \) and \( \hat{\lambda}_{\text{max}} \left( \frac{\hat{\sigma} + \hat{\sigma}^2}{2} \right) \leq 0.0237 \). Thus we obtain \( \hat{\lambda}^n = 27.24 \). We choose the time delay \( \tau(t) = 0.2e^{-t} \sqrt{3e^t} + \frac{0.1e^{2t}}{2 + e^{2t}} \). The time-varying scaling function matrix is chosen as...
Through simple computation, we get the initial value of the Lyapunov candidate function $V(t_0) = 270.7075$, where $t_0 = 1$.

The control parameters are chosen as $k = 50$, $\mu = 1$ and $\beta = 0.6$. If we choose the traditional nonlinear feedback control method that means $\eta_i(t) = -ke_i(t) - \mu\|e_i(t)\|^\alpha \text{sign}(e_i(t))$ in controller (13), finite-time GFMPLS between two coupled dynamical networks is achieved in the finite time $T_i = 9.8033$, where the control inputs are turned on at $Time = 1$.

If we choose the double power function nonlinear feedback control method, that means $\eta_i(t) = -ke_i(t) - \mu\left(\|e_i(t)\|^{\alpha} + \|e_i(t)\|^{\beta}\right)\text{sign}(e_i(t))$ in controller (13), and the control parameters are chosen as $k = 50$, $\mu = 1$, $\alpha = 1.5$ and $\beta = 0.6$, then finite-time GFMPLS between two coupled dynamical networks is achieved in the finite time $T = 1.4737$, where the control inputs are turned on at $Time = 1$. Finite-time GFMPLS between two coupled dynamical networks can be realized as displayed in figures 3–5. These three figures display the time evolution curves of the finite-time synchronization errors $e_i(t)$ and the total finite-time synchronization error $E(t)$, which indicates that GFMPLS between two coupled dynamical networks is achieved in the finite time $T = 1.4737$, and $T < T_i$.

Figures 4 and 5 display the different effects of the finite-time synchronization speed with different control parameters $k$ and $\mu$, respectively. The larger the control parameters $k$ and $\mu$, the faster the synchronization speed. Figure 6 displays the time evolution curves of the double power function nonlinear feedback controller $u_i(t)$. Figure 7 displays the finite-time $T$ decreases with increasing parameter $a$. Moreover, judging from the figures 8–10, the new method (double power function nonlinear feedback method) is better than the traditional method. The finite-time $T$, by using the new method, is shorter than the finite-time $T_i$ by using the traditional method. The percentage of finite-time reduction is defined as
The green line indicates the percentage of finite-time reduction with different parameters \( \beta \), \( k \) and \( \mu \) in figures 8–10. In short, the synchronization speed and the finite-time \( T \) are sensitively influenced by the parameters \( k \), \( \mu \), \( \alpha \) and \( \beta \).

Figure 7. The evolution curves of the finite-time \( T \) with different \( \alpha \) via the new method, where \( k = 50 \), \( \mu = 1 \), \( \beta = 0.6 \), \( t_0 = 1 \) and \( V(t_0) = 270.7075 \).

Figure 8. The evolution curves of the finite-time \( T \) and \( T_1 \) with different \( \beta \) via the new method and traditional method, respectively, where \( k = 50 \), \( \alpha = 1.5 \), \( t_0 = 1 \) and \( V(t_0) = 270.7075 \). (Green line: the percentage of finite-time reduction.)

Figure 9. The evolution curves of the finite-time \( T \) and \( T_1 \) with different \( k \) via the new method and traditional method, respectively, where \( \mu = 1 \), \( \alpha = 1.5 \), \( \beta = 0.6 \), \( t_0 = 1 \) and \( V(t_0) = 270.7075 \). (Green line: the percentage of finite-time reduction.)

Figure 10. The evolution curves of the finite-time \( T \) and \( T_1 \) with different \( \mu \) via the new method and traditional method, respectively, where \( k = 50 \), \( \alpha = 1.5 \), \( \beta = 0.6 \), \( t_0 = 1 \) and \( V(t_0) = 270.7075 \). (Green line: the percentage of finite-time reduction.)

The above example is simulated to verify the effectiveness of the double power function nonlinear feedback control method in this paper. From the above numerical simulation results, the limit of the total finite-time synchronization error \( E(t) \) approaches zero in the finite-time \( T \). That is to say, finite-time GFMPPLS between two completely different coupled dynamical networks with different dimensions of network nodes can be accomplished by the proposed double power function nonlinear feedback controller (13). Theoretical analysis and simulation results show that the double power function nonlinear feedback control method makes significant improvements over the traditional feedback control method. The proposed finite-time synchronization method in this paper further improves the synchronization efficiency, and shortens the the finite-time \( T \).

5. Conclusions

In this paper, based on the finite-time stability theory, we investigate finite-time generalized function matrix projective lag synchronization between two completely different coupled dynamical networks with different dimensions of network nodes. The double power function nonlinear feedback control method is proposed in this paper to guarantee that the state trajectories of the response network converge to the state trajectories of the drive network, according to a function matrix in a given finite-time \( T \). Moreover, an example is simulated to verify the effectiveness of the double power function nonlinear feedback control method. The limit of the finite-time synchronization errors \( e_i(t) \) and the total finite-time synchronization error \( E(t) \) approaches zero in the finite-time \( T \). Furthermore, theoretical analysis and simulation results show that the double power function nonlinear feedback control method is better than the traditional method, and the finite-time \( T_1 \) by using the new method, is shorter than the finite-time \( T_0 \) by using the traditional method, through the strict proof in the appendix. The double power function
nonlinear feedback control method can improve the synchronization efficiency, and shortens the finite-time \( T \). The synchronization speed and the finite-time \( T \) are sensitively influenced by the control parameters \( k, \mu, \alpha \) and \( \beta \).

The main content of this paper is theoretical research. Although the controller seems a little complicated, it can realize the finite-time synchronization between two different coupled dynamical networks with non-identical nodes and different dimensions, in theory. In addition, it is valuable to give some possible research topics for our future work. The first one is to simplify the complex controller, as a simple controller can be applied in practice more easily [41]. The second interesting topic would be to study finite-time distributed synchronization between two different coupled dynamical networks [42, 43]. A distributed controller could improve the control efficiency, and reduce the control cost. The third topic is that of finite-time synchronization between fractional-order and integer-order coupled dynamical networks, which will be studied in our future work [44, 45].

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Appendix A. Proof of Theorem 2

Proof

a) For \( V(t_0) \in (1, + \infty), T = t_0 + \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) + \frac{1}{a(1 - \theta_2)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) \)

Define \( \zeta_1 = \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) \) and \( \zeta_2 = \frac{1}{a(1 - \theta_2)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) \), then \( T = t_0 + \zeta_1 + \zeta_2 \).

\[
\frac{\partial \zeta_1}{\partial a} = \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) = \frac{1}{a^2(\theta_1 - 1)} \left[ \frac{aV^{\theta_1}(t_0) + b}{a + b} \right] - \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right)
\]

\[
\frac{\partial \zeta_2}{\partial a} = \frac{1}{a^2(\theta_1 - 1)(a + b)V^{\theta_1}(t_0) + b}
\]

\[
(\zeta_1)^{2} = -\left[ \frac{aV^{\theta_1}(t_0) + b}{a + b} \right] \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right) < 0.
\]

Because \( \frac{d\omega_1}{da} < 0 \), this means that \( \omega_1 \) is a monotone decreasing function with \( a \in (0, + \infty) \).

Then \( \omega_1 < \lim_{a \to 0} \omega_1 = 0 \). From equation (A.2), we have \( \frac{\partial \omega_1}{\partial a} < 0 \), which means \( \omega_1 \) is a monotone decreasing function with \( a \in (0, + \infty) \). Therefore,

\[
\zeta_1 < \lim_{a \to 0} \zeta_1 = \lim_{a \to 0} \frac{1}{a(\theta_1 - 1)} \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right)
\]

\[
= \lim_{a \to 0} \left( \frac{aV^{\theta_1}(t_0) + b}{a} \right) \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right)
\]

\[
= \frac{1}{(\theta_1 - 1)} \left[ \frac{aV^{\theta_1}(t_0) + b}{a + b} \right] - \ln \left( \frac{a + b}{aV^{\theta_1}(t_0) + b} \right)
\]

\[
= \frac{1}{(\theta_1 - 1)(a + b)V^{\theta_1}(t_0) + b}
\]
\begin{equation}
\frac{\partial \zeta_2}{\partial a} = \frac{1}{(1 - \theta_2)} \left( c \left( \frac{a + c}{c} \right) - \frac{1}{c} \cdot a - \ln \left( \frac{a + c}{c} \right) \right)
\end{equation}
\begin{equation}
\frac{\partial \omega_2}{\partial a} = -\ln \left( \frac{a + c}{c} \right) < 0.
\end{equation}

Define $\omega_2 = a - (a + c) \ln \left( \frac{a + c}{c} \right)$, then
\begin{equation}
\frac{\partial \zeta_2}{\partial a} = \frac{\omega_2}{a^2(a + c)(1 - \theta_2)}.
\end{equation}

Because $\frac{\partial \omega_2}{\partial a} < 0$, that means $\omega_2$ is a monotone decreasing function with $a \in (0, +\infty)$.

\begin{equation}
\frac{\partial \omega_2}{\partial a} = 1 - \left( \ln \left( \frac{a + c}{c} \right) + (a + c) \cdot \frac{c}{a + c} \cdot \frac{1}{c} \right) = -\ln \left( \frac{a + c}{c} \right) < 0.
\end{equation}

Define $\zeta_1 = \frac{1}{a(1 - \theta_2)} \ln \left( \frac{aV^{\frac{1}{a}}(t_0)}{c} \right)$, then $T = t_0 + \zeta_1$.

\begin{equation}
\frac{\partial \zeta_1}{\partial a} = \frac{1}{a(1 - \theta_2)} \left( c \right),
\end{equation}

\begin{equation}
\frac{\partial \omega_1}{\partial a} = V^{\frac{1}{a}}(t_0) - \left( \frac{1}{aV^{\frac{1}{a}}(t_0) + c} \right) \left( \frac{aV^{\frac{1}{a}}(t_0)}{c} \right)
\end{equation}

\begin{equation}
\frac{\partial \omega_1}{\partial a} = -V^{\frac{1}{a}}(t_0) \ln \left( \frac{aV^{\frac{1}{a}}(t_0) + c}{c} \right) < 0.
\end{equation}

Because $\frac{\partial \omega_1}{\partial a} < 0$, that means $\omega_1$ is a monotone decreasing function with $a \in (0, +\infty)$. Then $\omega_1 \lim \omega_2 = 0$. From equation (A.11), we have $\frac{\partial \omega_1}{\partial a} < 0$, which means $\zeta_1$ is a monotone decreasing function with $a \in (0, +\infty)$. Therefore,

\begin{equation}
\zeta_1 < \lim \zeta_1 = \lim_{a \to 0} \frac{1}{a(1 - \theta_2)} \ln \left( \frac{aV^{\frac{1}{a}}(t_0)}{c} \right)
\end{equation}

\begin{equation}
\frac{\partial \omega_1}{\partial a} = V^{\frac{1}{a}}(t_0) - \left( \frac{1}{aV^{\frac{1}{a}}(t_0) + c} \right) \left( \frac{aV^{\frac{1}{a}}(t_0)}{c} \right)
\end{equation}

\begin{equation}
\frac{\partial \omega_1}{\partial a} = -V^{\frac{1}{a}}(t_0) \ln \left( \frac{aV^{\frac{1}{a}}(t_0) + c}{c} \right) < 0.
\end{equation}

In summary, the proof is completed from inequalities (A.9) and (A.14).
Appendix B. Proof of Theorem 3

Proof

(1) For $V(t_0) \in (1, +\infty)$, theorem 2 shows that

$$T < t_0 + \frac{1 - e^{-\alpha t_0}}{\delta(\theta_1 - 1)} + \frac{1}{\delta(1 - \theta_2)}.$$  

$$\Delta T = T - T > \Delta T = \left( t_0 + \frac{V^{1-\theta_1}(t_0)}{c(1 - \theta_1)} \right) - \left( t_0 + \frac{1 - V^{1-\theta_1}(t_0)}{b(\theta_1 - 1)} + \frac{1}{c(1 - \theta_2)} \right)$$

$$= \frac{V^{1-\theta_1}(t_0)}{c(1 - \theta_2)} - \frac{1 - V^{1-\theta_1}(t_0)}{b(\theta_1 - 1)}.$$  

By substituting

$$b = \mu 2^\alpha \in (0, +\infty),$$

$$c = \mu 2^\beta \in (0, +\infty),$$

$$\theta_1 = \frac{1 + \alpha}{\alpha} \in (1, +\infty)$$

and

$$\theta_2 = \frac{1 + \beta}{\beta} \in (\frac{1}{2}, 1)$$

into equation (B.1), $\Delta T$ is obtained as follows:

$$\Delta T = \frac{1}{\mu(1 - \beta)(\alpha - 1)2^{\frac{a-1}{2}}(\frac{a-1}{2})} \left[ \frac{a}{2} \frac{1}{2}(\alpha - 1) \left( \frac{1}{V^{1-\theta_1}(t_0)} - 1 \right) - 2^{\frac{\beta-1}{2}}(1 - \beta) \left( 1 - V^{1-\theta_1}(t_0) \right) \right].$$  

(B.2)

Define $\xi = 2^{\frac{a}{2}}(\alpha - 1) \left( \frac{1}{V^{1-\theta_1}(t_0)} - 1 \right) - 2^{\frac{\beta-1}{2}}(1 - \beta)$, then

$$\Delta T = \frac{\xi}{\mu(1 - \beta)(\alpha - 1)2^{\frac{a-1}{2}}(\frac{a-1}{2})}.$$  

(B.3)

In order to write this more conveniently, we will replace $V(t_0)$ with $V_0$, then

$$\xi = 2^{\frac{a}{2}}(\alpha - 1) \left( V_0^{1-\theta_1} - 1 \right) - 2^{\frac{\beta-1}{2}}(1 - \beta) \left( 1 - V_0^{1-\theta_1} \right).$$  

(B.4)

Because $\mu(1 - \beta)(\alpha - 1)2^{\frac{a-1}{2}}(\frac{a-1}{2}) > 0$, if $\xi > 0$, then $\Delta T > 0$. Because $\Delta T > \Delta T$, we have $\Delta T > 0$. We spell out the details below.

$$\frac{\partial \xi}{\partial V_0} = 2^{\frac{a-1}{2}}(\alpha - 1) \frac{1 - \beta}{2} \left( V_0^{1-\theta_1} \right) - 2^{\frac{\beta-1}{2}}(1 - \beta) \frac{1 - \alpha}{2} \left( V_0^{1-\theta_1} \right)$$

$$= \frac{1}{2}(\alpha)(1 - \beta) \left( 2^{\frac{a-1}{2}} V_0^{1-\theta_1} - 2^{\frac{\beta-1}{2}} V_0^{1-\theta_1} \right).$$  

(B.5)

Because $\alpha > 1, 0 < \beta < 1$ and $V_0 > 1$, then $2^{\frac{a-1}{2}} > 2^{\frac{\beta-1}{2}}$ and $V_0^{1-\theta_1} > V_0^{1-\theta_1}$. We have $2^{\frac{a-1}{2}} V_0^{1-\theta_1} - 2^{\frac{\beta-1}{2}} V_0^{1-\theta_1} > 0$, therefore, $\frac{\partial \xi}{\partial V_0} > 0$.

Because of $\frac{\partial \xi}{\partial \alpha} > 0$, that means $\xi$ is a monotone increasing function with $V_0 \in (1, +\infty)$.

(2)

We cannot judge whether $\frac{\partial \xi}{\partial \alpha}$ is a plus or a minus, therefore we need to find $\frac{\partial^2 \xi}{\partial \alpha^2}, \frac{\partial^2 \xi}{\partial \beta^2}$ and $\frac{\partial^2 \xi}{\partial \alpha \partial \beta}$

(a)
\[ \begin{align*}
\frac{\partial^2 \xi}{\partial \alpha \partial \beta} &= -\frac{1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left( \frac{\alpha - 1}{2} \ln 2 + 1 \right) \\
- V_0^{-\alpha_1} \left( \ln V_0 \right) \left[ \left( \frac{1}{2} \frac{\beta - \beta^0}{\alpha - 1} \ln 2 \right) \times (1 - \beta) + \left( -\frac{\beta - \beta^0}{\alpha - 1} \right) \right] \\
= -\frac{\alpha_1 - 1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left[ \left( 2^{\frac{\beta - \beta^0}{\alpha - 1}} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \right) \ln 2 \right] \\
= -\left( \ln V_0 \right) \left[ \frac{2^{\frac{\beta - \beta^0}{\alpha - 1}} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \ln 2}{V_0^{1-\beta} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \ln 2} \right].
\end{align*} \]

Therefore, we have
\[ \frac{\partial^2 \xi}{\partial \beta \partial \alpha} < 0, \quad \text{which means } \frac{\partial \xi}{\partial \beta} \text{ is a monotone decreasing function with } \beta \in (0, 1). \]

(b)
\[ \begin{align*}
\frac{\partial^2 \xi}{\partial \alpha \partial \beta} &= \frac{1}{2} \frac{\alpha_1 - 1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left( \frac{\alpha - 1}{2} \ln 2 + 1 \right) \\
- V_0^{-\alpha_1} \left( \ln V_0 \right) \left[ \left( \frac{1}{2} \frac{\beta - \beta^0}{\alpha - 1} \ln 2 \right) \times (1 - \beta) + \left( -\frac{\beta - \beta^0}{\alpha - 1} \right) \right] \\
= \frac{\alpha_1 - 1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left[ \left( 2^{\frac{\beta - \beta^0}{\alpha - 1}} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \right) \ln 2 \right] \\
= -\left( \ln V_0 \right) \left[ \frac{2^{\frac{\beta - \beta^0}{\alpha - 1}} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \ln 2}{V_0^{1-\beta} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \ln 2} \right].
\end{align*} \]

Because \( \alpha > 1, 0 < \beta < 1 \) and \( V_0 > 1 \), then \( 2^{\frac{\beta - \beta^0}{\alpha - 1}} > 2^{\frac{\beta - \beta^0}{\alpha - 1}} \) and \( V_0^{1-\beta} > V_0^{-\alpha_1} \). We have \( \frac{\alpha_1 - 1}{2} \frac{\alpha - 1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left( \frac{\alpha - 1}{2} \ln 2 + 1 \right) \)

(c)
\[ \begin{align*}
\frac{\partial^2 \xi}{\partial \alpha \partial \beta} &= \frac{1}{2} \frac{\alpha_1 - 1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left( \frac{\alpha - 1}{2} \ln 2 + 1 \right) \\
- V_0^{-\alpha_1} \left( \ln V_0 \right) \left[ \left( \frac{1}{2} \frac{\beta - \beta^0}{\alpha - 1} \ln 2 \right) \times (1 - \beta) + \left( -\frac{\beta - \beta^0}{\alpha - 1} \right) \right] \\
= \frac{\alpha_1 - 1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left[ \left( 2^{\frac{\beta - \beta^0}{\alpha - 1}} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \right) \ln 2 \right] \\
= \left( 1 - \beta \right) \left[ \frac{2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \ln 2}{V_0^{1-\beta} + 2^{\frac{\beta - \beta^0}{\alpha - 1}} V_0^{-\alpha_1} (1 - \beta) \ln 2} \right].
\end{align*} \]

Because \( \alpha > 1, 0 < \beta < 1 \) and \( V_0 > 1 \), then \( 2^{\frac{\beta - \beta^0}{\alpha - 1}} > 2^{\frac{\beta - \beta^0}{\alpha - 1}} \) and \( V_0^{1-\beta} > V_0^{-\alpha_1} \). We have \( \frac{\alpha_1 - 1}{2} \frac{\alpha - 1}{2} V_0^{1-\beta} \left( \ln V_0 \right) \left( \frac{\alpha - 1}{2} \ln 2 + 1 \right) \)

Therefore, \( \frac{\partial^2 \xi}{\partial \beta \partial \alpha} > 0, \) which means \( \frac{\partial \xi}{\partial \beta} \) is a monotone increasing function with \( V_0 \in (1, + \infty) \).

In summary, because \( \frac{\partial \xi}{\partial \alpha} \) is a monotone increasing function with \( \alpha \in (1, + \infty) \) and \( V_0 \in (1, + \infty) \), and \( \frac{\partial \xi}{\partial \beta} \) is a monotone decreasing function with \( \beta \in (0, 1) \), we have

\[ \frac{\partial \xi}{\partial \alpha} > 0, \quad \frac{\partial \xi}{\partial \beta} < 0. \]
\[
\frac{\beta - 1}{2} \left( \ln 2 \right) > 0
\]
and
\[
\frac{\beta - 1}{2} \left( \ln 2 \right) \left( \frac{1 - \alpha}{\beta} \ln 2 - 1 \right) < 0.
\]
We have \( \frac{\delta^2 \xi}{\delta \beta \delta \alpha} > 0 \), which means \( \frac{\delta \xi}{\delta \alpha} \) is a monotone increasing function with \( \beta \in (0, 1) \).

(a)
\[
\frac{\partial^2 \xi}{\partial \beta \partial V_0} = -\frac{1}{2} \left( \alpha - 1 \right) \times \left( \frac{1 - \beta}{2} V_0^{-1 - \beta} \ln V_0 + \frac{1}{V_0} V_0^{-1 - \beta} \right) - \frac{\beta - 1}{2}
\]
\[
\times \left( \frac{1 - \beta}{2} \ln 2 - 1 \right) \left( 1 - \frac{\alpha}{2} V_0^{-1 - \alpha} \right)
\]
\[
= -\left( \alpha - 1 \right) \times \left( \frac{1}{2} \left( \frac{\alpha - 3}{2} V_0^{-1 - \beta} + \frac{\beta - 1}{2} V_0^{-1 - \alpha} \right) \ln V_0
\]
\[
- \frac{\beta - 1}{2} V_0^{-1 - \beta} + \frac{\beta - 1}{2} V_0^{-1 - \alpha} \right) \right).
\]
(B.13)

Because \( \alpha > 1 \), \( 0 < \beta < 1 \) and \( V_0 > 1 \), then \( \frac{\alpha - 3}{2} > \frac{\beta - 1}{2} \) and \( V_0^{-1 - \beta} > V_0^{-1 - \alpha} \). We have \( \frac{\alpha - 3}{2} V_0^{-1 - \beta} - \frac{\beta - 1}{2} V_0^{-1 - \alpha} > 0 \), therefore \( \frac{\partial^2 \xi}{\delta \beta \delta \alpha} > 0 \), that means \( \frac{\delta \xi}{\delta \beta} \) is a monotone increasing function with \( V_0 \in (1, + \infty) \).

In summary, because \( \frac{\delta \xi}{\delta \alpha} \) is a monotone decreasing function with \( \alpha \in (1, + \infty) \) and \( V_0 \in (1, + \infty) \), and \( \frac{\delta \xi}{\delta \beta} \) is a monotone increasing function with \( \beta \in (0, 1) \), we have
\[
\max \left( \frac{\delta \xi}{\delta \beta} \right) < \lim_{\alpha \to 1, \beta \to 1, V_0 \to 1} \left( \frac{\delta \xi}{\delta \alpha} \right) = 0.
\]
Therefore, \( \frac{\delta \xi}{\delta \beta} < 0 \), which means \( \xi \) is a monotone decreasing function with \( \beta \in (0, 1) \).

To sum up, because \( \xi \) is a monotone increasing function with \( \alpha \in (1, + \infty) \) and \( V_0 \in (1, + \infty) \), and \( \xi \) is a monotone decreasing function with \( \beta \in (0, 1) \), we have
\[
\min \left( \xi \right) > \lim_{\alpha \to 1, \beta \to 1, V_0 \to 1} \left( \xi \right) = 0.
\]
Therefore, \( \xi > 0 \). From equation (B.3), we have \( \Delta T > 0 \). Because \( \Delta T = T_1 - T > \Delta T \), therefore \( \Delta T > 0 \), in other words \( T_1 > T \).

(2) For \( V(t_0) \in (0, 1] \), theorem 2 shows that
\( T < t_0 + \frac{v^2(t_0)}{v(t_0) + v^2_{t_0}} \). Because \( T_1 = t_0 + \frac{v^2(t_0)}{v(t_0) + v^2_{t_0}} \), then we have \( T_1 > T \).

To sum up, \( \Delta T = T_1 - T > 0 \), and the proof is completed.

References
[25] Sun Y Z, Li W and Zhao D H 2012 Chaos 22 023152
[26] He P, Ma S H and Fan T 2012 Chaos 22 043151
[29] Cai N, Li W Q and Jing Y W 2011 Nonlinear Dyn. 64 385
[41] Chen D Y, Shi L, Chen H T and Ma X Y 2012 Nonlinear Dyn. 67 1745