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Laplacian coefficients
Laplacian matrix
Wiener index
Matchings
Incidence energy

1. Introduction

Let $G = (V, E)$ be a simple undirected graph with $n = |V|$ vertices. The Laplacian characteristic polynomial $P(G, \mu)$ of $G$ is the characteristic polynomial of its Laplacian matrix $L(G) = D(G) - A(G)$,

$$P(G, \mu) = \det(\mu I_n - L(G)) = \sum_{k=0}^{n}(-1)^k c_k \mu^{n-k}.$$ 

The Laplacian matrix $L(G)$ has nonnegative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. From Viète’s formulas, $c_k = \sigma_k(\mu_1, \mu_2, \ldots, \mu_n)$ is a symmetric polynomial of order $n-1$. In particular, $c_0 = 1, c_1 = 2n, c_n = 0$ and $c_{n-1} = n \tau(G)$, where $\tau(G)$ denotes the number of spanning trees of $G$. If $G$ is a tree, the coefficient $c_{n-2}$ is equal to its Wiener index, which is a sum of distances between all pairs of vertices.

$$c_{n-2}(T) = W(T) = \sum_{u, v \in V} d(u, v),$$

while $c_{n-3}$ is its modified hyper-Wiener index, introduced by Gutman in [2]. The Wiener index is considered as one of the most used topological indices with high correlation with many physical and chemical properties of molecular compounds. A huge majority of chemical applications of the Wiener index deal with acyclic organic molecules. For recent results and applications of Wiener index see [3–6].

Let $m_k(G)$ be the number of matchings of $G$ containing exactly $k$ independent edges. In particular, $m_0(G) = 1$, $m_1(G) = |E(G)|$ and $m_k(G) = 0$ for $k > \frac{n}{2}$. A vertex $v$ is matched if it is incident to an edge in the matching; otherwise the vertex is unmatched. A vertex is said to be perfectly matched if it is matched in all maximum matchings of $G$.
The subdivision graph $S(G)$ of $G$ is obtained by inserting a new vertex of degree two on each edge of $G$. Zhou and Gutman [7] proved that for every acyclic graph $T$ with $n$ vertices holds

$$c_k(T) = m_k(S(T)), \quad 0 \leq k \leq n.$$  \hspace{1cm} (1)

The signless Laplacian matrix of the graph $G$ is defined as $Q(G) = D(G) + A(G)$. Matrix $Q(G)$ also has real and nonnegative eigenvalues $\mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_n \geq 0$ (see [8,9] for more details). Gutman et al. in [10,11] recently introduced the Incidence energy $IE(G)$ of a graph $G$, defining it as the sum of the singular values of the Incidence matrix. It turns out that

$$IE(G) = \sum_{k=1}^{n} \sqrt{\mu'_k}.$$  

The Laplacian-energy-like invariant of graph $G$ [12], LEL for short, is defined as follows:

$$\text{LEL}(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k}.$$  

This concept was introduced in [13] where it was shown that it has similar features as molecular graph energy, defined by Gutman [14]. In [15] it was shown that LEL describes well the properties which are accounted by the majority of molecular descriptors (octane number, entropy, volume, AF parameter, boiling point, melting point, log $P$). In a set of polycyclic aromatic hydrocarbons, LEL was proved to be as good as the Randic $\chi$ index and better than the Wiener index. Various results on the Laplacian-energy-like invariant and Laplacian coefficients of trees and unicyclic graphs were given in [16,17].

In particular, if $G$ is a bipartite graph, the spectra of $Q(G)$ and $L(G)$ coincide, and we have $IE(G) = \text{LEL}(G)$. In [18] the authors pointed out some further relations for IE and LEL, and established several lower and upper bounds for IE, including those that pertain to the line graph of $G$.

The energy is a graph parameter stemming from the Hückel molecular orbital approximation for the total $\pi$-electron energy and it is defined as the sum of the absolute values of all eigenvalues of adjacency matrix of a graph (for recent survey on molecular graph energy see [19]). The energy of $T$ is also expressible in terms of the Coulson integral formula [20] as

$$E(T) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{x^2} \ln \left( 1 + \sum_{k=1}^{[n/2]} m_k(T) \cdot x^{2k} \right) dx.$$  

The fact that $E(T)$ is a strictly monotonically increasing function of each matching number $m_k(T)$ defines a quasi-ordering over the set of all acyclic graphs for comparing their energies.


**Theorem 1.1.** Let $G$ and $H$ be two $n$-vertex graphs. If $c_k(G) \leq c_k(H)$ for $k = 1, 2, \ldots, n-1$ then $\text{LEL}(G) \leq \text{LEL}(H)$. Furthermore, if a strict inequality $c_k(G) < c_k(H)$ holds for some $1 \leq k \leq n-1$, then $\text{LEL}(G) < \text{LEL}(H)$.

Recently, in [22] the authors corrected the original proof of Theorem 1.1, and provided a necessary condition for function $F(\mu_1, \mu_2, \ldots, \mu_n)$ to support a partial ordering based on Laplacian coefficients.

Let $P_n$ be the path with $n$ vertices, and let $S_n$ be the star on $n$ vertices. Let $n$ and $m$ be positive integers and $n \geq 2m$. Define a tree $A(n, m)$ with $n$ vertices as follows: $A(n, m)$ is obtained from the star graph $S_{n-m+1}$ by attaching a pendant edge to each of certain $m - 1$ non-central vertices of $S_{n-m+1}$. We call $A(n, m)$ a spur and note that it has an $m$-matching (see Fig. 1). The center of $A(n, m)$ is the center of the star $S_{n-m+1}$.

In [23] the authors prove that if $T$ is $n$-vertex tree with an $m$-matching with $n \geq 2m$, and $T \neq A(n, m)$, then

$$\lambda_1(T) < \lambda_1(A(n, m)) = \frac{1}{2} \left( \sqrt{n-m+1 - 2\sqrt{n-2m+1}} + \sqrt{n-m+1 + 2\sqrt{n-2m+1}} \right),$$  

where $\lambda_1$ is the greatest eigenvalue of adjacency matrix $A(G)$.  

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**Fig. 1.** The spur $A(13, 6)$. 

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The Hosoya index of a graph $G$ is defined as the total number of its matchings,

$$Z(G) = \sum_{k=0}^{m} m_k(G).$$

It is a topological parameter that studies the relation between molecular structure and physical and chemical properties of certain hydrocarbon compounds. Many related results and the latest progress can be found in [24–27]. Hou in [28] proved the following

**Theorem 1.2.** Let $T$ be an $n$-vertex tree with an $m$-matching, where $m \geq 1$. Then

$$Z(T) \geq 2^{m-2}(2n - 3m + 3),$$

with equality if and only if $T$ is the spur graph $A(n, m)$.

In [29] the authors proved that the graph energy $E(T)$ is minimal for the graph $A \left( n, \frac{n}{2} \right)$ in the class of trees with $n$ vertices which have perfect matching. Recently, Guo in [30] generalized this result and found trees with third and fourth minimal energy in the same class of trees.

The Wiener index was used to order trees in [31], while in [32] the authors further supported the ordering of trees based on the Laplacian coefficients.

Motivated by results from [33], our goal here is to characterize the trees with given matching number which simultaneously minimize all Laplacian coefficients—and consequently strengthen the ordering of trees based on Incidence energy.

We also add some further evidence to support the use of Laplacian coefficients and Incidence energy as a measure of branching in alkanes. A topological index acceptable as a measure of branching must satisfy the inequalities [34]

$$TI(P_n) < TI(X_n) < TI(S_n) \quad \text{or} \quad TI(P_n) > TI(X_n) > TI(S_n),$$

for $n = 4, 5, \ldots$, where $P_n$ is the path, and $S_n$ is the star on $n$ vertices. For example, the first relation is obeyed by the largest graph eigenvalue and Estrada index, while the second relation is obeyed by the Wiener index, Hosoya index and graph energy. It is proven in [35] that for arbitrary tree $T \neq P_n$, $S_n$ it holds

$$c_k(P_n) > c_k(T) > c_k(S_n),$$

for all $2 \leq k \leq n - 2$. We further refine this relation, by showing a long chain of inequalities

$$c_k(A(n, 1)) \leq c_k(A(n, 2)) \leq \cdots \leq c_k \left( A \left( n, \left\lfloor \frac{n}{2} \right\rfloor \right) \right) \quad (2)$$

and consequently,

$$IE(A(n, 1)) \leq IE(A(n, 2)) \leq \cdots \leq IE \left( A \left( n, \left\lfloor \frac{n}{2} \right\rfloor \right) \right). \quad (3)$$

The plan of the paper is as follows. In Section 2 we introduce $\rho$ transformation of trees, such that all Laplacian coefficients are monotone under this transformation. In Section 3 we recall the linear algorithm for constructing a maximum cardinality matching in a tree and prove that the spur tree $A(n, m)$ minimizes all Laplacian coefficients among $n$-vertex trees with maximum matching $m$. In particular, $A(n, m)$ minimizes the Wiener index, the modified hyper-Wiener index and Incidence energy in the same class of trees. We also introduce the long chain of inequalities (2) and (3), supporting Laplacian coefficients and Incidence energy as a measure of branching. Finally, in Section 4 we illustrate on examples with the Wiener index, the modified hyper-Wiener index and Incidence energy that the opposite problem of simultaneously maximizing all Laplacian coefficients has no solution.

### 2. Tree Transformations

In this section we consider the trees on $n$ vertices with the matching number $m$. For $m = 1$, the star $S_n$ is the unique tree with maximum matching equal to 1. For even $n$ and $m = \frac{n}{2}$, the tree has a perfect matching.

The union $G = G_1 \cup G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. If $G$ is a union of two paths of lengths $a$ and $b$, then $G$ is disconnected and has $a + b$ vertices and $a + b - 2$ edges. Let $m_k(a, b)$ be the number of $k$-matchings in $G = P_a \cup P_b$.

The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest path between them. The eccentricity $e(v)$ of a vertex $v$ is the maximum distance from $v$ to any other vertex. The diameter $d(G)$ of a graph $G$ is the maximum eccentricity over all vertices in a graph, and the radius $r(G)$ is the minimum eccentricity over all $v \in V(G)$.

Vertices of minimum eccentricity form the center (see [3]). A tree $T$ has exactly one or two adjacent center vertices. In what follows, if a tree has a bicenter, then our considerations apply to any of its center vertices.
Let us divide the set of coefficients $k$ set $k$.

**Theorem 2.2.** Let $w$ be a vertex of the nontrivial connected graph $G$ and for nonnegative integers $p$ and $q$, let $G(p, q)$ denote the graph obtained from $G$ by attaching pendant paths $P = wv_1v_2\cdots v_p$ and $Q = wu_1u_2\cdots u_q$ of lengths $p$ and $q$, respectively, at $w$. If $p \geq q \geq 1$, then

$$c_k(G(p, q)) \leq c_k(G(p + 1, q - 1)), \quad k = 0, 1, 2, \ldots, n.$$ 

We introduce the following graph transformation.

**Definition 2.1.** Let $T$ be an arbitrary tree, rooted at a center vertex and let $v$ be a vertex with degree $m + 1$ that is not a center of the tree $T$. Suppose that $w$ is a parent of $v$ in tree $T$ and that $T_1, T_2, \ldots, T_m$ are subtrees under $v$ with root vertices $v_1, v_2, \ldots, v_m$, such that the tree $T_m$ is actually a path. We form a tree $T'$ by removing the edges $vv_1, vv_2, \ldots, vv_m$ from $T$ and adding new edges $wv_1, wv_2, \ldots, wv_m$. We say that $T'$ is a $\rho$ transformation of $T$.

This transformation preserves the number of pendant vertices in a tree $T$, and does not increase the diameter.

**Theorem 2.3.** For the $\rho$ transformation tree $T' = \rho(T, v)$ and $0 \leq k \leq n$ holds

$$c_k(T) \geq c_k(T').$$

**Proof.** Coefficients $c_0, c_1, c_{n-1}$ and $c_n$ are constant for all trees on $n$ vertices, as stated before. Therefore, we can assume that $2 \leq k \leq n - 2$. Let $G$ be the part of trees $T$ and $T'_n$ that is obtained by deleting vertices from the trees $T_1, T_2, \ldots, T_m$ (see Fig. 2).

Let $u$ and $u_1, u_2, \ldots, u_m$ be the subdivision vertices of the edges $vw$ and $wv_1, wv_2, \ldots, wv_m$, respectively. We will construct an injection of the set $\mathcal{M}'$ of $k$-matchings of the tree $S(T')$ into the set $\mathcal{M}$ of $k$-matchings of the tree $S(T)$. Let us divide the set of $k$-matchings of the subdivision graph $S(T')$ in two disjoint subsets $\mathcal{M}_1'$ and $\mathcal{M}_2'$. The set $\mathcal{M}_1'$ contains $k$-matchings without the edges from the set $\{wu_1, wu_2, \ldots, wu_m\}$, while the set $\mathcal{M}_2'$ contains all other $k$-matchings from $S(T')$. Analogously, divide the set of $k$-matchings of the subdivision graph $S(T)$ in two disjoint subsets $\mathcal{M}_1$ and $\mathcal{M}_2$. The set $\mathcal{M}_1$ contains $k$-matchings without the edges from the set $\{vu_1, vu_2, \ldots, vu_m\}$, while the set $\mathcal{M}_2$ contains all other $k$-matchings from $S(T)$.
AssumethatwehaveabranchingvertexT
with
numberofedges,differentfromthoseinvinthegraphG⊇\{v\}∈G. If
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}containedgevthenweputtheedgewv to be in M. After taking these edges in (k − 1)-matchings, we have the same components in both graphs, and again a trivial bijection. Now, we have reduced the problem to the following one: the number of k-matchings in the union (G∪M) is greater than or equal to the number of k-matchings in the graph (G ∩ M).

Consider the longest path in the graph (G ∩ M) that starts from the vertex w and ends in some pendant vertex in (G ∩ M). Let this path be wv1v2v3...vpy, where y1, y2, ..., yp are the subdivision vertices of degree two. From the assumption, this path is longer than or equal to the path Tm, or equivalently p ≥ q. The equality is achieved if and only if both vertices v and w are centers of the tree T.

If the edge uu1 does not belong to M, then we can construct a corresponding matching M by taking the same edges as those in M'. This way we do not take any of the edges in M adjacent to w in the graph G. If we use some of these extra edges, different from uu1, we will get the strict inequality in (4)—because there are more matchings in (S(T) than in S(T')).

Therefore, assume that the edge uu1 belongs to M and take the edge uy1 to be in M. We have to consider two different cases concerning the edge uy1. If uy1 ∉ M', then take the same set of edges in M, like in the previous case. Otherwise, the edge uy1 belongs to M' and then we set uu1 ∈ M.

Using this algorithm, we get a problem with the smaller dimensions. Since p > q, we will reach the state with graphs S(G') ∪ P2 and (S(G') \ {x}) ∪ P3, where the root vertex of S(G') is x1. If the first edge from P3 belongs to the matching M, then set x1y1+1 ∈ M. If y1+1x1+1 does not belong to M', we have an injection; otherwise y1+1x1+1 ∈ M' and we take the only edge from P2 in M. Now, it is obvious that the number of k-matchings in the graph S(G') \ {x, y1+1} is greater than or equal to the number of k-matchings in S(G') \ {x1, y1+1}.

This completes the inductive proof of ck(T) ≥ ck(T').

3. Trees with minimal Laplacian coefficients

The linear algorithm for constructing a matching of maximum cardinality in a tree T is greedy and based on mathematical induction. Namely, take an arbitrary pendant vertex v and match it to its parent w. Remove both from the tree and solve the resulting problem by induction. We need to prove that the edge vw belongs to a maximum matching. Let M be a matching of maximum cardinality in T. If M does not contain vw, then the vertex v is not matched. If vw is in the maximum matching M, then simply replace it with vw. It is still a matching, and it has the same cardinality. For more implementation details and different approaches of constructing maximal matching in graphs see [36].

Assume that there is a pendant path of length p > 2 attached to vertex v in the tree T. We can consider new tree T' that has two pendant paths attached at v, with lengths 2 and p − 2. The matching number of trees T and T' is the same according to the described algorithm, since we can remove an arbitrary pendant edge from the tree at any step. Using Theorem 2.2, we get that ck(T') < ck(T) for k = 0, 1, ..., n.

It is easy to prove by induction that a perfect matching of a tree is unique when it exists. We can use the ρ and π transformations in order to preserve the matching number of trees. From the above consideration, assume that we have a branching vertex v with attached p pendant paths P2 (pendent edges) and q pendant paths P3. Let u denotes the parent of v in the tree T. Based on the matching property of the vertex w, we have

- q = 0 and vertex w is perfectly matched—apply one ρ transformation at v, and get p − 1 pendant paths P2 and one pendant path P3 attached at w;
- q = 0 and vertex w is not perfectly matched—apply one ρ transformation at v and then one π transformation, and get p + 1 pendant paths P2 and q pendant path P3 attached at w;
- p = 0—apply one ρ transformation at v and then one π transformation, and get q pendant paths P3 and one pendant path P2 attached at w;
- p > 0, q > 0 and w is perfectly matched—apply one ρ transformation at v, and get p − 1 pendant paths P2 and q + 1 pendant paths P3 attached at w;
- p > 0, q > 0 and w is not perfectly matched—apply one ρ transformation at v and then one π transformation, and get p + 1 pendant paths P2 and q pendant paths P3 attached at w.

These combinations of transformations simultaneously decrease all Laplacian coefficients, and preserve the matching number.
Theorem 3.1. Among trees on n vertices and matching number $1 \leq m \leq \frac{n}{2}$, the tree $A(n, m)$ has minimal Laplacian coefficient $c_k$ for every $k = 0, 1, \ldots, n$.

**Proof.** Let $T$ be the extremal rooted $n$-vertex tree with matching number $m$ and minimal Laplacian coefficient $c_k$, for some $2 \leq k \leq n - 2$. Assume that $T$ has diameter strictly greater than 4. Let $v$ be an arbitrary pendant vertex furthest from the center vertex. From the assumption $d(T) > 4$, we conclude that the distance from $v$ to a center vertex is at least 3. This means that either there is a pendent path of length at least 3 or some branch vertex different from the center of $T$. In both cases, we can perform transformations described above using **Theorem 2.2** or **Theorem 2.3** and obtain new tree $T'$ such that $c_k(T') < c_k(T)$, while preserving the matching number.

Therefore, we can assume that the diameter of $T$ is equal 3 or 4. If $d(T) = 3$, the matching number is equal to 2 and $T$ is a double star. By applying once again $\rho$ transformation, we get $A(n, 2, 2)$. Otherwise, the diameter is equal to 4 and again we apply $\rho$ transformation to every vertex on distance one from the center of the tree and with degree greater than two to obtain $A(n, m)$ as the extremal graph. □

We can apply the previous theorem for the Wiener index and get the result from [33].

**Corollary 3.2.** Let $T$ be an $n$-vertex tree with an $m$-matching, where $1 \leq m \leq \frac{n}{2}$. Then

$$W(T) \geq 4 + (m + n)(n - 3).$$

**Proof.** We will prove that $W(A(n, m)) = 4 + (m + n)(n - 3)$. There are four types of vertices in the tree $A(n, m)$. Denote with $D(v)$ the sum of all distances from $v$ to all other vertices.

- For the center vertex $D(v) = n + m - 2$.
- For each pendant vertex attached to the center vertex $D(v) = 2n + m - 4$.
- For each vertex of degree 2, different from the center vertex $D(v) = 2n + m - 6$.
- For each pendant vertex not attached to the center vertex $D(v) = 3n + m - 8$.

After summing above contributions to the Wiener index, we get

$$W(T) = \frac{1}{2} \sum_{v \in A(n, m)} D(v) = (n + m - 2) + (n - 2m + 1)(2n + m - 4) + (m - 1)(2n + m - 6) + (m - 1)(3n + m - 8) = 4 + (m + n)(n - 3).$$

Since $W(T) \geq W(A(n, m))$, the inequality follows. □

**Corollary 3.3.** Among trees on $n$ vertices and with matching number $m$, $A(n, m)$ has the minimal modified hyper-Wiener index $WW(T)$.

Using **Theorem 1.1** and the fact that the starlike trees are determined by their Laplacian spectrum [37], we have

**Theorem 3.4.** Among trees on $n$ vertices and with matching number $m$, $A(n, m)$ is the unique tree that has minimal Incidence energy.

By the previous theorem, we have the following

**Corollary 3.5.** Among trees on $n \geq 6$ vertices and with perfect matching, it follows

$$\text{IE}(G) \geq \sqrt{2} + \left(\frac{n}{2} - 2\right) \left(\sqrt{\frac{3 + \sqrt{5}}{2}} + \sqrt{\frac{3 - \sqrt{5}}{2}}\right) + \frac{\sqrt{n + 4 + \sqrt{n^2 + 16}}}{2} + \frac{\sqrt{n + 4 - \sqrt{n^2 + 16}}}{2},$$

with equality if and only if $G \cong A(n, n/2)$.

**Proof.** From **Theorem 3.4**, it follows that $\text{IE}(G) \geq \text{IE}(A(n, n/2))$ with equality if and only if $G \cong A(n, n/2)$. By simple determinant manipulations of the Laplacian matrix of $A(n, n/2)$, one can establish the recurrent formula

$$P(A(n, n/2), \mu) = \begin{vmatrix}
\mu - 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & \mu - 2 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \mu - 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \mu - 2 & \cdots & 0 & 1 \\
& & & & & & \\
0 & 0 & 0 & 0 & \cdots & \mu - 1 & 1 \\
0 & 1 & 0 & 1 & \cdots & 1 & \mu - n/2 \\
\end{vmatrix}
= (\mu^2 - 3\mu + 1)P(A(n - 2, n/2 - 1), \mu) - (\mu^2 - 3\mu + 1)^{n/2 - 2}(\mu - 1)((\mu - 1)^2 - 1).$$
Fig. 3. Graphs with $n = 18$ and matching numbers $2 \leq m \leq 8$ with maximal IE.

For the initial values, we have $P(A(2, 1), \mu) = \mu(\mu - 2)$ and $P(A(4, 2), \mu) = \mu(\mu - 2)(\mu^2 - 4\mu + 2)$. By mathematical induction it follows that for every $n \geq 1$ holds:

$$P(A(n, n/2), \mu) = \mu(\mu - 2)\left(\frac{n}{2} + 2\right)\frac{\mu + n}{2} (\mu^2 - 3\mu + 1)^{n/2 - 2}.$$  

Finally, for $n \geq 6$ by solving the quadratic equations, we derive the value of $\text{IE}(A(n, n/2))$. □

The independence number of a graph $G$, denoted by $\alpha(G)$, is the size of a maximum independent set of $G$. Since in any bipartite graph, the sum of the independence number of $G$ and the matching number of $G$ is equal to the number of vertices $[36]$, we have

**Corollary 3.6.** Among trees on $n$ vertices and with independence number $\alpha$, $A(n, n - \alpha)$ is the unique tree that has minimal Incidence energy.

If $m < \left\lceil \frac{n}{2} \right\rceil$, we can apply the transformation from Theorem 2.2 at the vertex of degree greater than two in $A(n, m)$ to obtain $A(n, m + 1)$, and simultaneously decrease all Laplacian coefficients. Thus, for $2 \leq k \leq n - 2$ we get the chain of inequalities

$$c_k(A(n, 1)) \leq c_k(A(n, 2)) \leq \cdots \leq c_k\left(A\left(n, \left\lceil \frac{n}{2} \right\rceil\right)\right)$$

and consequently from Theorem 3.4

$$\text{IE}(A(n, 1)) \leq \text{IE}(A(n, 2)) \leq \cdots \leq \text{IE}\left(A\left(n, \left\lceil \frac{n}{2} \right\rceil\right)\right).$$

4. Concluding remarks

We proved that $A(n, m)$ is the unique graph that minimizes all Laplacian coefficients simultaneously among graphs on $n$ vertices with given matching number $m$. Naturally, one wants to solve the opposite problem and describe $n$-vertex graphs with fixed matching number with maximal Laplacian coefficients.

We have checked all trees up to 24 vertices and classified them based on the matching number. For every triple $(n, m, i)$ we found extremal graphs with $n$ vertices and fixed matching number $m$ that maximize coefficient $c_i$. Also, we have found the extremal $n$-vertex trees with matching number $m$ that maximize Incidence energy. The result is obvious—the extremal trees are different.

The dumbbell $D(n, a, b)$ consists of the path $P_{n-a-b}$ together with $a$ independent vertices adjacent to one pendent vertex of $P$ and $b$ independent vertices adjacent to the other pendent vertex. In [38] it is shown that

$$W(T) \leq W\left(D\left(n, \left\lceil \frac{n+1}{2} \right\rceil - m, \left\lceil \frac{n+1}{2} \right\rceil - m\right)\right),$$

with equality if and only if $G \cong D\left(n, \left\lceil \frac{n+1}{2} \right\rceil - m, \left\lceil \frac{n+1}{2} \right\rceil - m\right)$.

The graphs in Fig. 3 are extremal for Incidence energy, with $n = 18$ vertices and matching numbers from 2 to 8. We leave for future study to see whether these trees can be characterized.
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