On the ordering of trees by the Laplacian coefficients

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ABSTRACT
We generalize the results from [X.-D. Zhang, X.-P. Lv, Y.-H. Chen, Ordering trees by the Laplacian coefficients, Linear Algebra Appl. (2009), doi:10.1016/j.laa.2009.04.018] on the partial ordering of trees with given diameter. For two n-vertex trees $T_1$ and $T_2$, if $c_k(T_1) \leq c_k(T_2)$ holds for all Laplacian coefficients $c_k, k = 0, 1, \ldots, n$, we say that $T_1$ is dominated by $T_2$ and write $T_1 \preceq c T_2$. We proved that among n-vertex trees with fixed diameter $d$, the caterpillar $C_{n,d}$ has minimal Laplacian coefficients $c_k$, $k = 0, 1, \ldots, n$. The number of incomparable pairs of trees on $\leq 18$ vertices is presented, as well as infinite families of examples for two other partial orderings of trees, recently proposed by Mohar. For every integer $n$, we construct a chain $\{T_i\}_{i=0}^m$ of $n$-vertex trees of length $n^2$, such that $T_0 \cong S_n$, $T_m \cong P_n$ and $T_i \preceq c T_{i+1}$ for all $i = 0, 1, \ldots, m - 1$. In addition, the characterization of the partial ordering of starlike trees is established by the majorization inequalities of the pendent path lengths. We determine the relations among the extremal trees with fixed maximum degree, and with perfect matching and further support the Laplacian coefficients as a measure of branching.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph with $n = |V|$ vertices and $m = |E|$ edges. The Laplacian polynomial $P(G, \lambda)$ of $G$ is the characteristic polynomial of its Laplacian matrix $L(G) = D(G) - A(G)$,

$$P(G, \lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^{n} (-1)^k c_k \lambda^{n-k}.$$  

The Laplacian matrix $L(G)$ has non-negative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. From Viette’s formulas, $c_k = \sigma_k(\mu_1, \mu_2, \ldots, \mu_{n-1})$ is a symmetric polynomial of order $n - 1$. In particular,
we have $c_0 = 1, c_n = 0, c_1 = 2m, c_{n-1} = \tau(G)$, where $\tau(G)$ denotes the number of spanning trees of $G$ (see [1,14]). If $G$ is a tree, the coefficient $c_{n-2}$ is equal to its Wiener index, which is a sum of distances between all pairs of vertices

$$c_{n-2}(T) = W(T) = \sum_{u,v \in V} d(u,v),$$

while the coefficient $c_{n-3}$ is its modified hyper-Wiener index, introduced by Gutman in [9]. The Wiener index is considered as one of the most used topological indices with high correlation with many physical and chemical properties of molecular compounds (for recent results and applications of Wiener index see [2]).

Let $m_k(G)$ be the number of matchings of $G$ containing exactly $k$ independent edges. The subdivision graph $S(G)$ of $G$ is obtained by inserting a new vertex of degree two on each edge of $G$. Zhou and Gutman [26] proved that for every acyclic graph $T$ with $n$ vertices holds

$$c_k(T) = m_k(S(T)), \quad 0 \leq k \leq n. \tag{1}$$

Let $T_1$ and $T_2$ be two trees of order $n$. Denote by $r$ (respectively $s$) the smallest (respectively largest) integer such that $c_r(T_1) \neq c_r(T_2)$ (respectively $c_s(T_1) \neq c_s(T_2)$). Two partial orderings may be defined as follows. If $c_r(T_1) < c_r(T_2)$, we say that $T_1$ is smaller than $T_2$ and denote $T_1 \prec T_2$. If $c_s(T_1) < c_s(T_2)$, we say that $T_1$ is smaller than $T_2$ and denote $T_1 \prec^1 T_2$. We may now introduce relations $\prec_c$ and $\preceq_c$ on the set of $n$-vertex graphs by defining

$$G \preceq_c H \iff c_k(G) \leq c_k(H), \quad k = 0, 1, \ldots, n.$$

and

$$G \prec_c H \iff G \preceq_c H \quad \text{and} \quad c_k(G) < c_k(H) \quad \text{for some} \quad 1 \leq k \leq n - 1.$$ 

Recently, Mohar on his homepage [15] proposed some problems on ordering trees with the Laplacian coefficients.

**Problem 1.** Do there exist two trees $T_1$ and $T_2$ of order $n$ such that $T_1 \prec^1 T_2$ and $T_2 \prec^2 T_1$?

**Problem 2.** Do there exist two trees $T_1$ and $T_2$ of order $n$ such that $T_1 \prec^1 T_2$ and $T_1 \prec^2 T_2$, but there is an index $i$ such that $c_i(T_1) > c_i(T_2)$?

**Problem 3.** Let $\mathcal{T}_n$ be the set of all trees of order $n$. How large chains and antichains of pairwise non-Laplacian-cospectral trees are there?

**Problem 4.** Let us define $U(T, T')$ to be the set of all trees $Z$ of order $n = |T| = |T'|$ such that $Z$ majorizes $T$ and $T'$ simultaneously. For which trees $T$ and $T'$ has $U(T, T')$ only one minimal element up to cospectrality, i.e., when are all minimal elements in $U(T, T')$ cospectral?

Our goal here is also to add some further evidence to support the use of Laplacian coefficients as a measure of branching in alkanes. A topological index acceptable as a measure of branching must satisfy the inequalities [5]

$$TI(P_n) < TI(X_n) < TI(S_n) \quad \text{or} \quad TI(P_n) > TI(X_n) > TI(S_n),$$

for $n = 4, 5, \ldots$, where $P_n$ is the path, and $S_n$ is the star on $n$ vertices. For example, the first relation is obeyed by the largest graph eigenvalue and Estrada index, while the second relation is obeyed by the Wiener index, Hosoya index and graph energy. It is proven in [16] and [26] that for arbitrary tree $T \not\cong P_n, S_n$ holds

$$c_k(P_n) > c_k(T) > c_k(S_n),$$

for all $2 \leq k \leq n - 2$. We further refine this relation, by introducing long chain of inequalities.

Stevanović and Ilić in [19] investigated the properties of the Laplacian coefficients of unicyclic graphs. Guo in [9] presented the several tree orderings by the Laplacian spectral radius, while Dong
and Guo in [3] used Wiener index for ordering the trees. The authors in [10] generalized the recent results from [13,21], which proved that the caterpillar $C_{n,d}$ is the unique tree with $n$ vertices and diameter $d$, that minimizes Wiener index $w_{n,d}$. Zhang et al. in [25], proved that $C_{n,d}$ has minimal Laplacian coefficients only for the cases $d = 3$ and $d = 4$, while here we prove it for all $2 \leq d \leq n - 1$.

The paper is organized as follows. In Section 2 we revise two graph transformations, such that all Laplacian coefficients are monotone under these transformations. Also, we derive the partial ordering of incomparable pairs of trees based on two Mohar’s ordering and calculate the Laplacian coefficients for the special case $d = n - 3$. In Section 4, the number of incomparable pairs of trees for $n \leq 18$ vertices is presented, and we also derive a chain of inequalities of length $n$, that minimizes Wiener index $w_{n}$. The authors in [10] generalized the recent results from [13,21], which proved that the caterpillar $T = \delta(T, v)$ is a tree composed of the root $v$, and the $P$ paths incident with $v$. The number of vertices of the tree $T = \delta(T, v)$ is $n = n_1 + n_2 + \cdots + n_k + 1$. The starlike tree $BS_{n,k}$ is balanced if all paths have almost equal lengths, i.e., $|n_i - n_j| \leq 1$ for every $1 \leq i < j \leq k$.

2. Transformations and starlike trees

Mohar in [16] proved that every tree can be transformed into a star by a sequence of $\sigma$-transformations. Here we present the transformation from [11], that is a generalization of $\sigma$-transformation.

**Definition 5.** Let $v$ be a vertex of a tree $T$ of degree $m + 1$. Suppose that $P_1, P_2, \ldots, P_m$ are pendent paths incident with $v$, with lengths $n_i \geq 1, i = 1, 2, \ldots, m$. Let $w$ be the neighbor of $v$ distinct from the starting vertices of paths $v_1, v_2, \ldots, v_m$, respectively. We form a tree $T' = \delta(T, v)$ by removing the edges $vw_1, v_wv_2, v_wv_3, \ldots, v_wv_m$ from $T$ and adding $m - 1$ new edges $vw, vw_1, vw_2, \ldots, vw_m$ incident with $w$. We say that $T'$ is a $\delta$-transform of $T$.

This transformation preserves the number of pendent vertices in a tree $T$ and decreases all Laplacian coefficients.

**Theorem 6.** Let $T$ be an arbitrary tree, rooted at the center vertex. Let vertex $v$ be a vertex furthest from the center of tree $T$ among all branching vertices with degree at least three. Then, for $\delta$-transformation tree $T' = \delta(T, v)$ and $0 \leq k \leq n$ holds

$$c_k(T) \geq c_k(T').$$

Mohar in [16] proved that every tree can be transformed into a path by a sequence of $\pi$-transformations. Here we present the transformation from [10], that is a generalization of $\pi$-transformation.

**Theorem 7.** Let $w$ be a vertex of the nontrivial connected graph $G$ and for non-negative integers $p$ and $q$, let $G(p, q)$ denote the graph obtained from $G$ by attaching pendent paths $P = wv_1v_2v_3v_p$ and $Q = uw_1u_2u_3\cdots u_q$ of lengths $p$ and $q$, respectively, at $w$. If $p \geq q \geq 1$, then

$$c_k(G(p, q)) \leq c_k(G(p + 1, q - 1)), \quad k = 0, 1, 2, \ldots, n.$$
Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two integer arrays of length $n$. We say that $x$ majorize $y$ and write $x \prec y$ if elements of these arrays satisfy following conditions:

(i) $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$,
(ii) $x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k$, for every $1 \leq k < n$,
(iii) $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$.

**Theorem 8.** Let $p$ and $q$ be the arrays of length $k \geq 2$, such that $p \prec q$. Then

$$T(p) \preceq_c T(q). \quad (2)$$

**Proof.** Let $n$ denotes the number of vertices in trees $T(p)$ and $T(q)$, $n = p_1 + p_2 + \cdots + p_m = q_1 + q_2 + \cdots + q_k$. We will proceed by mathematical induction on the size of the array $k$. For $k = 2$, we can directly apply transformation from Theorem 7 on tree $T(q)$ several times, in order to get $T(p)$.

Assume that the inequality $(2)$ holds for all lengths less than or equal to $k$. If there exists index $1 \leq m < k$ such that $p_1 + p_2 + \cdots + p_m = q_1 + q_2 + \cdots + q_m$, we can apply inductive hypothesis on two parts $T(q_1, q_2, \ldots, q_m)$ and $T(q_{m+1}, q_{m+2}, \ldots, q_k)$ and get $T(p_1, p_2, \ldots, p_m)$ and $T(p_{m+1}, p_{m+2}, \ldots, p_k)$.

Otherwise, we have strict inequalities $p_1 + p_2 + \cdots + p_m < q_1 + q_2 + \cdots + q_m$ for all indices $1 \leq m < k$. We can transform tree $T(q_1, q_2, \ldots, q_k)$ into

$$T(q_1, q_2, \ldots, q_{s-1}, q_s - 1, q_{s+1}, \ldots, q_r - 1, q_r + 1, q_{r+1}, \ldots, q_k),$$

where $s$ is the largest index such that $q_1 = q_2 = \cdots = q_s$ and $r$ is the smallest index such that $q_r = q_{r+1} = \cdots = q_k$. The condition $p \prec q$ is preserved, and we can continue until the array $q$ transforms into $p$, while at every step we decrease the Laplacian coefficients. \hfill $\square$

A canonical example of majorization is

**Corollary 9.** Let $T \not\cong BT_{n,k}$ be an arbitrary starlike tree with $k$ pendent paths on $n$ vertices. Then

$$c_k(BT_{n,k}) \leq c_k(T), \quad k = 0, 1, \ldots, n.$$  

The broom $B_{n,\Delta}$ is a tree consisting of a star $S_{\Delta+1}$ and a path of length $n - \Delta - 1$ attached to an arbitrary pendent vertex of the star (see Fig. 1). It is proven in [12] that among trees with perfect matching and maximum degree equal to $\Delta$, the broom $B_{n,\Delta}$ uniquely minimizes the largest eigenvalue of adjacency matrix. Also it is shown that among trees with bounded degree $\Delta$, the broom has minimal Wiener index and Laplacian-like energy [20]. In [23,24] the broom has minimal energy among trees with fixed diameter or fixed number of pendent vertices.

For the maximum case, we have following

**Corollary 10.** Let $T \not\cong B_{n,\Delta}$ be an arbitrary tree on $n$ vertices with the maximum vertex degree $\Delta$. Then

$$c_k(B_{n,\Delta}) \geq c_k(T), \quad k = 0, 1, \ldots, n.$$  

We can refine the above relation using Theorem 7 applied on the vertex of degree greater than 2

$$c_k(S_n) = c_k(B_{n,n-1}) \leq c_k(B_{n,n-2}) \leq \cdots \leq c_k(B_{n,3}) \leq c_k(B_{n,2}) = c_k(P_n).$$
for every $k = 0, 1, \ldots, n$. It follows that $B_{n,3}$ has the second largest Laplacian coefficients among trees on $n$ vertices.

3. Laplacian coefficients of trees with given diameter

Let $C(a_1, a_2, \ldots, a_{d-1})$ be a caterpillar obtained from a path $P_d$ with vertices $\{v_0, v_1, \ldots, v_d\}$ by attaching $a_i$ pendent edges to vertex $v_i$, $i = 1, 2, \ldots, d - 1$. Clearly, $C(a_1, a_2, \ldots, a_{d-1})$ has diameter $d$ and $n = d + 1 + \sum_{i=1}^{d-1} a_i$. For simplicity, denote $C_{n,d} = C(0, \ldots, 0, a_{d/2}, 0, \ldots, 0)$. In [18] it is shown that caterpillar $C_{n,d}$ has minimal spectral radius (the greatest eigenvalue of adjacency matrix) among graphs with fixed diameter (see Fig. 2).

**Theorem 11.** Among trees on $n$ vertices and diameter $d$, caterpillar $C_{n,d}$ has minimal Laplacian coefficient $c_k$, for every $k = 0, 1, \ldots, n$.

In [10], the authors also considered the connected $n$-vertex graphs with fixed radius, and proved that $C_{n,2r-1}$ is the extremal graph with minimal Laplacian coefficients.

Here, we give an alternative proof of Theorem 11. Let $P = v_0v_1v_2 \cdots v_d$ be a path in tree $T$ of maximal length. Every vertex $v_i$ on the path $P$ is a root of a tree $T_i$ with $a_i + 1$ vertices, that does not contain other vertices of $P$. We apply $\sigma$-transformation (or combination of transformations from Theorems 6 and 7) on trees $T_1, T_2, \ldots, T_{d-1}$ to decrease coefficients $c_k$, as long as we do not get a caterpillar $C(a_0, a_1, a_2, \ldots, a_d)$.

Let $1 \leq r \leq d - 1$ be the smallest index such that $a_r > 0$, and analogously let $1 \leq s \leq d - 1$ be the largest index such that $a_s > 0$. We can perform $\delta$ transformation to vertex $v_r$ or vertex $v_s$ and get a caterpillar with smaller Laplacian coefficients by moving pendent vertices to the central vertex $v_{\lfloor d/2 \rfloor}$ of a path. After applying this algorithm, we finally get the extremal tree $C_{n,d}$.

If $d < n - 1$, we can apply the transformation from Theorem 7 at the central vertex of degree greater than 2 and obtain $C_{n,d+1}$. Therefore, we have

$$c_k(S_n) = c_k(C_{n,2}) \leq c_k(C_{n,3}) \leq \cdots \leq c_k(C_{n,n-2}) \leq c_k(C_{n,n-1}) = c_k(P_n),$$

for every $k = 0, 1, \ldots, n$. It follows that $C_{n,3}$ has the second smallest Laplacian coefficients among trees on $n$ vertices.

Naturally, one wants to describe $n$-vertex trees with fixed diameter with maximal Laplacian coefficients. We have checked all trees up to 18 vertices and for every triple $(n, d, k)$ we found extremal trees with $n$ vertices and fixed diameter $d$ that maximize coefficient $c_k$. The outcome is interesting – the extremal trees are not isomorphic (see Fig. 3).

For $d = n - 2$, the maximum Laplacian coefficients are achieved for $B_{n,3}$. For $d = n - 3$, we have three potential extremal trees depicted on Fig. 4 (based on transformations from Theorem 6 and Theorem 7).

It is easy to prove that $T_3 \preceq T_2$. Namely, consider two marked edges of the subdivision trees – $a \in E(S(T_2))$ and $b \in E(S(T_3))$. Let $M$ be an arbitrary $k$-matching of $S(T_3)$. We will construct a corresponding $k$-matching $M'$ of $S(T_2)$ and prove that $c_k(T_3) \leq c_k(T_2)$. If $M$ does not contain $b$, then $M' = M$ is also $k$-matching of the tree $S(T_2) \setminus \{a\}$. If $M$ contains the edge $b$, then we set the edge $a$ in the corresponding matching $M'$ of $S(T_2)$. After removing $a$ and $b$ with their neighboring edges from $S(T_2)$ and $S(T_3)$ respectively, the decomposed graph of $S(T_3)$ is a subgraph of the decomposed graph of $S(T_2)$.
Fig. 3. Graphs with \( n = 18 \) and \( d = 4 \) that maximize \( c_{16} \) and \( c_{15} \).

Notice that the red edge \( c \in E(S(T_2)) \) does not belong to any corresponding matching \( M' \), and it follows that in this case we have an injection from the set of \( k \)-matchings of \( S(T_3) \) that contain the edge \( b \) to the set of \( k \)-matchings of \( S(T_2) \) that contains the edge \( a \). Finally, we have \( c_k(T_3) < c_k(T_2) \) for \( n > 7 \) and \( 2 \leq k \leq n - 2 \).

The Laplacian coefficient \( c_2(T) \) is equal to (see [4]):

\[
c_2(T) = 2n^2 - 5n + 3 - \frac{1}{2} \sum_{i=1}^{n} d_i^2 = 2n^2 - 5n + 3 - \frac{1}{2} Z(T),
\]

where \( Z(T) \) is the first Zagreb index [8]. Clearly, we have

\[
Z(T_2) - Z(T_1) = (3^2 + 1^2) - (2^2 + 2^2) = 2,
\]

which means that \( c_2(T_1) > c_2(T_2) \) for \( n > 7 \). On the other hand, for the Wiener index \( W(T) \) we have

\[
W(T_2) - W(T_1) = (2(1 + 2 + \cdots + (n - 4) + (n - 3) + 2) + n - 3) - (2(1 + 2 + 3 + \cdots + (n - 5) + (n - 4) + 2 + 3) + n - 1)
\]

\[
= 2n - 14,
\]

which gives \( c_{n-2}(T_2) > c_{n-2}(T_1) \) for \( n > 7 \). For \( n > 7 \), the pairs \((T_1, T_2)\) represent an infinite family of examples for Problem 1.

We can calculate the Laplacian coefficients of trees \( T_1 \) and \( T_2 \), by considering several cases involving red edges on Fig. 4. In [10], the authors proved that for \( 0 \leq k \leq \lceil \frac{n}{2} \rceil \), the number of matchings with \( k \) edges for path \( P_n \) is \( m_k(P_n) = m(n, k) = \binom{n-k}{k} \). After taking some independent red edges in \( k \)-matching, the decomposed graphs are the union of one long path and some number of paths with lengths 2, 3 or 4. Using MATHEMATICA software [22], we get

Fig. 4. Extremal trees for \( d = n - 3 \) (squares represent subdivision vertices).
After some manipulations, the difference \( c_k(T_2) - c_k(T_1) \) is equal to
\[
2 \cdot \frac{(2n - 9 - k)!}{(k - 2)!(2n - 2k - 3)!} \cdot P(n, k),
\]
and the sign of \( c_k(T_2) - c_k(T_1) \) depends only on the following expression
\[
P(n, k) = -408 - 7888k - 120k^2 - 13k^3 + 3k^4 + 844n + 639kn + 81k^2n - 4k^3n - 4662n - 186kn^2 - 6k^2n^2 + 104n^3 + 16kn^3 - 8n^4.
\]
For large \( n \), we can substitute \( x = \frac{k}{n} \) and get the fourth degree polynomial
\[
P(x) = 3x^4 - 4x^3 - 6x^2 + 16x - 8.
\]
This polynomial has only one positive real root \( x_0 \approx 0.771748 \), and therefore for the Laplacian coefficient \( c_k \), we have: \( c_k(T_1) > c_k(T_2) \) for \( k < nx_0 \), and \( c_k(T_2) > c_k(T_1) \) for \( k > nx_0 \).

### 4. Further examples for Mohar’s problems

We will use the series of \( \delta \) transformations, in order to obtain a chain of \( n \)-vertex trees of length \( m \),
\[
S_n \cong T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_{m-1} \leq T_m \cong P_n.
\]
The main idea is to move one pendent vertex attached at the center vertex of caterpillar \( C_{n,d} \) to the end vertex \( v_0 \). This requires \( \left\lfloor \frac{d}{2} \right\rfloor \) transformations to get caterpillar \( C_{n,d+1} \), and at every step we decrease all Laplacian coefficients. Starting from the star \( S_n \) and ending with the path \( P_n \), we have
\[
m = \left\lceil \frac{2}{2} \right\rceil + \left\lceil \frac{3}{2} \right\rceil + \left\lceil \frac{4}{2} \right\rceil + \cdots + \left\lceil \frac{n-2}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil.
\]
For \( n = 2k \) it follows \( m = (k - 1)^2 \), while for \( n = 2k + 1 \) it follows \( m = k(k - 1) \). Finally, we conclude that the length of the chain is equal to \( m = \left\lceil \frac{n-1}{2} \right\rceil \cdot \left\lceil \frac{n-2}{2} \right\rceil \), which is proportional to \( \frac{n^2}{4} \).
Table 1
Computational results of the ordering of small trees.

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Fig. 5. Trees \( T_1 \) and \( T_2 \).

In Table 1, we present for every \( n \) between 3 and 18, the number of trees on \( n \) vertices, the number of pairs of trees that give affirmative answer to Problem 1 and Problem 2, the number of all incomparable pairs and the percentage of \( n \)-vertex tree pairs that are incomparable.

We conclude that the smallest pair of trees that gives the affirmative answer for Problem 1 appears on 8 vertices, while the smallest pair of trees for Problem 2 appears on 10 vertices. In [25] the authors did not present the smallest example for Problem 1. Let \( T_1 \) and \( T_2 \) be two trees of order 8 depicted on the Fig. 5. It follows \( T_1 \prec_1 T_2 \) and \( T_2 \prec_2 T_1 \), since

\[
P(T_1, \lambda) = -8x + 65x^2 - 190x^3 + 267x^4 - 196x^5 + 75x^6 - 14x^7 + x^8
\]

and

\[
P(T_2, \lambda) = -8x + 66x^2 - 188x^3 + 259x^4 - 190x^5 + 74x^6 - 14x^7 + x^8.
\]

Notice that the percent of pairs of incomparable trees grows rapidly. It would be of interest to determine the limiting ratio when \( n \) tends to infinity.

5. Laplacian coefficients of trees with perfect matchings

It is well known that if a tree \( T \) has a perfect matching, then the perfect matching \( M \) is unique. Namely, a pendent vertex \( v \) has to be matched with its unique neighbor \( w \), and then \( M - \{vw\} \) forms the perfect matching of tree \( T - v - w \).

Let \( A_{n,\Delta} \) be a \( \Delta \)-starlike tree \( T(n - 2\Delta, 2, 2, \ldots, 2, 1) \) consisting of a central vertex \( v \), a pendent edge, a pendent path of length \( n - 2\Delta \) and \( \Delta - 2 \) pendent paths of length 2, all attached at \( v \) (see Fig. 6).

**Theorem 12.** The tree \( A_{n,\Delta} \) has minimal Laplacian coefficients among trees with perfect matching and maximum degree \( \Delta \).

**Proof.** Let \( T \) be an arbitrary tree with perfect matching and let \( v \) be a vertex of degree \( \Delta \), with neighbors \( v_1, v_2, \ldots, v_\Delta \). Let \( T_1, T_2, \ldots, T_\Delta \) be the maximal subtrees rooted at \( v_1, v_2, \ldots, v_\Delta \), respectively, such that
neither of these trees contains \( v \). Then at most one of the numbers \( |T_1|, |T_2|, \ldots, |T_A| \) can be odd (if \( T_i \) and \( T_j \) have odd number of vertices, than the root vertices \( v_i \) and \( v_j \) will be unmatched – which is impossible). Actually, since the number of vertices in \( T \) is even, there exists exactly one tree among \( T_1, T_2, \ldots, T_A \) with odd number of vertices.

Using Theorem 7, we may transform each \( T_i \) into a pendent path attached at \( v \) – while simultaneously decreasing all Laplacian coefficients and keeping the existence of a perfect matching. Assume that \( T_A \) has odd number of vertices, while the remaining trees have even number of vertices. We apply similar transformation to the one in Theorem 7, but instead of moving one edge, we move two edges in order to keep the existence of a perfect matching. Therefore, if \( p \geq q \geq 2 \) then

\[
c_k(G(p, q)) \leq c_k(G(p + 2, q - 2)),
\]

for all \( k = 0, 1, \ldots, n \). Using this transformation, we may reduce \( T_A \) to one vertex, the trees \( T_2, \ldots, T_{A-1} \) to two vertices, leaving \( T_1 \) with \( n - 2 \cdot A \) vertices, and thus obtaining \( A_{n,A} \). Since we have been decreasing all Laplacian coefficients simultaneously, we conclude that \( A_{n,A} \) indeed has minimal Laplacian coefficients \( c_k, k = 0, 1, \ldots, n \), among the trees with perfect matching. \( \square \)

If \( A > 2 \), we can again apply Theorem 7 (by moving two vertices) at the vertex of degree \( A \) in \( A_{n,A} \) and obtain \( A_{n,A-1} \). Thus, it follows that

\[
c_k(F_n) = c_k(A_{n,n/2}) \leq c_k(A_{n,n/2-1}) \leq \cdots \leq c_k(A_{n,3}) \leq c_k(A_{n,2}) = c_k(P_n),
\]

holds for every \( k = 0, 1, \ldots, n \).

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