The signless Laplacian spectral radius of bicyclic graphs with prescribed degree sequences

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Abstract

In this paper, we characterize all extremal connected bicyclic graphs with the largest signless Laplacian spectral radius in the set of all connected bicyclic graphs with prescribed degree sequences. Moreover, the signless Laplacian majorization theorem is proved to be true for connected bicyclic graphs. As corollaries, all extremal connected bicyclic graphs having the largest signless Laplacian spectral radius are obtained in the set of all connected bicyclic graphs of order \( n \) (resp. all connected bicyclic graphs with a specified number of pendant vertices, and all connected bicyclic graphs with given maximum degree). © 2011 Elsevier B.V. All rights reserved.

1. Introduction

In this paper, only connected simple graphs are considered. Let \( G = (V, E) \) be a connected simple graph with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E \). If \( |E| = |V| + c - 1 \), then \( G = (V, E) \) is called a \( c \)-cyclic graph, where \( c \) is a nonnegative integer. Especially, if \( c = 0 \) (resp. \( c = 1 \) and \( 2 \)), then \( G \) is called a tree (resp. unicyclic and bicyclic graphs). Let \( u, v \in V(G) \). If \( u \) is adjacent to \( v \) in \( G \), then \( v \) is called the neighbor of \( u \). Let \( N(u) \) denote the neighbor set of \( u \). Then \( d(u) = d_G(u) = |N(u)| \) is called the degree of \( u \) in \( G \). A non-increasing sequence of nonnegative integers \( \pi = (d_1, d_2, \ldots, d_n) \) is called graphic if there exists a graph \( G \) having \( \pi \) as its vertex degree sequence (\( \pi \) is also called the degree sequence of \( G \)). For all other definitions, notations and terminologies on the spectral graph theory, not given here, see e.g. [1,4,6].

Denote by \( D(G) = \text{diag}(d(u), u \in V) \) the diagonal matrix of vertex degrees of \( G \) and \( A(G) \) the \((0,1)\)-adjacency matrix of \( G \). If \( G \) is connected, then the matrix \( Q(G) = D(G) + A(G) \) is called the signless Laplacian matrix of \( G \) (also called the unoriented Laplacian matrix of \( G \) in [12,18]), which may be first introduced in book [4] without giving a name. The signless Laplacian matrices of graphs have received much attention in recent years (see e.g. [3,5,7–10,12,13,18]). The largest eigenvalue of \( Q(G) \) is called the signless Laplacian spectral radius of \( G \) and is denoted by \( \mu(G) \). Note that \( Q(G) \) is an irreducible nonnegative matrix, then there exists only one unit positive eigenvector \( f = (f(u), v \in V(G)) \) with \( \|f\| = 1 \) such that \( Q(G)f = \mu(G)f \). Such an eigenvector \( f \) is called the Perron vector of \( Q(G) \), and \( f(u) \) is called the \( \mu \)-weight of vertex \( u \) [15].

For a prescribed graphic degree sequence \( \pi \), let \( \mathcal{G}_\pi = \{ G \mid G \text{ is simple, connected and with } \pi \text{ as its degree sequence} \} \). Motivated by the Brualdi–Solheid problem [2], Zhang put forward the determination of graphs maximizing (or minimizing) the signless Laplacian spectral radius in the set \( \mathcal{G}_\pi \) [21]. It has been proved in [20,21] that for a given degree sequence \( \pi \) of trees (resp. unicyclic graphs), there exists a unique tree (resp. unicyclic graph) that has the largest signless Laplacian spectral radius in \( \mathcal{G}_\pi \).

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Let \( \mathcal{B}_n \) denote the class of all bicyclic graphs with bicyclic degree sequence \( \pi \). In this paper, we determine the unique maximal graph in \( \mathcal{B}_n \) with respect to the signless Laplacian spectral radius of \( G \in \mathcal{B}_n \).

Moreover, recall the notion of majorization. Let \( \pi = (d_1, \ldots, d_n) \) and \( \pi' = (d'_1, \ldots, d'_n) \) be two non-increasing sequences. If \( \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} d'_i \) for \( k = 1, 2, \ldots, n-1 \) and \( \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i \), then the sequence \( \pi \) is said to be majorized by the sequence \( \pi' \), which is denoted by \( \pi \prec \pi' \).

Let \( \pi \) and \( \pi' \) be two different tree (resp. unicyclic) degree sequences with the same order. Let \( G \) and \( G' \) be the trees (resp. unicyclic graphs) with the largest signless Laplacian spectral radius in \( \mathcal{B}_n \) and \( \mathcal{B}_n' \), respectively. In [20] (resp. [21]), the author proved that if \( \pi \prec \pi' \), then \( \mu(G) < \mu(G') \). Such a theorem is called the signless Laplacian majorization theorem for trees (resp. unicyclic graphs). In Section 3, the signless Laplacian majorization theorem is proved to be true for bicyclic graphs. As corollaries, all extremal bicyclic graphs having the largest signless Laplacian spectral radius are obtained in the set of all bicyclic graphs of order \( n \) (resp. all bicyclic graphs with a specified number of pendant vertices, all bicyclic graphs with given maximum degree).

2. Preliminaries

In this section, we introduce some definitions and lemmas which are useful in the presentations and proofs of our main results.

Let \( G \) be a graph with a root vertex \( v_1 \in V(G) \). Denote by \( \text{dist}(v, v_1) \) the distance between \( v \in V(G) \) and \( v_1 \). Besides, the distance \( \text{dist}(v, v_1) \) is called the height of vertex \( v \) in \( G \), denoted by \( h(v) = \text{dist}(v, v_1) \).

**Definition 2.1** ([21]). Let \( G = (V, E) \) be a graph with a root \( v_1 \in V(G) \). A well-ordering \( \prec \) of the vertices is called a breadth-first-search ordering (BFS-ordering for short) if the following hold for all vertices \( u, v \in V \):

1. \( u \prec v \) implies \( h(u) \leq h(v) \);
2. \( u \prec v \) implies \( d(u) \geq d(v) \);
3. suppose \( uv, xy \in E, uy, xv \notin E \) with \( h(u) = h(x) = h(v) = 1 = h(y) = 1 \). If \( u \prec x \), then \( v \prec y \).

We call a graph that has a BFS-ordering of its vertices a BFS-graph.

**Lemma 2.2** ([21]). Let \( G = (V, E) \) be a simple connected graph having the largest signless Laplacian spectral radius in \( \mathcal{B}_n \) and \( f \) be the Perron vector of \( Q(G) \). If \( V = \{v_1, \ldots, v_n\} \) satisfies \( f(v_i) \geq f(v_j) \) for \( i < j \) (i.e., the vertices of \( V \) are denoted with respect to \( f(v) \) in non-increasing order), then \( d(v_i) \geq d(v_j) \) for \( i < j \). Moreover, if \( f(v_1) = f(v_2) \), then \( d(v_1) = d(v_2) \).

**Corollary 2.3.** Let \( G = (V, E) \) be a simple connected graph having the largest signless Laplacian spectral radius in \( \mathcal{B}_n \) and \( f \) be the Perron vector of \( Q(G) \). If \( d(u) > d(v) \), then \( f(u) > f(v) \), where \( u, v \in V \).

**Proof.** Suppose \( f(u) \geq f(v) \). By Lemma 2.2, \( d(u) \geq d(v) \), a contradiction. \( \square \)

**Lemma 2.4** ([21]). Let \( G = (V, E) \) be a simple connected graph having the largest signless Laplacian spectral radius in \( \mathcal{B}_n \). Then \( G \) has a BFS-ordering consistent with the Perron vector \( f \) of \( Q(G) \) in such a way that \( u \prec v \) implies \( f(u) \geq f(v) \).

From the Perron–Frobenius Theorem of nonnegative matrices [15], we have

**Lemma 2.5** ([15]). If \( H \) is a proper connected subgraph of a connected simple graph \( G \), then \( \mu(H) < \mu(G) \).

Suppose that \( uv \in E(G) \) (resp. \( uv \notin E(G) \)). Denote by \( G - uv \) (resp. \( G + uv \)) the graph obtained from \( G \) by deleting (resp. adding) the edge \( uv \).

**Lemma 2.6** ([20]). Let \( G = (V, E) \in \mathcal{B}_n \) and \( f \) be a Perron vector of \( Q(G) \).

1. Suppose \( uv_1 \in E \) and \( uv_i \notin E \) for \( i = 1, \ldots, k \). Let \( G' = G + uv_1 + \cdots + uv_k - uv_1 - \cdots - uv_k \). If \( G' \) is connected and \( f(v) \geq f(u) \), then \( \mu(G') \geq \mu(G) \).
2. If \( d(u) > d(v) \) and \( f(u) \leq f(v) \), then there exists a connected graph \( G' \in \mathcal{B}_n' \) such that \( \mu(G') > \mu(G) \).

**Lemma 2.7** ([20]). Let \( G = (V, E) \in \mathcal{B}_n \) and \( f \) be a Perron vector of \( Q(G) \). Assume that \( v_1u_1, v_2u_2 \in E, \) and \( v_1v_2, u_1u_2 \notin E \). Let \( G' = G + v_1v_2 + u_1u_2 - v_1u_1 - v_2u_2 \). If \( G' \) is connected and \( f(v_1) - f(u_1) \geq 0 \), then \( \mu(G') \geq \mu(G) \). Moreover, the last equality holds if and only if \( f(v_1) = f(u_2) \) and \( f(v_2) = f(u_1) \).

**Corollary 2.8.** Let \( G = (V, E) \) be a simple connected graph that has the largest signless Laplacian spectral radius in \( \mathcal{B}_n \) and \( f \) be a Perron vector of \( Q(G) \). Assume that \( v_1u_1, v_2u_2 \in E, \) and \( v_1v_2, u_1u_2 \notin E \). Let \( G' = G + v_1v_2 + u_1u_2 - v_1u_1 - v_2u_2 \). Then the degree sequence of \( G' \) is \( \pi \). Moreover,

1. if \( G' \) is connected and \( f(v_1) > f(u_2) \), then \( f(v_2) < f(u_1) \); (1)
2. if \( G' \) is connected and \( f(v_1) = f(u_2) \), then \( f(v_2) = f(u_1) \). (2)
Proposition 3.2. Obviously, the degree sequence of $G'$ is $\pi$. Now suppose that $f(v_2) \geq f(u_1)$ in (1), and $f(v_2) \neq f(u_1)$ in (2). Since $(f(v_1) - f(u_2))f(v_2) - f(u_1)) \geq 0$, by Lemma 2.7, we both have $\mu(G') > \mu(G)$, a contradiction. $\square$

An internal path of $G$ is a path (or cycle) $P$ with vertices $v_1, \ldots, v_k$ such that $d(v_i), d(v_k) \geq 3$ and $d(v_2) = \cdots = d(v_{k-1}) = 2$, where $k > 1$.

Lemma 2.9 ([14]). Let $G$ be a simple connected graph and $uv$ be an edge on an internal path of $G$. If $G_{uv}$ is obtained from $G$ by the subdivision of the edge $uv$ into the edges $uv$ and $wv$, then $\mu(G_{uv}) < \mu(G)$.

Let $G$ be a simple connected nontrivial graph and $u \in V(G)$. Let $G(k, l)$ be the graph obtained from $G$ by attaching two paths $u u_1 u_2 \cdots u_k$ and $u_1 v_2 \cdots v_l$ at vertex $u$ respectively, where $k, l \geq 1$.

Lemma 2.10 ([13]). If $k \geq l \geq 2$, then $\mu(G(k, l)) > \mu(G(k + 1, l - 1))$.

Lemma 2.11 ([11, 19]). Let $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ be two non-increasing graphic sequences. If $\pi < \pi'$, then there exist a series of graphic degree sequences $\pi_1, \ldots, \pi_k$ such that $(\pi =) \pi_0 < \pi_1 < \cdots < \pi_k < \pi_{k+1} (= \pi')$, and only two components of $\pi_i$ and $\pi_{i+1}$ are different from 1, where $i = 0, 1, \ldots, k$.

Lemma 2.12 ([16]). Let $m(v) = \sum_{u \in N(v)} d(u)/d(v)$. Then

$$\mu(G) \leq \max_{u \in E(G)} \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} \right\}.$$ 

Lemma 2.13 ([17]). Let $G$ be a connected graph with at least one edge. Then

$$\mu(G) \geq \lambda(G) \geq 1,$$

where $\lambda(G)$ is the Laplacian spectral radius of $G$, and $\Delta$ is the maximum degree of $G$. The first equality holds if and only if $G$ is bipartite, and the second equality holds if and only if $\Delta = n - 1$.

3. Main results

To begin with, we give a characterization of the degree sequences of connected bicyclic graphs as follows.

Proposition 3.1 ([1, 11]). Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence. Then $\pi$ is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min[k, d_i], \text{ where } 1 \leq k \leq n.$$

(1)

Proposition 3.2. Let $\pi = (d_1, d_2, \ldots, d_n)$ be a positive non-increasing integer sequence with even sum and $n \geq 4$. If $\pi$ is connected bicyclic graph (also called a bicyclic degree sequence), then $\sum_{i=1}^{n} d_i = 2n + 2$ and (1) holds.

Let $\theta(p, q, r)$ denote the $\theta$-graph, which is obtained from three vertex-disjoint paths (of orders $p + 1, q + 1$ and $r + 1$ respectively, where $p, q, r > 0$ and at most one of $p, q, r$ equals 1) by identifying the three initial (resp. terminal) vertices of them. Let $C(n_1, n_2)$ denote the graph obtained from two cycles $C_{n_1}$ and $C_{n_2}$ (of orders $n_1$ and $n_2$ respectively, where $n_1, n_2 \geq 1$) by identifying a vertex of $C_{n_1}$ with a vertex of $C_{n_2}$.

For a prescribed non-increasing bicyclic degree sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $n \geq 4$, by Proposition 3.2, $\pi$ should be one of the following four cases. Moreover, we construct a special bicyclic graph $B_\pi^n$ with degree sequence $\pi$ as follows.

(1) $d_n = 2$ and $d_1 = 4$. Then $d_i = 2 (2 \leq i \leq n - 1)$. Let $B_\pi^n = C(n, n - 2)$.

(2) $d_n = 2$ and $d_3 = 3$. Then $d_2 = 3$ and $d_i = 2 (3 \leq i \leq n - 1)$ by Proposition 3.2. Let $B_\pi^n = \theta(1, 2, n - 2)$.

(3) $d_n = 1$ and $d_2 = 2$. Then we have $d_i > 4, d_i = 2 (i = 3, 4, 5)$, and $d_i < 2 (6 \leq i \leq n - 1)$. Let $B_\pi^n = \theta(2, 4, n - 2)$.

(4) $d_n = 1$ and $d_2 = 3$. Then $d_1 > d_2 \geq 3$, and we use the breadth-first-search method to define a special bicyclic graph $B_\pi^n$ with degree sequence $\pi$ as follows. Select a vertex $v_0$ as a root and begin with $v_0$ of the zeroth layer. Put $s_1 = d_1$ and select $s_1$ vertices $\{v_1, \ldots, v_{s_1}\}$ of the first layer such that they are adjacent to $v_0$, and $v_{s_1}$ is adjacent to $v_1$ and $v_{s_1}$. Thus $d(v_0) = s_1 = d_1$. Next we construct the second layer. Put $d(v_0) = d_i + 1 (i = 1, \ldots, s_1)$ and select $s_2$ vertices $\{v_2, \ldots, v_{s_2}\}$ of the second layer such that $d(v_1) - 3$ vertices are adjacent to $v_1$, $d(v_2) - 2$ vertices are adjacent to $v_2$, and $d(v_3) - 1$ vertices are adjacent to $v_3$. In general, assume that all vertices of the $t$th $(t \geq 2)$ layer have been constructed and are denoted by $\{v_{t}, \ldots, v_{t,s_t}\}$. Now using the induction
hypothesis, we construct all the vertices of the \((t + 1)\)st layer. Put \(d(v_i) = d_{i+1} + \sum_{j=1}^{t-1} y_j\) (\(i = 1, \ldots, s_t\)) and select \(s_{t+1} = \sum_{i=1}^{s_t} d(v_i) - s_t\) vertices \(\{v_{t+1,1}, \ldots, v_{t+1,s_{t+1}}\}\) of the \((t + 1)\)st layer such that \(d(v_i) - 1\) vertices are adjacent to \(v_i\) for \(i = 1, \ldots, s_t\). In this way, we obtain only one connected bicyclic graph with degree sequence \(\pi\).

For example, for a given bicyclic degree sequence \(\pi = (6, 5, 4, 3, 2, 1^{10})\), \(B^*_\pi\) is the bicyclic graph of order 13 shown in Fig. 1.

It can be seen that for a prescribed bicyclic degree sequence \(\pi\), the graph \(B^*_\pi\) constructed above has a BFS-ordering. Now we show that \(B^*_\pi\) is the only connected bicyclic graph that has the largest signless Laplacian spectral radius in \(\mathcal{B}_\pi\).

**Lemma 3.3.** Let \(\pi = (d_1, d_2, \ldots, d_n)\) be a non-increasing bicyclic degree sequence with \(d_n = 2\). Then \(B^*_\pi\) is the only connected bicyclic graph that has the largest signless Laplacian spectral radius in \(\mathcal{B}_\pi\).

**Proof.** Let \(G = (V, E)\) be a connected graph that has the largest signless Laplacian spectral radius in \(\mathcal{B}_\pi\), and \(f\) be the Perron vector of \(Q(G)\). By **Lemma 2.4**, there exists a BFS-ordering of \(G\) with root \(v_1\) such that

\[
\begin{align*}
 v_1 &< v_2 < \cdots < v_n, \\
 f(v_1) &\geq f(v_2) \geq \cdots \geq f(v_n), \\
 h(v_1) &\leq h(v_2) \leq \cdots \leq h(v_n), \quad \text{and} \quad d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n).
\end{align*}
\]

Let \(V_i = \{v \in V \mid h(v) = i\}\) for \(i = 0, 1, \ldots, h(v_{\pi}) (=p)\). Hence we can relabel the vertices of \(G\) in such a way that \(V_i = \{v_1, \ldots, v_{s_i}\}\) with \(f(v_i) \geq f(v_{i+1}) \geq \cdots \geq f(v_{s_i})\) and \(d(v_i) \geq d(v_{i+1}) \geq \cdots \geq d(v_{s_i})\) for \(0 \leq i < r \leq p, 1 \leq i \leq j \leq s_i\) and \(1 \leq k \leq s_i\). Clearly, \(s_1 = d(v_1) = d(v_0) = d_1\).

On the other hand, note that \(d_n = 2\) and \(\sum_{i=1}^{n} d_i = 2n + 2\) by **Proposition 3.2**. Then we need to consider the following two cases.

**Case 1.** \(\pi = (4, 2, \ldots, 2)\), where the frequency of \(2\) in \(\pi\) is \(n - 1\). Hence \(d(v_0) = 4\) and \(d(v_{ij}) = 2\) for \(1 \leq i \leq p\) and \(1 \leq j \leq s_i\).

If \(v_{ij}v_{ij} \notin E\) for all \(1 \leq i < j \leq 4\), it follows from \(\pi = (4, 2, \ldots, 2)\) that we can suppose there exist two cycles \((v_{01}, v_{11}, v_{12}, v_{13}, v_{14}, v_{10})\) and \((v_{01}, v_{13}, u_1, \ldots, u_{n_2}, v_{14}, v_{10})\) without loss of generality, where \(n_1, n_2 \geq 1\) and \(n_1 + n_2 + 5 = n\). Since \(d(v_{01}) > d(v_{11})\), by **Corollary 2.3**, we have \(f(v_{01}) > f(v_{11})\). It follows from **Corollary 2.8** that \(f(v_{13}) > f(w_{1})\). Let \(G_1 = G - v_{11}w_{1} - v_{11}u_{1} + v_{11}v_{13} + w_{1}u_{1}\). Note that \(G_1 \in \mathcal{B}_\pi, f(v_{11}) \geq f(u_1)\) (\(u_1 \in V_2\)) and \(f(v_{13}) > f(w_1)\), then by **Lemma 2.7**, we have \(\mu(G_1) > \mu(G)\), a contradiction.

If there exists an edge \(v_{ij}v_{ij} \in E\) (\(1 \leq i < j \leq 4\)), according to the degree sequence \(\pi = (4, 2, \ldots, 2)\), then \(G\) is isomorphic to \(C(3, n - 2)\), that is, \(B^*_\pi\).

**Case 2.** \(\pi = (3, 3, 2, \ldots, 2)\), where the frequency of \(2\) in \(\pi\) is \(n - 2\). Thus \(d(v_0) = d(v_{11}) = 3\) and \(d(v_{12}) = d(v_{13}) = d(v_{ij}) = 2\) for \(1 \leq i \leq p\) and \(1 \leq j \leq s_i\).

If \(v_{11}v_{12} \in E\) (or \(v_{11}v_{13} \in E\)), then it follows from \(\pi = (3, 3, 2, \ldots, 2)\) that \(G\) is isomorphic to \(\theta(1, 2, n - 2)\), namely, \(G \cong B^*_\pi\).

If \(v_{11}v_{12} \notin E\), according to the degree sequence \(\pi = (3, 3, 2, \ldots, 2)\), we consider the following two subcases.

**Subcase 2.1.** There exist two cycles \((v_{01}, v_{12}, u_1, \ldots, u_{n_1}, v_{13}, v_{10})\) and \((v_{01}, v_{12}, u_1, \ldots, u_{n_2}, v_{12}, v_{11})\), where \(n_1, n_2 \geq 0\) and \(n_1 + n_2 + 2 = n\). Since \(d(v_{11}) > d(w_{1})\), by **Corollary 2.3**, we have \(f(v_{11}) > f(w_{1})\). Note that \(w_1v_{12}, v_{11}v_{12} \notin E\). Let \(G_1 = G - v_{11}w_{1} - v_{12}w_{1} + v_{11}v_{12} + v_{23}w_{1}\). Then \(G_1 \in \mathcal{B}_\pi\). Since \(f(v_{11}) > f(w_{1})\) and \(f(v_{12}) \geq f(v_{23})\), by **Lemma 2.7**, we get \(\mu(G_1) > \mu(G)\). This contradicts \(G\) having the largest signless Laplacian spectral radius in \(\mathcal{B}_\pi\).

**Subcase 2.2.** There exist two cycles \((v_{01}, u_{11}, u_1, \ldots, u_{n_1}, v_{12}, v_{13}, v_{10})\) and \((v_{01}, u_{11}, u_1, \ldots, u_{n_2}, v_{13}, v_{10})\), where \(n_1, n_2 \geq 1\) and \(n_1 + n_2 + 2 = n\). Since \(d(v_{11}) > d(u_{n_2})\), it follows from **Corollary 2.3** that \(f(v_{11}) > f(u_{n_2})\). Note that \(w_{11}u_{n_2}, v_{11}v_{13} \notin E\). Let \(G_1 = G - v_{11}w_{1} - v_{13}u_{n_2} + v_{13}v_{13} + u_{n_2}w_{1}\). Then \(G_1 \in \mathcal{B}_\pi\). Since \(f(v_{11}) > f(u_{n_2})\) and \(f(v_{13}) \geq f(w_{1})\) (\(w_{1} \in V_2\)), then by **Lemma 2.7**, \(\mu(G_1) > \mu(G)\), which is a contradiction.

All in all, we conclude that \(G\) is isomorphic to \(B^*_\pi\). □

**Lemma 3.4.** Let \(\pi = (d_1, d_2, \ldots, d_n)\) be a non-increasing bicyclic degree sequence with \(d_n = 1\) and \(d_2 = 2\). Then \(B^*_\pi\) is the only connected bicyclic graph that has the largest signless Laplacian spectral radius in \(\mathcal{B}_\pi\).

**Proof.** Let \(H\) be a connected bicyclic graph that has the largest signless Laplacian spectral radius in \(\mathcal{B}_\pi\). By **Proposition 3.2**, we have \(d_1 > 4\) and \(d_2 \leq 2\) (\(2 \leq i \leq n\)). Then \(H\) must be the graph of order \(n\) obtained from \(C(n_1, n_2)\) and \(d_1 - 4\) paths \(P_i\) of order \(p_i\), for \(i = 1, 2, \ldots, d_1 - 4\) by identifying the maximum degree vertex of \(C(n_1, n_2)\) with one end vertex of each \(P_i\).
Lemma 2.9. Let $G$ be a graph of order $n - 1$ obtained from $H$ by the contraction of an edge of $C_n$ (or $C_{n+1}$) in $H$. By Lemma 2.9, we have $\mu(H_1) > \mu(H)$. Let $H_2$ be a graph of order $n$ obtained from $H_1$ by adding a new vertex $v$ and adding a new edge incident to $v$ and a pendant vertex of $H_1$. It follows from Lemma 2.5 that $\mu(H_2) > \mu(H_1)$, which implies $\mu(H_2) > \mu(H)$. Note that $H_2 \notin \mathcal{B}_\pi$, a contradiction. Hence $n_1 = n_2 = 3$.

Claim 2. If $p_1 - p_2 \leq 1$ for all $i, j \leq d_1 - 4$. If there exist $s, t$ such that $p_s - p_t \geq 2$, then $H$ can be expressed as $G(p_s, p_t)$. Hence by Lemma 2.10, $\mu(H) = \mu(G(p_s, p_t)) < \mu(G(p_s - 1, p_t + 1))$ and $G(p_s - 1, p_t + 1) \notin \mathcal{B}_\pi$, a contradiction. Hence the $d_1 - 4$ paths have almost equal lengths. Consequently, $H$ is isomorphic to $B_{\pi}^*$. □

Let $uv$ be a cut edge of $G = (V, E)$. If one component of $G - uv$ is a tree $T$ (suppose $u \in V(T)$), then the induced subgraph $G[V(T) \cup \{v\}]$ is called a hanging tree on vertex $v$.

Lemma 3.5. Let $G = (V, E)$ be a connected graph with pendant vertices that has the largest signless Laplacian spectral radius in $\mathcal{B}_\pi$. Let $P = w_0w_1 \cdots w_kw_{k+1}$ $(k \geq 0)$ be a path with $d(w_0) > d(w_{k+1}) = 1$ and $u_1u_2 \in E$ be an edge of a cycle. If $u_1w_1$ $(0 \leq j \leq k)$ and $u_1w_1, u_2w_i \notin E$ for $j < t \leq k + 1$, then $f(u_2) > f(u_j) > f(w_{j+1})$.

Moreover, let $T$ be a hanging tree on a vertex $v$ and $u_1u_2 \in E$ $(u_1, u_2 \neq v)$ be an edge of a cycle. If $u_1v \notin E$, then $f(u_2) > f(v)$.

Proof. Note that $d(w_{k+1}) = 1 < d(w_j)$, where $0 \leq j \leq k$. By Corollary 2.3, we have $f(w_{k+1}) < f(w_j)$. Now we show that $f(u_2) > f(u_j)$.

By contradiction, suppose $f(u_2) < f(u_j)$. Let $G' = G - w_jw_{j+1} - u_1u_2 + u_1w_j + u_2w_{j+1}$, then $G' \notin \mathcal{B}_\pi$. If $f(w_{j+2}) < f(u_2)$, then by Lemma 2.7, we have $\mu(G') > \mu(G)$, a contradiction. Hence $f(w_{j+2}) > f(u_2)$. Let $G'' = G - w_{j+1}w_{j+2} - u_1u_2 + u_1w_{j+2} + u_2w_{j+1}$, then we get $f(w_{j+2}) > f(u_2)$ by Lemma 2.7 and the fact that $G'' \notin \mathcal{B}_\pi$. By repeating similar discussion as above, we arrive at $f(w_{k+1}) = \min\{f(u_1), f(u_2)\}$. It follows from Corollary 2.3 that $f(u_2) > f(u_j) > f(w_{j+1}) \geq 1$. Note that $u_1v \notin E$. Hence by the previous discussion, we immediately get $f(u_2) > f(v)$. □

Corollary 3.6. Let $G = (V, E)$ be a connected graph with pendant vertices that has the largest signless Laplacian spectral radius in $\mathcal{B}_\pi$. Then the vertex of $G$ having the largest $\mu$-weight lies on a cycle.

Proof. Let $v_1$ be the vertex of $G$ having the largest $\mu$-weight. By contradiction, suppose $v_1$ does not lie on any cycles. By Lemma 2.2, we have $d(v_1) \geq 3$, and then there exists a hanging tree on $v_1$. Note that there is an edge $u_1u_2 \in E$ of a cycle such that $u_1v_1 \notin E$. Hence by Lemma 3.5, $f(u_2) > f(v_1)$, a contradiction. □

Lemma 3.7. Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing bicyclic degree sequence with $d_1 \geq d_2 \geq 3$ and $d_n = 1$. Then $B_{\pi}^*$ is the only connected bicyclic graph that has the largest signless Laplacian spectral radius in $\mathcal{B}_\pi$.

Proof. Let $G = (V, E)$ be a connected graph that has the largest signless Laplacian spectral radius in $\mathcal{B}_\pi$, and $f$ be the Perron vector of $Q(G)$. By Lemma 2.4, there exists a BFS-ordering of $G$ with root $v_i$ such that

$$v_1 < v_2 < \cdots < v_n,$$

$$f(v_1) \geq f(v_2) \geq \cdots \geq f(v_n),$$

$$h(v_1) \leq h(v_2) \leq \cdots \leq h(v_n),$$

and $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$.

Let $V_i = \{v \in V \mid h(v) = i\}$ for $i = 0, 1, \ldots, h(v_n) (=p)$. Hence we can relabel the vertices of $G$ in such a way that $V_i = \{v_1, \ldots, v_{s_i}\}$ with $f(v_0) \geq f(v_{s_i}) = f(v_{s_0})$. Let $d(v_0) \geq d(v_{s_i}) \geq d(v_{s_0})$ for $0 \leq i < r \leq p$, $1 \leq l < j \leq s_i$, and $1 \leq k \leq s_r$. Clearly, $s_1 = d(v_1) = d(v_{s_0}) = d_1 \geq d(v_1) \geq 3$, and $d(v_{s_0}) \geq d(v_{s_1}) \geq 2$.

Since $G$ is a connected bicyclic graph, let $C_1, C_2$ be two cycles of $G$ (and $C_2$ may have some common vertices) and $P$ (if exists) be the unique path joining $C_1$ and $C_2$. By Corollary 3.6, assume that $v_01 \in V(C_1)$ without loss of generality.

Claim 1. $|V(C_1) \cap V(C_2)| \geq 2$. By contradiction, we have the following two cases.

Case 1. $|V(C_1) \cap V(C_2)| = 0$.

If there exists a hanging tree on $v_01$, we can take $u_1u_2 \in E(C_2)$ such that $u_1v_01 \notin E$. By Lemma 3.5, we get that $f(u_2) > f(v_01)$, a contradiction.

If there exists a hanging tree on $v_01$, observe that there is an edge $u_1u_2$ of a cycle such that $u_1, u_2 \neq v_0$ and $u_1v_01 \notin E$, then by Lemma 3.5, we have $f(u_2) > f(v_01)$, which is a contradiction.

Otherwise, there exist no hanging trees on $v_01$ and $v_1$. Hence $v_01 \in V(C_1) \cap V(P), v_1 \in V(C_2) \cap V(P)$, and $d(v_01) = d(v_01) = d(v_{s_0}) = d(v_{s_0}) = 3$ (since $d_n = 1$). Thus there is a hanging tree on $v_{s_0}$. Without loss of generality, suppose $v_02 \notin V(C_0)$. Then there is an edge $u_1u_2 \in E(C)$ such that $u_1, u_2 \neq v_1$ and $u_1v_12 \notin E$. By Lemma 3.5, we have $f(u_2) > f(v_12)$, also a contradiction.

Case 2. $|V(C_1) \cap V(C_2)| = 1$. If $V(C_1) \cap V(C_2) = \{v_01\}$ (or $V(C_1) \cap V(C_2) \neq \{v_01\}$), then there is a hanging tree on $v_01$ (or $v_01$), and an edge $u_1u_2 \in E(C)$ such that $u_1, u_2 \neq v_0$ and $u_1v_01 \notin E$ (or $u_1v_01 \notin E$). By Lemma 3.5, $f(u_2) > f(v_11)$ or $f(u_2) > f(v_01)$, a contradiction.
From Claim 1, we conclude that $G$ contains a $\theta$-graph $\theta(p, q, r)$ as its induced subgraph. Let $x, y$ be the vertices such that $d_{\theta(p,q,r)}(x) = d_{\theta(p,q,r)}(y) = 3$. If $v_0 \notin \{x, y\}$ (or $v_1 \notin \{x, y\}$), then there is a hanging tree on $v_0$ (or $v_1$), and an edge $u_1u_2 \in E(C_1)$ such that $u_1, u_2 \notin v_0$ and $u_1u_0 \notin E$ (or $u_1u_1 \notin E$). By Lemma 3.5, $f(u_2) > f(v_0)$ (or $f(u_2) > f(v_1)$), which is a contradiction. Hence $\{v_0, v_1\} = \{x, y\}$.

Claim 2. $|V(C_1)| = |V(C_2)| = 3$. By contradiction, suppose $|V(C_i)| > 3$ ($i = 1$ or 2). Note that there exists a hanging tree on $v_0$ (or on $v_1$), or a hanging tree on $v_{12}$ since $d_2 = 1$. Take an edge $u_1u_2 \in E(C_i)$ such that $u_1, u_2 \notin v_0, u_2 \notin v_1$, and $u_1u_0 \notin E$ (or $u_1u_1 \notin E$, or $u_1u_1 \notin E$). By Lemma 3.5, $f(u_2) > f(v_0)$ (or $f(u_2) > f(v_1)$, or $f(u_2) > f(v_{12})$), a contradiction.

It follows from Claim 2 that $G$ contains $\theta(1, 2, 2)$ as its induced subgraph. Finally, we need to show that $u_{12}, u_{13} \in \theta(1, 2)$ by contradiction. Suppose $u_0 \notin \theta(1, 2, 2)$ ($h = 2$ or 3). Then there is a hanging tree on $v_1$. Take an edge $u_{12} \in \theta(1, 2, 2)$ such that $u_2 \notin v_0, v_1$ and $u_1u_{1h} \notin E$. Then by Lemma 3.5, $f(u_2) > f(v_{1h})$, which is a contradiction.

Therefore, $v_{11}, v_{12}, v_{13} \in E$. Combining this with the BFS-ordering of $G$ given above, we obtain that $G \cong B_n^*$. □

The main result of this section follows from Lemmas 3.3, 3.4 and 3.7.

**Theorem 3.8.** Let $\pi$ be a bicyclic degree sequence. Then $B_n^*$ is the only connected bicyclic graph that has the largest signless Laplacian spectral radius in $\mathcal{B}_n$.

Now the signless Laplacian majorization theorem is proved to be true for bicyclic graphs in the following theorem.

**Theorem 3.9.** Let $\pi$ and $\pi'$ be two different non-increasing bicyclic degree sequences with the same order. If $\pi < \pi'$, then $\mu(B_n^*) < \mu(B_n'^*)$.

**Proof.** By Lemma 2.11, without loss of generality, assume that $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ with $d_i = d'_i$ ($i \neq p, q$), and $d_p = d'_p - 1$, $d_q = d'_q + 1$ ($1 \leq p < q \leq n$). Let $f$ be the Perron vector of $Q(B_n^*)$. By Lemma 2.4, there exists a BFS-ordering of $B_n^*$ with root $v_1$ such that

$$v_1 < v_2 < \cdots < v_n,$$

$$f(v_1) \geq f(v_2) \geq \cdots \geq f(v_n),$$

$$h(v_1) \leq h(v_2) \leq \cdots \leq h(v_n), \quad \text{and} \quad d(v_i) = d_i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

If $\pi = (4, 2, \ldots, 2)$, then $\pi' = (5, 2, \ldots, 2, 1)$ or $(4, 3, 2, \ldots, 2, 1)$; if $\pi = (3, 3, 2, \ldots, 2)$, then $\pi' = (3, 3, 3, 2, \ldots, 2, 1)$ or $(4, 3, 2, \ldots, 2, 1)$ or $(4, 2, \ldots, 2)$. By Theorem 3.8, it can be directly checked that there exists a vertex $v_k$ ($k \neq p, q$) such that $v_kv_k \in E(B_n^*)$ and $v_kv_k \notin E(B_n'^*)$. Let $B_1 = B_n^* + v_kv_k - v_kv_k$. Then by Lemma 2.6, $\mu(B_n^*) < \mu(B_1)$. Since $B_1 \in \mathcal{B}_n'$, it follows from Theorem 3.8 that $\mu(B_n^*) < \mu(B_1) = \mu(B_n'^*)$.

In the following, we may assume that $\pi \neq (4, 2, \ldots, 2)$ and $(3, 3, 2, \ldots, 2)$. Note that $d'_q \geq \begin{cases} 1, & \text{if} 5 \leq q \leq n, \\ 2, & \text{if} \ 2 \leq q \leq 4. \end{cases}$

Case 1. $d_q \geq \begin{cases} 1, & \text{if}\ 6 \leq q \leq n, \\ 2, & \text{if} \ 2 \leq q \leq 4. \end{cases}$

By Theorem 3.8, there exists a vertex $v_k$ with $k > q$ such that $v_kv_k \in E(B_n^*)$ and $v_kv_k \notin E(B_n'^*)$. Let $B_1 = B_n^* + v_kv_k - v_kv_k$. Then by Lemma 2.6, $\mu(B_n^*) < \mu(B_1)$. Since $B_1 \in \mathcal{B}_n'$, by Theorem 3.8, we have $\mu(B_n'^*) < \mu(B_1) = \mu(B_n'^*)$.

Case 2. $q = 2$ and $d_2 = 3$. Then $d_q = 2$. 

Subcase 2.1. $d_q = 4$. Then $d_1 = 3$ and $n \geq 5$. Hence $\pi' = (4, 2, \ldots, 2)$, and then $\pi = (3, 3, 2, \ldots, 2)$. Such a subcase has been proved.

Subcase 2.2. $d_q > 4$. Suppose that $B_n^*$ has $k$ pendant vertices. It follows that $\pi' = (k + 4, 2, \ldots, 2, 1, \ldots, 1)$, and then $\pi = (k, 3, 2, \ldots, 2, 2, 1, \ldots, 1)$. By Theorem 3.8 and Lemmas 2.12 and 2.13, we obtain that

$$\mu(B_n'^*) \leq \max_{u \in E(B_n')} \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} \right\}$$

$$\leq k + 5 = \Delta(B_n'^*) + 1 \leq \mu(B_n'^*).$$

Case 3. $q = 5$ and $d_3 = 2$. Then $d_q = 1$.

Subcase 3.1. $d_4 = 1$ and $d_2 = 2$. If $p = 1$, then $d_1 > d_1 > 4$ and $d_1 = d_2 = 2$ ($2 \leq i \leq 4$), but there does not exist such graph with degree sequence $\pi'$ by Proposition 3.2. Hence $p = 2$, and then $d'_2 = d_2 + 1 = 3$. By Theorem 3.8, $v_qv_q \in E(B_n^*)$ but $v_kv_k \notin E(B_n^*)$. Let $B_1 = B_n^* + v_qv_q - v_kv_q$. Then by Lemma 2.6, $\mu(B_n^*) < \mu(B_1)$. Since $B_1 \in \mathcal{B}_n'$, by Theorem 3.8, $\mu(B_n'^*) < \mu(B_1) = \mu(B_n'^*)$.

Subcase 3.2. $d_n = 1$ and $d_3 \geq 3$. By Theorem 3.8, there exists a vertex $u$ such that $u \in E(B_n^*)$ but $u \notin E(B_n'^*)$. Let $B_1 = B_n^* + u_kv_q - v_kv_q$. Then by Lemma 2.6, $\mu(B_n^*) < \mu(B_1)$. Since $B_1 \in \mathcal{B}_n'$, by Theorem 3.8, $\mu(B_n'^*) < \mu(B_1) = \mu(B_n'^*)$. □

Let $\mathcal{B}_n(s)$ denote the set of all connected bicyclic graphs of order $n$. Denote by $\mathcal{B}_n(s)$ the set of all connected bicyclic graphs of order $n$ with $s$ pendant vertices, and $\mathcal{B}_n(\Delta)$ the set of all connected bicyclic graphs of order $n$ with maximum degree $\Delta$, respectively.
Corollary 3.10. Let $G \in \mathcal{B}_n$, where $n \geq 4$. Then $\mu(G) \leq \mu(B_{n\pi_1}^*)$ with equality if and only if $G$ is isomorphic to $B_{n\pi_1}^*$, where $\pi_1 = (n - 1, 3, 2, 2, 1, \ldots, 1)$, where the frequency of 1 in $\pi_1$ is $n - 4$. (In other words, $B_{n\pi_1}^*$ is obtained from $\theta(1, 2, 2)$ and $(n - 4)$ $K_2$ by identifying one of the maximum degree vertices of $\theta(1, 2, 2)$ with one vertex of each $K_2$.)

Proof. Let $G \in \mathcal{B}_n$ with the non-increasing degree sequence $\pi$. If $\pi = \pi_1$, then by Theorem 3.8, $\mu(G) \leq \mu(B_{n\pi_1}^*)$ with equality if and only if $G$ is isomorphic to $B_{n\pi_1}^*$. If $\pi \neq \pi_1$, it is easy to see that $\pi$, $\pi_1$ are two different bicyclic degree sequences, and $\pi \prec \pi_1$. By Theorems 3.8 and 3.9, we have

$$\mu(G) \leq \mu(B_{n\pi_1}^*) \leq \mu(B_{n\pi_1}^*)$$

with equality if and only if $G$ is isomorphic to $B_{n\pi_1}^*$. □

Corollary 3.11. Let $G \in \mathcal{B}_n(s)$, where $n \geq 4$.

(1) If $n = 4$, then $G \cong \theta(1, 2, 2)$.
(2) If $s = 0$ and $n \geq 5$, then $\mu(G) \leq \mu(C(C, n - 3))$ with equality if and only if $G \cong C(C, n - 2)$.
(3) If $n = 5$ and $s \geq 1$, then $\mu(G) \leq \mu(G^*)$ with equality if and only if $G$ is isomorphic to $G^*$, where $G^*$ is obtained from $\theta(1, 2, 2)$ and $K_2$ by identifying one of the maximum degree vertices of $\theta(1, 2, 2)$ with one vertex of each $K_2$.
(4) If $1 \leq s \leq n - 5$, then $\mu(G) \leq \mu(B_{n\pi_2}^*)$ with equality if and only if $G$ is isomorphic to $B_{n\pi_2}^*$ with $\pi_2 = (s + 4, 2, \ldots, 2, 1, \ldots, 1)$, where the frequency of 2 in $\pi_2$ is $n - s - 1$, and the frequency of 1 in $\pi_2$ is $s$. (In other words, suppose $n - 5 = sq + t$, $0 \leq t < s$, and $B_{n\pi_2}^*$ is obtained from $C(C, 3)$, $t$ paths of order $q + 2$ and $s - t$ paths of order $q + 1$ by identifying the maximum degree vertex of $C(3, 3)$ with one end of each path of the $s$ paths.)

Proof. The result of (1) is obvious. Now let $G \in \mathcal{B}_n(s)$ with the non-increasing degree sequence $\pi$. If $s = 0$ and $n \geq 5$, then $\pi = (4, 2, \ldots, 2)$ or $(3, 3, 2, \ldots, 2)$. If $n = 5$ and $s \geq 1$, then $\pi = (4, 3, 2, 2, 1)$ or $(3, 3, 2, 2, 1)$. Note that $(3, 3, 2, \ldots, 2) \prec (4, 2, \ldots, 2)$, and $(3, 3, 2, 2, 1) \prec (4, 3, 2, 2, 1)$. By Theorems 3.8 and 3.9, we obtain the results of (2) and (3) as desired.

(4) If $1 \leq s \leq n - 5$, let $\pi = (d_1, \ldots, d_n)$ be the non-increasing degree sequence of $G \in \mathcal{B}_n(s)$. Then $d_{n-s} > 1$ and $d_{n-s+1} = \cdots = d_n = 1$. If $\pi = \pi_2$, then by Theorem 3.8, $\mu(G) \leq \mu(B_{n\pi_2}^*)$ with equality if and only if $G$ is isomorphic to $B_{n\pi_2}^*$. If $\pi \neq \pi_2$, it is easy to see that $\pi$, $\pi_2$ are two different bicyclic degree sequences, and $\pi \prec \pi_2$. By Theorems 3.8 and 3.9, we have

$$\mu(G) \leq \mu(B_{n\pi_2}^*) \leq \mu(B_{n\pi_2}^*)$$

with equality if and only if $G \cong B_{n\pi_2}^*$. Hence the assertion holds. □

Corollary 3.12. Let $G \in \mathcal{B}_{n, \Delta}$, where $n - 1 \geq \Delta \geq 3$. Then $\mu(G) \leq \mu(B_{n\pi_3}^*)$ with equality if and only if $G \cong B_{n\pi_3}^*$, where $\pi_3$ is defined as follows:

(1) If $\left\lceil \frac{n+2}{2} \right\rceil \leq \Delta \leq n - 1$, then $\pi_3 = (\Delta, n - \Delta + 2, 2, 1, \ldots, 1)$, where the frequency of 1 in $\pi_3$ is $n - 4$.
(2) If $\left\lceil \frac{n+4}{3} \right\rceil \leq \Delta \leq \left\lceil \frac{n+2}{2} \right\rceil$, then $\pi_3 = (\Delta, n - 2\Delta + 4, 2, 1, \ldots, 1)$, where the frequency of 1 in $\pi_3$ is $n - 4$.
(3) If $\Delta < \left\lceil \frac{n+4}{3} \right\rceil$, then let $p = \left\lceil \log_{\Delta-1} \frac{n(\Delta-2)}{2n-\Delta-4} \right\rceil$. Denote by

$$\frac{n - (\Delta - 1)p - [\Delta(\Delta - 1) - 4] + 2}{\Delta - 2} = \begin{cases} (\Delta - 1)r - 4 + q, & \text{if } p = 1, \\ (\Delta - 1)r + q, & \text{if } p \geq 2, \end{cases}$$

where $0 \leq q < \Delta - 1$. Hence $\pi_3 = (\Delta, \ldots, \Delta, q + 1, 1, \ldots, 1)$, where the frequency of $\Delta$ in $\pi_3$ is $m = \begin{cases} 1 + r, & \text{if } p = 1, \\ (\Delta - 1)p - 2[\Delta(\Delta - 1) - 4] + 2 + r, & \text{if } p \geq 2. \end{cases}$

Proof. Let $\pi$ be the non-increasing degree sequence of $G \in \mathcal{B}_{n, \Delta}$. If $\pi = \pi_3$, then by Theorem 3.8, $\mu(G) \leq \mu(B_{n\pi_3}^*)$ with equality if and only if $G \cong B_{n\pi_3}^*$. If $\pi \neq \pi_3$, it is easy to see that $\pi$, $\pi_3$ are two different bicyclic degree sequences, and

(1) If $\left\lceil \frac{n+2}{2} \right\rceil \leq \Delta \leq n - 1$, then $\pi \prec \pi_3 = (\Delta, n - \Delta + 2, 2, 1, \ldots, 1)$;
(2) If $\left\lceil \frac{n+4}{3} \right\rceil \leq \Delta \leq \left\lceil \frac{n+2}{2} \right\rceil$, then $\pi \prec \pi_3 = (\Delta, n - 2\Delta + 4, 2, 1, \ldots, 1)$.

Then for (1) and (2), by Theorems 3.8 and 3.9, we have

$$\mu(G) \leq \mu(B_{n\pi_3}^*) \leq \mu(B_{n\pi_3}^*)$$

with equality if and only if $G \cong B_{n\pi_3}^*$.

(3) If $\Delta < \left\lceil \frac{n+4}{3} \right\rceil$, by Theorem 3.8, $B_{n\pi_3}^*$ can be constructed as follows. Assume that $B_{n\pi_3}^*$ has $(p + 2)$ layers. Then there is one vertex in the layer 0 (i.e., the root vertex); there are $\Delta$ vertices in layer 1; there are $\Delta(\Delta - 1) - 4$ vertices in layer 2;
there are \([\Delta(\Delta - 1) - 4](\Delta - 1)^{p-2}\) vertices in layer \(p\) \((p \geq 2)\); there are at most \([\Delta(\Delta - 1) - 4](\Delta - 1)^{p-1}\) vertices in layer \(p + 1\). Hence for \(p \geq 1\),
\[
\frac{(\Delta - 1)^{p-1}[\Delta(\Delta - 1) - 4] + 2}{\Delta - 2} < n \leq \frac{(\Delta - 1)^{p}[\Delta(\Delta - 1) - 4] + 2}{\Delta - 2}.
\]

Let \(p = \lceil \log_{\Delta-1} \left\lceil \frac{n(\Delta-2) - 2}{\Delta(\Delta - 1) - 4} \right\rceil \rceil\). Since \(\Delta < \left\lceil \frac{n+\Delta}{3} \right\rceil\), then \(p \geq 1\), and there exist an integer \(r\) and \(0 \leq q < \Delta - 1\) such that
\[
n = \frac{(\Delta - 1)^{p-1}[\Delta(\Delta - 1) - 4] + 2}{\Delta - 2} = \begin{cases} (\Delta - 1)r - 4 + q, & \text{if } p = 1, \\ (\Delta - 1)r + q, & \text{if } p \geq 2. \end{cases} \quad (*)
\]

Therefore, the degrees of all vertices from layer 0 to layer \(p - 1\) are \(\Delta\), and there are \(r\) vertices in layer \(p\) with degree \(\Delta\). Let
\[
m = \begin{cases} 1 + r, & \text{if } p = 1, \\ \frac{(\Delta - 1)^{p-2}[\Delta(\Delta - 1) - 4] + 2}{\Delta - 2} + r, & \text{if } p \geq 2. \end{cases}
\]

Then there are \(m\) vertices with degree \(\Delta\) in \(B_{\pi_3}^*\). Therefore, by Equality \((*)\), \(\pi_3 = (\Delta, \ldots, \Delta, q + 1, 1, \ldots, 1)\), where the frequency of \(\Delta\) in \(\pi_3\) is \(m\). Moreover, it can be seen that \(\pi, \pi_3\) are two different bicyclic degree sequences and \(\pi \prec \pi_3\). It follows from Theorems 3.8 and 3.9 that
\[
\mu(G) \leq \mu(B_{\pi_3}^*) \leq \mu(B_{\pi_3}^*)
\]
with equality if and only if \(G \cong B_{\pi_3}^*\). \(\square\)

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**References**