The least eigenvalue of a graph with a given domination number

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ABSTRACT

In this paper, we characterize the unique graph whose least eigenvalue achieves the minimum among all graphs with n vertices and domination number \( \gamma \). Thus we can obtain a lower bound on the least eigenvalue of a graph in terms of the domination number.

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1. Introduction

Throughout this paper all graphs are finite and simple. Readers are suggested to refer to [4] for graph theoretical terminologies not specified here.

Let \( G = (V(G), E(G)) \) be a simple graph with \( n \) vertices and \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Denote by \( N(v) \) (or \( N_G(v) \) for short) the set of all neighbors of \( v \) in \( G \). The adjacency matrix of \( G \) is \( A(G) = (a_{ij})_{n \times n} \), where \( a_{ij} = 1 \) if two vertices \( v_i \) and \( v_j \) are adjacent in \( G \) and \( a_{ij} = 0 \) otherwise. All eigenvalues of \( A(G) \) are real and can be arranged in order as \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \) since it is a real symmetric matrix. The largest eigenvalue \( \lambda_1(G) \) of \( A(G) \) is called the spectral radius of \( G \), denoted by \( \rho(G) \). In addition, by the Perron–Frobenius Theorem, we know that the spectral radius \( \rho(G) \) is simple and has a unique (up to a multiplication by a scalar) positive eigenvector if \( G \) is connected. We shall refer to such an eigenvector as the Perron vector of \( A(G) \). If \( x \) is a unit Perron vector of \( A(G) \), then we have
\[ \rho(G) = \max_{y \in \mathbb{R}^n, \|y\| = 1} y^T A(G) y = x^T A(G) x = \sum_{v_i v_j \in E(G)} 2x_{v_i}x_{v_j}. \]  

The least eigenvalue \( \lambda_{\min}(G) \) is now denoted by \( \lambda_{\min}(G) \), and the corresponding eigenvectors are called the least vectors of \( G \). Assume that \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) and \( \mathbf{x} \) is a unit least vector of \( G \). Then by the Rayleigh–Ritz Theorem,

\[ \lambda_{\min}(G) = \min_{y \in \mathbb{R}^n, \|y\| = 1} y^T A(G) y = x^T A(G) x = \sum_{v_i v_j \in E(G)} 2x_{v_i}x_{v_j} \]  

and

\[ \lambda_{\min}(G)x_v = \sum_{u \in N_G(v)} x_u \quad \text{for each} \quad v \in V(G). \]  

Eq. (1.3) is also called an eigenvalue equation for the vertex \( v \) of \( G \). It is known that \( \lambda_{\min}(G) = -\rho(G) \) for a bipartite graph \( G \) (see [7]).

Recall that a set \( D \) of vertices of a graph \( G \) is said to be dominating if every vertex of \( V(G) \setminus D \) is adjacent to a vertex of \( D \), and the domination number \( \gamma(G) \) \( (\gamma, \text{ for short}) \) is the minimum number of vertices of a dominating set in \( G \). If \( G \) has no isolated vertices, then \( \gamma \leq \frac{n}{2} \) (see [17]).

The investigation on the spectrum of graphs is an important topic in the theory of graph spectra. Brualdi and Solheid [5] proposed the following problem concerning the spectral radii: Given a set of graphs \( \mathcal{G} \), find an upper bound for the spectral radii of graphs in \( \mathcal{G} \) and characterize the graphs in which the maximal spectral radius is attained. This problem has been well studied, see [3, 10, 13, 20] for example. Recently, researchers have begun to pay attention to the least eigenvalues of graphs with a given value of some well-known integer graph invariant: for instance: order and size [1, 2, 8, 18], unicyclic graphs with a given number of pendant vertices [14], matching number and independence number [21], number of cut vertices [22], connectivity [23], chromatic number [9]. On the other hand, there are some bounds on the least eigenvalue. For example: Constantine [6] obtained that

\[ \lambda_{\min}(G) \geq -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil} \]

for any graph of order \( n \), where equality holds if and only if \( G = K_{\left\lfloor \frac{n}{2} \right\rfloor , \left\lceil \frac{n}{2} \right\rceil} \). Powers [19] gave a similar result

\[ \lambda_{\min}(G) \geq -\sqrt{m}, \]

where \( m \) is the size of the graph \( G \). Hoffman [12] showed that

\[ \lambda_{\min}(G) \geq \frac{-\rho(G)}{\chi(G) - 1}, \]

where \( \chi(G) \) is the chromatic number of \( G \). For \( K_{r+1} \)-free graphs \( G \) of order \( n \) and size \( m \), Nikiforov [15] obtained the upper bound

\[ \lambda_{\min}(G) < -\frac{2}{r} \left( \frac{2m}{n^2} \right)^r n. \]

Some other relevant bounds are also obtained by Godsil and Newman [11] and Nikiforov [16].

For convenience, a graph is called minimizing in a certain graph class if its least eigenvalue achieves the minimum among all graphs in the class. Denote by \( \mathcal{G}_{n, \gamma} \) (respectively, \( \mathcal{B}_{n, \gamma} \)) the set of all graphs (respectively, bipartite graphs) with \( n \) vertices and the domination number \( \gamma \).
In this paper, we consider the structure of a minimizing graph in $G_{n, \gamma}$ for the fixed $n$ and $\gamma$. In [20], Stevanović et al. characterized the graphs having the maximal spectral radius in $G_{n, \gamma}$. In contrast, we will characterize the minimizing graph(s) in $G_{n, \gamma}$.

2. Graphs in $B_{n, \gamma}$ with the maximal spectral radius

In this section, we will characterize the graphs in $B_{n, \gamma}$ with the maximal spectral radius. In order to prove our results, we need the following lemma.

**Lemma 1** [3, Lemma 2.1]. Let $G$ be a connected graph on $n$ vertices with vertex degrees $d_1, d_2, \ldots, d_n$ and $\rho(G)$ be the spectral radius of $G$. Then
\[ \rho(G) \leq \max_{uv \in E(G)} \sqrt{d_ud_v}. \] (2.1)

Moreover, equality holds if and only if $G$ is regular or bipartite semi-regular.

**Theorem 1.** Let $G \in B_{n, \gamma}$. If $G$ has the maximal spectral radius, then we have

(i) $G \cong K_{1, n-1}$ for $\gamma = 1$,
(ii) $G \cong K_{\lceil \frac{n-\gamma+2}{2} \rceil, \lceil \frac{n-\gamma+2}{2} \rceil} \cup (\gamma - 2)K_1$ for $\gamma \geq 2$.

**Proof.** Assume that $G \in B_{n, \gamma}$ and $G$ has the maximal spectral radius and let $G_1, G_2, \ldots, G_k$ be its connected components. Then $\rho(G) = \max_{1 \leq i \leq k} \rho(G_i)$. It is trivial for $\gamma = 1$. Thus, we assume $\gamma \geq 2$ in the following. Note that $\max_{uv \in E(G)} d_u + d_v \leq n - \gamma + 2$; otherwise there exists a dominating set in $G$ with fewer than $\gamma$ vertices, a contradiction. Consequently, we have
\[ \rho(G) \leq \max_{uv \in E(G)} \sqrt{d_ud_v} \leq \sqrt{\left\lfloor \frac{n-\gamma+2}{2} \right\rfloor \left\lceil \frac{n-\gamma+2}{2} \right\rceil} \]
with the equality in the first inequality if and only if $G$ is a bipartite semi-regular graph in view of Lemma 1, the equality in the second inequality if and only if $d_u = \left\lfloor \frac{n-\gamma+2}{2} \right\rfloor$ and $d_v = \left\lceil \frac{n-\gamma+2}{2} \right\rceil$ for some edge $uv$. Hence $G \cong K_{\left\lfloor \frac{n-\gamma+2}{2} \right\rfloor, \left\lceil \frac{n-\gamma+2}{2} \right\rceil} \cup (\gamma - 2)K_1$. \[\square\]

3. Minimizing graphs in $B_{n, \gamma}$

In this section, we will determine the minimizing graphs in $B_{n, \gamma}$.

**Lemma 2** [14, Lemma 2.6]. Let $A$ be an $n \times n$ real symmetric matrix and $\lambda$ be the least eigenvalue of $A$. If $\lambda = x^T A x$, where $x \in \mathbb{R}^n$ is a unit vector, then $A x = \lambda x$.

**Lemma 3.** Let $G^*$ be a connected graph with two nonadjacent vertices $u$, $v$ and let $G$ be the graph obtained from $G^*$ by adding the edge $uv$. Assume that $x$ and $y$ are the unit least vectors of $G$ and $G^*$, respectively. Then

(i) $\lambda_{\min}(G^*) \leq \lambda_{\min}(G)$ if $x_u = 0$ or $x_v = 0$, and the equality holds if and only if $x$ is a least vector of $G$ and $x_u = x_v = 0$.

(ii) $\lambda_{\min}(G) \leq \lambda_{\min}(G^*)$ if $y_u = 0$ or $y_v = 0$, and the equality holds if and only if $y$ is a least vector of $G$ and $y_u = y_v = 0$.

(iii) $\lambda_{\min}(G) < \lambda_{\min}(G^*)$ if $y_u y_v < 0$. 
Proof. (i) By Eq. (1.2), we have
\[
\lambda_{\text{min}}(G^*) \leq x^T A(G^*)x = x^T A(G)x - 2x_u x_v = \lambda_{\text{min}}(G) - 2x_u x_v.
\]
If \(x_u = 0\) or \(x_v = 0\), then \(\lambda_{\text{min}}(G^*) \leq \lambda_{\text{min}}(G)\). If the equality holds, then \(\lambda_{\text{min}}(G^*) = x^T A(G^*)x\). Thus, by Lemma 2, we have \(A(G^*)y = \lambda_{\text{min}}(G^*)y\) and so \(x\) is also a least vector of \(G^*\). Comparing the eigenvalue equation (1.3) of \(G\) and \(G^*\), for the vertex \(u\) or \(v\), we get \(x_u = x_v = 0\).

Similarly, (ii) and (iii) can be proved. □

Denote by \(\lambda_{n, \gamma}\) the minimum of the least eigenvalue of the graphs in \(\mathcal{G}_{n, \gamma}\), or equivalently the least eigenvalue of a minimizing graph in \(\mathcal{G}_{n, \gamma}\).

Lemma 4. Let \(\lambda_{n, \gamma}\) denote the minimum of the least eigenvalues of the graphs in \(\mathcal{G}_{n, \gamma}\) and \(\gamma \geq 2\). Then we obtain that \(\lambda_{n, \gamma}\) is strictly decreasing with respect to \(n\), and is strictly increasing with respect to \(\gamma\).

Proof. Let \(G\) be a minimizing graph in \(\mathcal{G}_{n, \gamma}\), and let \(x\) be a unit least vector of \(G\). Note that there exists a vertex \(u\) in any dominating set \(D\) of \(G\) with \(x_u \neq 0\); otherwise we have
\[
-\sqrt{\left[\frac{n - \gamma + 2}{2}\right] \left[\frac{n - \gamma + 2}{2}\right]} \geq \lambda_{n, \gamma} = x^T A(G)x = x^T A(G - D)x \geq \lambda_{\text{min}}(G - D)
\]
and
\[
\lambda_{\text{min}}(G - D) \geq -\sqrt{\left[\frac{n - \gamma}{2}\right] \left[\frac{n - \gamma}{2}\right]},
\]
a contradiction. Let \(G^*\) be obtained from \(G\) by adding a new vertex \(v\) with the edge \(vu\). It is clear that \(G^* \in \mathcal{G}_{n+1, \gamma}\). Let \(y \in \mathbb{R}^{n+1}\) with \(y_v = 0\) and \(y_s = x_s\) for any vertex \(s\) of \(G\). Then we have \(\lambda_{\text{min}}(G^*) \leq y^T A(G^*)y = x^T A(G) = \lambda_{\text{min}}(G)\). Further, we obtain \(\lambda_{\text{min}}(G^*) < \lambda_{\text{min}}(G)\) since \(y_u \neq 0\) and \(y_v = 0\). Thus, \(\lambda_{n+1, \gamma} < \lambda_{n, \gamma}\) follows from \(\lambda_{n+1, \gamma} < \lambda_{n, \gamma}\).

In what follows, we will show that \(\lambda_{n, \gamma}\) is strictly increasing with respect to \(\gamma\). This result is obvious if \(G \cong K_{\left\lfloor \frac{n - \gamma + 2}{2} \right\rfloor, \left\lceil \frac{n - \gamma + 2}{2} \right\rceil} \cup (\gamma - 2)K_1\), since
\[
\lambda_{n, \gamma} = -\sqrt{\left[\frac{n - \gamma + 2}{2}\right] \left[\frac{n - \gamma + 2}{2}\right]} > -\sqrt{\left[\frac{n - (\gamma - 1) + 2}{2}\right] \left[\frac{n - (\gamma - 1) + 2}{2}\right]} \geq \lambda_{n, \gamma - 1}.
\]

Thus, we assume \(G \neq K_{\left\lfloor \frac{n - \gamma + 2}{2} \right\rfloor, \left\lceil \frac{n - \gamma + 2}{2} \right\rceil} \cup (\gamma - 2)K_1\) in the following. Let \(V^+ = \{v \in V(G) : x_v > 0\}\), \(V^- = \{v \in V(G) : x_v < 0\}\) and \(V^0 = \{v \in V(G) : x_v = 0\}\). We claim that there exists two nonadjacent vertices \(u\) and \(v\) with \(x_u x_v < 0\). Assume, to the contrary, that each vertex of \(V^+\) is adjacent to each vertex of \(V^-\) and if some vertex of \(V^0\) is adjacent to one vertex of \(V^+\), then it has to be adjacent to at least one vertex of \(V^-\). Thus, we have \(|V^0| \geq \gamma - 2\), where equality holds if and only if \(V_0\) is an independent set of \(G\) and no edges connect the vertices of \(V^0\) and those of \(V^+ \cup V^-\). Consequently, we have
\[
\lambda_{n, \gamma} = \lambda_{\text{min}}(G) = x^T A(G)x = x^T A(G - V^0)x \geq -\sqrt{\left[\frac{n - \gamma + 2}{2}\right] \left[\frac{n - \gamma + 2}{2}\right]},
\]
where equality holds if and only if \(V^0\) is an independent set of order \(\gamma - 2\) and \(G - V^0 \cong K_{\left\lfloor \frac{n - \gamma + 2}{2} \right\rfloor, \left\lceil \frac{n - \gamma + 2}{2} \right\rceil}\). On the other hand, we have
Let $G$ be a minimizing graph in $\mathcal{G}_{n,\gamma}$ with $\gamma \geq 2$. Assume for the contradiction that there exists a vertex $v$ of $G$ such that $x_v = 0$ and $x_u \neq 0$ for some edge $e = uv$. Let $G^* = G - e$, and let $x$ be a least vector of $G$. Consequently, we have $G - e \in \mathcal{G}_{n,\gamma'}$, where $\gamma' \geq \gamma$. Thus, by Lemma 3, we obtain that
\[ \lambda_{n,\gamma'} = \lambda_{\min}(G^*) \geq x^T A(G)x > \lambda_{\min}(G^*) \geq \lambda_{n,\gamma'}, \]
which is a contradiction since $\lambda_{n,\gamma'} > \lambda_{n,\gamma}$ in view of Lemma 4. Hence, $x_v \neq 0$ for any $v \in V(G)$ with $d_v > 0$. □

**Theorem 2.** Let $G$ be a minimizing graph in $\mathcal{G}_{n,\gamma}$. Then we have

(i) $G \cong K_1 \lor K_{\lceil \frac{n-1}{2} \rceil, \lceil \frac{n-1}{2} \rceil}$ for $\gamma = 1$ and $n \geq 6$.

(ii) $G \cong K_{\lceil \frac{n-\gamma+2}{2} \rceil, \lceil \frac{n-\gamma+2}{2} \rceil} \lor (\gamma - 2)K_1$ for $\gamma \geq 2$.

**Proof.** Let $G$ be a minimizing graph in $\mathcal{G}_{n,\gamma}$, and let $x$ be a unit least vector of $G$. Let $V^+ = \{ v \in V(G) : x_v > 0 \}$, $V^- = \{ v \in V(G) : x_v < 0 \}$, and $V^0 = \{ v \in V(G) : x_v = 0 \}$.

(i) If $\gamma = 1$. It is easy to see that each vertex in $V^+$ and each vertex in $V^-$ are adjacent; otherwise, we have $\lambda_{n,1} < \lambda_{\min}(G + e) < \lambda_{\min}(G) = \lambda_{n,1}$ by adding a such edge $e$ to $G$, a contradiction. Note that $\lambda_{n,1} \leq \lambda_{\min}(K_{1,n-1}) < -2$ for $n \geq 6$.

**Case 1.** If $V^0 \neq \emptyset$, then each vertex in $V^0$ is adjacent to all the vertices in $V^+ \cup V^-$ by Lemma 3. In addition, both $V^+$ and $V^-$ are independent sets; otherwise, we have a graph $G^* \in \mathcal{G}_{n,1}$ obtained from $G$ by deleting edges within $V^+$ or $V^-$ with $\lambda_{\min}(G^*) < \lambda_{\min}(G)$, a contradiction. Let $x^*$ be a subvector of $x$ by deleting the entries corresponding to vertices in $V^0$. Thus,
\[ \lambda_{\min}(G) = x^T A(G)x = x^* A(G^*)x^* \geq \lambda_{\min}(G - V^0) \geq \sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil}. \]

Hence $\lambda_{\min}(G) \geq -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil}$, where equality holds if and only if $|V^0| = 1$, $|V^+| = \lceil \frac{n-1}{2} \rceil$, and $|V^-| = \lfloor \frac{n-1}{2} \rfloor$. Furthermore, using the eigen-equation at the vertex of $V^0$, we have $G \cong K_1 \lor K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ for $\gamma = 1$ only if $n$ is odd.

**Case 2.** If $V^0 = \emptyset$, then we assume that $|V^+| = a + 1$ and $|V^-| = b$ for $a + b = n - 1$. Without loss of generality, let $d_u = n - 1$ for some vertex $u \in V^+$ since $\gamma = 1$. Then $V^+ \setminus \{u\}$ and $V^-$ are independent sets, respectively; otherwise, we have a graph $G^* \in \mathcal{G}_{n,1}$ obtained from $G$ by deleting edges within $V^+ \setminus \{u\}$ or $V^-$ with $\lambda_{\min}(G^*) < \lambda_{\min}(G)$, a contradiction. Thus, $G \cong K_1 \lor C_{a-b}$ and $a \neq b$ since
$V^O = \emptyset$, where $O_a$ is an empty graph of order $a$. It is easy to obtain that $\lambda_{\min}(G)$ is the least zeros of the polynomial $f(a, b, x) = x^3 - (n - 1 + ab)x - 2ab$. Since

$$f(a, b, x) - f(a + 1, b - 1, x) = (b - a - 1)(x + 2) \leq 0$$

for $a < b$ and $x < -2$, $\lambda_{\min}(G)$ is achieved only at $K_n \cup K_{\frac{n-1}{2}} \cup K_{\frac{n-1}{2}}$ and $n$ is even.

Combining Case 1 and Case 2, we have shown that (i) holds.

(ii) By Theorem 1, it suffices to show that $G$ is a bipartite graph. In view of Lemma 5, $x_v \neq 0$ for any $v \in V(G)$ with $d_v > 0$. Assume that $G$ is not a bipartite graph. Consequently, we have $x_vx_v > 0$ for some edge $e = uv$ in some odd cycle and $G - e \in \mathcal{G}_{n,y'}$, where $y' \geq y$. However, we have

$$\lambda_{n,y'} = x^T A(G)x = x^T A(G - e)x + 2x_u x_v > x^T A(G - e)x \geq \lambda_{\min}(G - e) \geq \lambda_{n,y'},$$

which is a contradiction since $\lambda_{n,y'} < \lambda_{n,y'}$ by virtue of Lemma 4. Thus, we obtain that $G$ is a bipartite graph. The proof is complete. □

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