The vertex (edge) independence number, vertex (edge) cover number and the least eigenvalue of a graph

Ying-Ying Tan \(^a\), Yi-Zheng Fan \(^b\),\(^*\)

\(^a\) Department of Mathematics and Physics, Anhui University of Architecture, Hefei 230601, PR China
\(^b\) School of Mathematical Sciences, Anhui University, Hefei 230039, PR China

ARTICLE INFO

Article history:
Received 4 October 2009
Accepted 4 April 2010
Available online 8 May 2010
Submitted by X. Zhan

AMS classification:
05C50
15A18

Keywords:
Graph
Adjacency matrix
Vertex (edge) independence number
Vertex (edge) cover number
Least eigenvalue

ABSTRACT

In this paper we characterize the unique graph whose least eigenvalue attains the minimum among all graphs of a fixed order and a given vertex (edge) independence number or vertex (edge) cover number, and get some bounds for the vertex (edge) independence number, vertex (edge) cover number of a graph in terms of the least eigenvalue of the graph.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

Let \( G = (V, E) \) be a simple graph of order \( n \) with vertex set \( V = V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = E(G) \). The adjacency matrix of \( G \) is defined to be a \((0, 1)\)-matrix \( A(G) = [a_{ij}] \), where \( a_{ij} = 1 \) if \( v_i \) is adjacent to \( v_j \) and \( a_{ij} = 0 \) otherwise. The eigenvalues of the graph \( G \) are referred to the eigenvalues of \( A(G) \), which are arranged as \( \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G) \). One can find that \( \lambda_n(G) \), denoted by \( \rho(G) \), is

\(^*\) Corresponding author.
E-mail addresses: tansusan1@aiai.edu.cn (Y.-Y. Tan), fanyz@ahu.edu.cn (Y.-Z. Fan).
2. Results

Some bounds for these parameters in terms of the least eigenvalue, fixed vertex (edge) independence number or vertex (edge) cover number. Consequently we obtain all graphs in the class. In this paper we determine the unique minimizing graph among all graphs with eigenvalue (respectively, spectral radius) of \( G \). There are many results in literature about the spectral radius of the adjacency matrix of a graph; see e.g. [8,9] for basis results.

The least eigenvalue \( \lambda_1(G) \) is now denoted by \( \lambda_{\min}(G) \), and the corresponding eigenvectors are called the least vectors of \( G \). Relative to spectral radius, the least eigenvalue is received less attention. In the past the main work on the least eigenvalue of a graph is about its bounds; see e.g. [10,16]. Recently, the work on minimizing the least eigenvalues of graphs subject to one or more given parameters is received more and more attention [1,2,11,17,18,20,21,24].

In this paper we concern the relation between the least eigenvalue of a graph and its vertex (edge) independence number, vertex (edge) cover number. We recall some notions as follows; or see [3].

- A vertex (edge) independent set of a graph \( G \) is a set of vertices (edges) such that any two vertices (edges) of the set are not adjacent (incident). The vertex (edge) independence number of \( G \), denoted by \( \alpha(G) \) (\( \alpha'(G) \)), is the maximum of the cardinalities of all vertex (edge) independent sets.
- A vertex (edge) cover of a graph \( G \) is a set of vertices (edges) such that each edge (vertex) of \( G \) is incident with at least one vertex (edge) of the set. The vertex (edge) cover number of \( G \), denoted by \( \beta(G) \) (\( \beta'(G) \)), is the minimum of the cardinalities of all vertex (edge) covers.

An edge independent set (edge independence number) is usually called a matching (matching number). When we consider an edge cover of a graph, we always assume that the graph contains no isolated vertices. It is known that for a graph \( G \) of order \( n \), \( \alpha(G) + \beta(G) = n \); and if in addition \( G \) has no isolated vertices, then \( \alpha'(G) + \beta'(G) = n \). For a bipartite graph, \( \alpha'(G) = \beta(G) \), and if in addition \( G \) has no isolated vertices then \( \alpha(G) = \beta'(G) \).

Many results concern the relation between eigenvalues and vertex independence number or edge independence number (matching number) of a graph. A lower bound of the vertex independence number of a regular graph, in terms of the least eigenvalue and the corresponding eigenvectors, is given by Wilf [22]. An upper bound of the vertex independence number of a regular graph, due to Delsarte and Hoffman, in terms of the least eigenvalue, is extended to a general graph by Haemers [14], Godsil and Newman [13]. The minimum spectral radius of graphs with given vertex independence number is discussed by Xu et al. [23]. Brouwer and Haemers [4] give a sufficient condition of a regular graph having perfect matchings in terms of the third largest eigenvalue. The result has been generalized in [5,6] to a general graph. In addition, some bounds of the larger eigenvalues in terms of matching number are discussed in many papers; see, e.g. [12,15,19]. However, less results can be found on the vertex (edge) cover number in terms of eigenvalues.

A graph \( G \) is called minimizing (respectively, maximizing) in a certain class of graphs if the least eigenvalue (respectively, spectral radius) of \( G \) attains the minimum (respectively, maximum) among all graphs in the class. In this paper we determine the unique minimizing graph among all graphs with fixed vertex (edge) independence number or vertex (edge) cover number. Consequently we obtain some bounds for these parameters in terms of the least eigenvalue.

2. Results

We introduce some preliminary knowledge and notations. A vector \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) can be considered as a function defined on the vertices of a graph \( G \) on vertices \( v_1, v_2, \ldots, v_n \), which are given by \( x(v_i) = x_i \) for each \( i = 1, 2, \ldots, n \). Thus the quadratic form \( x^T A(G) x \) can be written as

\[
    x^T A(G) x = 2 \sum_{uv \in E(G)} x(u)x(v) \quad (2.1)
\]

The eigenvector equation \( A(G)x = \lambda x \) can be interpreted as

\[
    \lambda x(v) = \sum_{u \in N_C(v)} x(u), \quad \text{for each } v \in V(G), \quad (2.2)
\]
which is called an \((\lambda, \alpha)\)-eigenequation of the graph \(G\), where \(N_G(v)\) denotes the neighborhood of \(v\) in \(G\). In addition, for an arbitrary unit vector \(x \in \mathbb{R}^n\),
\[
\lambda_{\min}(G) \leq x^T A(G)x,
\]
with equality if and only if \(x\) is a least vector of \(G\).

We first discuss the minimizing graph(s) subject to fixed matching number. To characterize such graph, it is necessary for us to discuss the maximizing bipartite graph(s) subject to fixed matching number. By the relations between the vertex (edge) independence number and the vertex (edge) cover number, we then characterize the the minimizing graphs subject to fixed vertex independence number or vertex (edge) cover number.

Denote by \(K_{p,q}\) a complete bipartite graph of which the bipartition of the vertex set consists of a subset of \(p\) vertices and a subset of \(q\) vertices. For a graph \(G\) of order \(n\), its matching number holds that \(0 \leq \alpha'(G) \leq \lceil n/2 \rceil\). By [7] or [24], the graph \(K_{\lceil n/2 \rceil,\lfloor n/2 \rfloor}\) is the unique minimizing graph among all graphs of order \(n\). The result also tells us that \(K_{\lceil n/2 \rceil,\lfloor n/2 \rfloor}\) is the unique minimizing graph of all graphs of order \(n\) with matching number \(\lfloor n/2 \rfloor\). If \(\alpha'(G) = 0\), then \(G\) is an empty graph (containing no edges). In this paper we do not consider the empty graph \(G\), for which \(\alpha'(G) = 0\), \(\alpha(G) = n\), \(\beta(G) = 0\).

**Theorem 2.1.** The graph \(K_{\alpha',n-\alpha'}\) is the unique maximizing graph among all bipartite graphs of order \(n\) with matching number \(\alpha'\).

**Proof.** Suppose that \(G\) is a maximizing graph among all bipartite graphs of order \(n\) with matching number \(\alpha'\). Let \((A, B)\) be a bipartition of the vertex set of \(G\) such that \(|A| \geq |B| \geq \alpha'\), and let \(M\) be a maximum matching of \(G\) of size \(\alpha'\). If \(|B| = \alpha'\), then surely \(G = K_{\alpha',n-\alpha'}\) since the spectral radius of a graph is increasing when adding edges.

Now assume that \(|B| > \alpha'\). We will get a contradiction to show this case cannot occur. Let \(A_M, B_M\) be the sets of vertices of \(A, B\) which are incident to the edges of \(M\), respectively. Note that \(|A_M| = |B_M| = \alpha'\), and \(G\) contains no edges between \(A - A_M\) and \(B - B_M\). Let \(x\) be a unit nonnegative eigenvector of \(A(G)\) corresponding to the spectral radius. Without loss of generality, assume \(\sum_{v \in A_M} x(v) \geq \sum_{v \in B_M} x(v)\). Removing all edges (if they exist) between \(A - A_M\) and \(B_M\), and adding all possible edges between \(A - A_M\) and \(A_M\), we get a bipartite graph \(G'\) with the bipartition \((A_M, B \cup (A - A_M))\). In the graph \(G'\), adding all possible edges between \(A_M\) and \(B\) \((A - A_M)\), we obtain a complete bipartite graph \(G''\), namely \(K_{\alpha',n-\alpha'}\). By (2.1) we get
\[
\rho(G) = x^T A(G)x \leq x^T A(G')x \leq x^T A(G'')x \leq \rho(G'').
\]
As \(G\) is maximizing, \(\rho(G') = \rho(G'') =: \rho\), and above all inequalities hold as equalities. So \(x\) is also a Perron vector of \(A(G'')\). As \(G''\) is connected, \(x\) has all entries being positive. Considering the \((\rho, x)\)-eigenequations (2.2) of the graphs \(G\) and \(G''\) on one (arbitrarily chosen) vertex of \(A_M\) respectively, we get a contradiction. The result follows. \(\Box\)

**Corollary 2.2.** The graph \(K_{\gamma,n-\gamma}\) is the unique maximizing graph among all bipartite graphs of order \(n\) with vertex cover number, or vertex independence number, or edge cover number, being \(\gamma\).

**Proof.** Noting that for a bipartite graph \(G\), \(\beta(G) = \alpha'(G)\), \(\alpha(G) = n - \beta(G)\), and \(\beta'(G) = \alpha(G)\) (assuming \(G\) contains no isolated vertices), the result follows by Theorem 2.1. \(\Box\)

**Theorem 2.3.** The graph \(K_{\alpha',n-\alpha'}\) is the unique minimizing graph among all graphs of order \(n\) with matching number \(\alpha'\).

**Proof.** Let \(G\) be a minimizing graph among all graphs of order \(n\) with matching number \(\alpha'\). Let \(x\) be a unit least vector of \(A(G)\). Denote \(V_+ = \{v : x(v) \geq 0\}\), \(V_- = \{v : x(v) < 0\}\), both being nonempty. Deleting all edges within \(V_+\) and within \(V_-\), we get a bipartite graph \(G_0\) with \(\alpha'(G_0) =: \alpha'_0 \leq \alpha'\). Then by (2.1) and (2.3)
\[ \lambda_{\min}(K_{\alpha',n-\alpha'}) \geq \lambda_{\min}(G) = x^TA(G)x \geq x^TA(G_0)x \geq \lambda_{\min}(G_0) = \lambda_{\min}(K_{\alpha'_0,n-\alpha'_0}), \]

where the first inequality holds as \( G \) is a minimizing graph with matching number \( \alpha' \), and the last inequality holds by Theorem 2.1 as \( G_0 \) is bipartite and \( \lambda_{\min}(G_0) = -\rho(G_0) \). However, as \( \alpha'_0 \leq \alpha' \leq n/2, \)

\[-\sqrt{\alpha'(n - \alpha')} = \lambda_{\min}(K_{\alpha',n-\alpha'}) \leq \lambda_{\min}(K_{\alpha'_0,n-\alpha'_0}) = -\sqrt{\alpha'_0(n - \alpha'_0)}.\]

So, all inequalities above hold as equalities, \( \alpha' = \alpha'_0 \), and \( x \) is also a least vector of \( K_{\alpha'_0,n-\alpha'_0} \), which implies \( x \) contains no zero entries and hence \( V_+ \) contains no vertices with zero values given by \( x \). By Theorem 2.1, we also get \( G_0 = K_{\alpha'_0,n-\alpha'_0} = K_{\alpha',n-\alpha'}. \) By (2.1) and the equality \( x^TA(G)x = x^TA(G_0)x \), we get \( G = G_0 = K_{\alpha',n-\alpha'}. \) The result follows. \( \square \)

**Corollary 2.4.** The graph \( K_{\beta',n-\beta'} \) is the unique minimizing graph among all graphs of order \( n \) with edge independence number \( \beta' \).

**Proof.** The result follows by Theorem 2.3 as \( \beta'(G) = n - \alpha'(G). \) \( \square \)

**Corollary 2.5.** The graph \( K_{\beta,n-\beta} \) is the unique minimizing graph among all graphs of order \( n \) with vertex cover number \( \beta \leq n/2. \)

**Proof.** Let \( G \) be a minimizing graph among all graphs of order \( n \) with vertex cover number \( \beta \leq n/2. \) Let \( x \) be a least vector of \( A(G) \). Define \( V_+ \) and \( V_- \) as those in Theorem 2.3. Deleting all edges within \( V_+ \) and within \( V_- \), we get a bipartite graph \( G_0 \) with \( \beta(G_0) =: \beta_0 \leq \beta \leq n/2. \) Then by Corollary 2.2,

\[ \lambda_{\min}(K_{\beta,n-\beta}) \geq \lambda_{\min}(G) = x^TA(G)x \geq x^TA(G_0)x \geq \lambda_{\min}(G_0) = \lambda_{\min}(K_{\beta_0,n-\beta_0}). \]

By a similar discussion as in the proof of Theorem 2.3, we get \( G = K_{\beta,n-\beta}. \) \( \square \)

**Corollary 2.6.** The graph \( K_{\alpha,n-\alpha} \) is the unique minimizing graph among all graphs of order \( n \) with vertex independence number \( \alpha \geq n/2. \)

**Proof.** The result follows by Corollary 2.5 as \( \alpha(G) = n - \beta(G) \) for a graph \( G \) of order \( n. \) \( \square \)

By Theorem 2.3 and Corollaries 2.4–2.6, we get relations between the least eigenvalue of a graph and the matching number (edge independence number), or edge cover number, or vertex cover number, or vertex independence number.

**Theorem 2.7.** Let \( G \) be a graph of order \( n \) with \( \alpha'(G) \), or \( \beta'(G) \), or \( \alpha(G) \), or \( \beta(G) \), being \( \gamma \), where \( \alpha(G) \geq n/2 \) and \( \beta(G) \leq n/2. \) Then

\[ \lambda_{\min}(G) \geq -\sqrt{\gamma(n - \gamma)}, \]

with equality if and only if \( G = K_{\gamma,n-\gamma}. \)

**Theorem 2.8.** Let \( G \) be a graph of order \( n. \) Then

\[ \alpha'(G) \leq \frac{n - \sqrt{n^2 - 4\lambda_{\min}(G)^2}}{2}, \quad \beta'(G) \leq \frac{n + \sqrt{n^2 - 4\lambda_{\min}(G)^2}}{2}, \]

\[ \alpha(G) \leq \frac{n + \sqrt{n^2 - 4\lambda_{\min}(G)^2}}{2}, \quad \beta(G) \geq \frac{n - \sqrt{n^2 - 4\lambda_{\min}(G)^2}}{2}, \]

all with equalities if and only if \( G = K_{\gamma,n-\gamma}, \) where \( \gamma \) equals \( \alpha'(G), \beta'(G), \alpha(G), \beta(G) \), respectively.
Proof. The bounds for $\alpha'(G), \beta'(G)$ are obtained from Theorem 2.7. In the case of $\alpha(G) \geq n/2$, we get the bound as listed. However, the bound surely holds if $\alpha(G) < n/2$. So we need no limitation of $\alpha(G)$. The bound for $\beta(G)$ is obtained by a similar discussion. □

Some Remarks:

(1) For a bipartite graph, its least eigenvalue is the minus of its spectral radius. So the graphs in Theorem 2.1 and Corollary 2.2 are respectively the unique minimizing graphs among all bipartite graphs of order $n$ with corresponding matching number, vertex cover number, vertex independence number, and edge cover number.

(2) By Theorem 2.1 and Corollary 2.2, we get relations between the spectral radius of a bipartite graph and the matching number, edge cover number, vertex cover number, and vertex independence number, which is similar to Theorem 2.7 and Theorem 2.8 by a little modification. Theorem 2.7 is modified as $\rho(G) \leq \sqrt{\gamma(n - \gamma)}$, and Theorem 2.8 is modified only by replacing $\lambda_{\min}(G)$ by $\rho(G)$.

(3) Corollary 2.5 does not hold for $\beta > n/2$. For example, if a graph $G$ of order $n \geq 3$ holds $\beta(G) = n - 1$, then $G$ is necessarily a complete graph. Similarly, Corollary 2.6 does not hold for $\alpha < n/2$.

(4) Finally we compare the bounds of vertex dependence number in Theorem 2.8 with some known bounds. The Delsarte–Hoffman bound of vertex dependence number of a regular graph of order $n$ with degree $k$ is given as (see [13])

$$\alpha(G) \leq n - \frac{\lambda_{\min}(G)}{k - \lambda_{\min}(G)}.$$  

(2.4)

The bound is extended to a general graph $G$ of order $n$ with minimum degree $\delta$ by Haemous [14] as

$$\alpha(G) \leq n - \frac{\rho(G)\lambda_{\min}(G)}{\delta^2 - \rho(G)\lambda_{\min}(G)}.$$  

(2.5)


**Theorem 2.9 ([13]).** Let $G$ be a graph of order $n$. Then for any independent set $S$ of size $s$,

$$s \leq n - \frac{\lambda_{\min}(G)}{k_S - \lambda_{\min}(G)},$$  

(2.6)

where $k_S = 2\bar{d}_S - \frac{1}{n} \sum_{v \in V(G)} d_v, \bar{d}_S = \frac{1}{s} \sum_{v \in S} d_v, d_v$ denotes the degree of the vertex $v$.

As our bound is obtained from a minimizing graph which is a complete bipartite graph, it will behave well for bipartite graphs. For example, consider the complete bipartite graphs $K_{a,b}$ with $a \leq b$. Clearly $\alpha(K_{a,b}) = b$ and our bound for $\alpha(K_{a,b})$ is exact by Theorem 2.8. If using Haemous’s result, the bound (2.5) for the graph is also exact, while using Godsil-Newman’s bound (2.6) we get

$$\alpha(K_{a,b}) \leq \frac{(a + b)^2 \sqrt{ab}}{(a + b) \sqrt{ab} + 2a^2} = b + \frac{a\sqrt{ab}(a + b - 2\sqrt{ab})}{(a + b)\sqrt{ab} + 2a^2},$$

with equality if and only if $a = b$.

Next we give a comparison of the bounds (2.5) and (2.6) and ours for a non-bipartite graph. Let $G$ be a graph of order $n \geq 4$ obtained from a triangle by appending $(n - 3)$ pendant edge at some vertex.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha(G)$</th>
<th>Bound (2.5)</th>
<th>Bound (2.6)</th>
<th>Our bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>3.0509</td>
<td>2.3879</td>
<td>3.3439</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>9.0178</td>
<td>9.2098</td>
<td>9.0634</td>
</tr>
<tr>
<td>100</td>
<td>98</td>
<td>99.0002</td>
<td>99.7952</td>
<td>99.0019</td>
</tr>
</tbody>
</table>

Fig. 2.1. A numerical comparison between the bounds (2.5) and (2.6) and ours for a non-bipartite graph.
We get the numerical values with approximation in Fig. 2.1, from which we find our bound will behave better when \( n \) increases or the vertex independence number get larger relative to the order of the graph. In addition, Haemous’s bound is always better in above examples. The reason may be that his bound is involved with another eigenvalue, namely the spectral radius.

However, our bound is always greater than or equal to \( n/2 \), it will be useless for a graph with small vertex independence number. For example, the bounds (2.5) and (2.6) are exact for a complete graph \( K_n \) \( (n \geq 3) \). But our bound is \( \left( n + \sqrt{n^2 - 4} \right) / 2 > 1 = \alpha(K_n) \).

References