On the spectral radius of graphs with a given domination number

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Abstract

We characterize the graphs which achieve the maximum value of the spectral radius of the adjacency matrix in the sets of all graphs with a given domination number and graphs with no isolated vertices and a given domination number.

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1. Introduction

All graphs in this note are simple and undirected. For a graph $G$, let $A(G)$ denote its adjacency matrix and $\rho(G)$ denote the spectral radius of $A(G)$. The number of vertices of $G$ is, as usual, denoted by $n$. For two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, we define $G \cup H$ to be a graph...
with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$, while $kG$ denotes $G \cup G \cup \cdots \cup G$. Next, $\overline{G}$ denotes the complement of $G$. For other undefined notions, the reader is referred to [3] for general graph theory, and to [6,7] for spectral graph theory topics.

Brualdi and Solheid [4] proposed the following general problem, which became one of the classic problems of spectral graph theory:

*Given a set $\mathcal{G}$ of graphs, find $\min\{\rho(G): G \in \mathcal{G}\}$ and $\max\{\rho(G): G \in \mathcal{G}\}$, and characterize the graphs which achieve the minimum or maximum value.*

The maximization part of this problem has been solved for a number of graph classes so far, although it is interesting that it has been solved only recently for the sets of connected graphs which have a given value of some well-known integer graph invariant: for example, the number of cut vertices [2], the matching number [8], the chromatic number [9], or the clique number [13]. We should also note that the graphs with maximum spectral radius received much more attention in the literature than the graphs with minimum spectral radius (moreover, only the maximization part of the above problem is usually cited).

Here we are interested to find the maximum spectral radius among graphs with a given value of domination number $\gamma$. To recall, a set $S$ of vertices of a graph $G$ is said to be dominating if every vertex of $V(G) \setminus S$ is adjacent to a vertex of $S$, and the domination number $\gamma(G)$ is the minimum number of vertices of a dominating set in $G$. If $G$ has no isolated vertices, then $\gamma \leq \frac{n}{2}$ [14]. We will deal with two sets of graphs separately: all graphs with domination number $\gamma$, and graphs with domination number $\gamma$ having no isolated vertices.

We should state here that prior to formulating our main results, the examples of graphs, with no isolated vertices and a given domination number, maximizing the spectral radius were found using the computer system AutoGraphiX [1,5]. It is our opinion that, due to its versatility, this system may become an indispensable tool when it comes to looking for examples of extremal graphs.

The *surjective split graph* $SSG(n, k; a_1, \ldots, a_k)$, defined for positive integers $n, k, a_1, \ldots, a_k$, $n \geq k \geq 3$, satisfying $a_1 + \cdots + a_k = n - k, a_1 \geq \cdots \geq a_k$, is a split graph on $n$ vertices formed from a clique $K$ with $n - k$ vertices and an independent set $I$ with $k$ vertices, in such a way that the $i$th vertex of $I$ is adjacent to $a_i$ vertices of $K$, and that no two vertices of $I$ have a common neighbor in $K$. See Fig. 1 for examples of surjective split graphs. Note that $\gamma(SSG(n, k; a_1, \ldots, a_k)) = k$.

![Fig. 1. The surjective split graphs $SSG(15, 5; 2, 2, 2, 2, 2)$ and $SSG(15, 5; 6, 1, 1, 1, 1)$.](image-url)
It is known that the surjective split graphs have maximum number of edges among graphs with no isolated vertices and a given domination number \( \gamma \geq 3 \) (see [16]).

Our main results are the following.

**Theorem 1.** If \( G \) is a graph on \( n \) vertices with domination number \( \gamma \), then
\[
\rho(G) \leq n - \gamma.
\]
Equality holds if and only if \( G \cong K_{n-\gamma+1} \cup (\gamma - 1)K_1 \) or, when \( n - \gamma \) is even, \( G \cong \frac{n-\gamma+2}{2}K_2 \cup (\gamma - 2)K_1 \).

**Theorem 2.** If \( G \) is a graph on \( n \) vertices with no isolated vertices and domination number \( \gamma \), then:

(i) if \( \gamma = 1 \), then \( \rho(G) \leq \rho(K_n) \), with equality if and only if \( G \cong K_n \);

(ii) if \( \gamma = 2 \) and \( n \) is even, then \( \rho(G) \leq \rho(\frac{n}{2}K_2) \), with equality if and only if \( G \cong \frac{n}{2}K_2 \).

(iii) if \( \gamma = 2 \) and \( n \) is odd, then \( \rho(G) \leq \rho(\frac{n}{2} - 1)K_2 \cup P_3 \), with equality if and only if
\[
G \cong \left(\frac{n}{2} - 1\right)K_2 \cup P_3.
\]

(iv) if \( 3 \leq \gamma \leq \frac{n}{2} \), then
\[
\rho(G) \leq \rho(SSG(n, \gamma; n - 2\gamma + 1, 1, 1, \ldots, 1))
\]
with equality if and only if \( G \cong SSG(n, \gamma; n - 2\gamma + 1, 1, 1, \ldots, 1) \).

2. Lemmas

In order to prove Theorem 2, we have to find a rather tight estimate on the spectral radius of \( SSG(n, \gamma; n - 2\gamma + 1, 1, 1, \ldots, 1) \) and to show the existence of several vertices of maximum degree in a graph whose spectral radius is sufficiently close to its maximum degree. These useful results, crucial for the proof of Theorem 2, are given next.

**Lemma 3.** \( \rho(SSG(n, \gamma; n - 2\gamma + 1, 1, 1, \ldots, 1)) \geq n - \gamma - 1 + \frac{1}{n-\gamma} + \frac{(n-2\gamma+1)(n-2\gamma)}{(n-\gamma)^2} \).

**Proof.** Let \( S^* = SSG(n, \gamma; n - 2\gamma + 1, 1, 1, \ldots, 1) \). Let \( \lambda \) denote the value on the right-hand side of the above inequality. It is easy to see that
\[
\lambda = n - \gamma - \frac{(2n - 3\gamma)(\gamma - 1)}{(n-\gamma)^2} \leq n - \gamma.
\]

Let \( I' = \{u_0\} \) be the subset of the independent set \( I \) of \( S^* \) containing the vertex of degree \( n - 2\gamma + 1 \), and let \( K' \) be the subset of the clique \( K \) of \( S^* \) containing the \( n - 2\gamma + 1 \) vertices adjacent to \( u_0 \). Set \( I'' = I \setminus I' \) and \( K'' = K \setminus K' \). From the definition of \( S^* \), each vertex of \( K'' \) is adjacent to a unique vertex of \( I'' \), and \( |K''| = |I''| = \gamma - 1 \).

Now, let \( y = (y_u)_{u \in V(G)} \) be the vector defined in the following way:
\[
y_u = \begin{cases} 
a = (n - 2\gamma + 1) \left( 1 + \frac{n-2\gamma}{(n-\gamma)^2} \right), & u \in I', \\
b = \lambda \left( 1 + \frac{n-2\gamma}{(n-\gamma)^2} \right), & u \in K', \\
c = \lambda, & u \in K'', \\
d = 1, & u \in I''.
\end{cases}
\]
For $A = A(S^*)$, we have

$$
(Ay)_u = \begin{cases} 
(n - 2\gamma + 1)b, & u \in I', \\
 a + (n - 2\gamma)b + (\gamma - 1)c, & u \in K', \\
(n - 2\gamma + 1)b + (\gamma - 2)c + d, & u \in K'', \\
c, & u \in I''. 
\end{cases}
$$

Let us show that for this particular vector the inequality $Ay \geq \lambda y$ holds componentwise. Actually, for $u \in I$ we have equality:

$$
(Ay)_u = \lambda(n - 2\gamma + 1) \left(1 + \frac{n - 2\gamma}{(n - \gamma)^2}\right) = \lambda y_u, \quad u \in I'.
$$

Next, for $u \in K'$ we have

$$
(Ay)_u = n - 2\gamma + 1 + \lambda(n - \gamma - 1) + \frac{n - 2\gamma}{(n - \gamma)^2}(n - 2\gamma + 1 + (n - 2\gamma)\lambda)
\geq n - 2\gamma + \lambda \left(\frac{1}{n - \gamma} + n - \gamma - 1\right) + \frac{n - 2\gamma}{(n - \gamma)^2}(n - 2\gamma + 1 + (n - 2\gamma)\lambda)
= \lambda \left(\frac{(n - 2\gamma + 1)(n - 2\gamma)}{(n - \gamma)^2} + \frac{n - 2\gamma}{(n - \gamma)^2}(n - \gamma)^2 + n - 2\gamma + 1 - \lambda\right)
\geq \lambda^2 \left(1 + \frac{n - 2\gamma}{(n - \gamma)^2}\right) = \lambda y_u,
$$

where in the first inequality above we used the relation $1 \geq \frac{\lambda}{n - \gamma}$, and the second inequality, based on $(n - \gamma)^2 + n - 2\gamma + 1 - \lambda > \lambda^2$, follows from

$$
(n - \gamma)^2 - \lambda^2 + n - \gamma - \lambda = (n - \gamma - \lambda)(n - \gamma + \lambda + 1)
\geq \frac{(2n - 3\gamma)(\gamma - 1)}{(n - \gamma)^2} \cdot 2(n - \gamma) > \gamma - 1,
$$

thanks to the fact that $\lambda > n - \gamma - 1$ and $2(2n - 3\gamma) > n - \gamma$.

Finally, for $u \in K''$ we have

$$
(Ay)_u = (n - 2\gamma + 1)\lambda \left(1 + \frac{n - 2\gamma}{(n - \gamma)^2}\right) + (\gamma - 2)\lambda + 1
= \lambda \left(n - \gamma - 1 + \frac{(n - 2\gamma + 1)(n - 2\gamma)}{(n - \gamma)^2}\right) + 1
= \lambda \left(\frac{1}{n - \gamma}\right) + 1 = \lambda^2 - \frac{\lambda}{n - \gamma} + 1 \geq \lambda^2 = \lambda y_u.
$$
Therefore, we may now see that
\[ \rho(S^*) = \sup_{x \neq 0} \frac{x^T A x}{x^T x} \geq \frac{y^T A y}{y^T y} \geq \frac{y^T (\lambda y)}{y^T y} = \lambda. \]
\[ \square \]

\textbf{Lemma 4.} If a graph \( G \) with maximum degree \( \Delta \) satisfies
\[ \rho(G) \geq \Delta - 1 + \frac{k}{\Delta}, \quad 1 \leq k < \Delta, \]
then \( G \) contains at least \( k + 1 \) vertices of degree \( \Delta \).

\textbf{Proof.} Suppose the contrary, i.e., that \( G \) contains \( l \) vertices with degree \( \Delta, l \leq k \), while the degree of any other vertex is at most \( \Delta - 1 \). Let \( d_v \) denotes the degree of a vertex \( v \in V(G) \). Define a vector \( y = (y_v)_{v \in V(G)} \) by
\[ y_v = \begin{cases} \frac{1}{\Delta}, & \text{if } d_v = \Delta, \\ 1, & \text{if } d_v \leq \Delta - 1. \end{cases} \]
Then the adjacency matrix \( A = A(G) \) is such that when \( d_v = \Delta \)
\[ (Ay)_v \leq \Delta + \frac{l - 1}{\Delta} \leq \Delta + \frac{k - 1}{\Delta} + \frac{k}{\Delta^2} = \left( \Delta - 1 + \frac{k}{\Delta} \right) y_v, \]
while when \( d_v \leq \Delta - 1 \)
\[ (Ay)_v \leq d_v + \frac{k}{\Delta} \leq \Delta - 1 + \frac{k}{\Delta} = \left( \Delta - 1 + \frac{k}{\Delta} \right) y_v. \]
Thus, for positive vector \( y \), the inequality
\[ Ay \leq \left( \Delta - 1 + \frac{k}{\Delta} \right) y \]
holds componentwise.

It is well known that for a connected graph \( G \), there is a unique positive eigenvector corresponding to \( \rho(G) \), usually called the Perron vector. Let \( x \) be the Perron vector of \( G \). Then \( x^T y > 0 \) and we have
\[ \rho(G) x^T y = x^T A y \leq \left( \Delta - 1 + \frac{k}{\Delta} \right) x^T y \]
from where it follows that \( \rho(G) \leq \Delta - 1 + \frac{k}{\Delta} \).

Note that \( \rho(G) \), as an algebraic integer, is either an integer or an irrational number. So, it may not hold \( \rho(G) = \Delta - 1 + \frac{k}{\Delta} \). Therefore, \( \rho(G) < \Delta - 1 + \frac{k}{\Delta} \), which is a contradiction. \[ \square \]

3. Proofs

\textbf{Proof of Theorem 1.} Let \( G \) be a graph with domination number \( \gamma \), and let \( G_1, G_2, \ldots, G_t \) be its connected components. Then \( \rho(G) = \max_{1 \leq i \leq t} \rho(G_i) \). The maximum vertex degree \( \Delta(G) \) of \( G \) satisfies
\[ \Delta(G) \leq n - \gamma, \quad (1) \]
since if a vertex \( u \) has more than \( n - \gamma \) neighbors, then \( u \) and its nonneighbors form a dominating set with fewer than \( \gamma \) vertices, a contradiction. Next, from the well-known bound \( \rho(G) \leq \Delta(G) \) (see [6]) it easily follows that
\[ \rho(G) \leq n - \gamma. \]  \hspace{1cm} \text{(2)}

Suppose that equality holds in (2). Then we have \( \rho(G) = \Delta(G) = n - \gamma \). The first of these equalities holds if and only if one of the components containing a vertex of degree \( \Delta(G) \), say \( G_1 \), is \( \Delta(G) \)-regular, while for the other components, \( 2 \leq i \leq t \), it holds that \( \rho(G_i) \leq \Delta(G) \).

Let \( u \) be a vertex of \( G_1 \). It has \( n - \gamma \) neighbors (in \( G_1 \)) dominated by \( u \) and \( \gamma - 1 \) nonneighbors (which include \( G_2, \ldots, G_t \)) which dominate itself. If any two nonneighbors \( v, w \) of \( u \) are adjacent, then we can form a dominating set in \( G \) of size less than \( \gamma \) by choosing \( u \), one of \( v, w \) only and then the remaining nonneighbors of \( u \). Thus, the nonneighbors of \( u \) must not be adjacent. In particular, the components \( G_2, \ldots, G_t \) consist of isolated vertices. As a consequence, we get that \( G_1 \) has \( n - t + 1 \) vertices.

If \( u \) is adjacent to all vertices of \( G_1 \), then \( G_1 \) as a \( (n - \gamma) \)-regular graph on \( n - \gamma + 1 \) vertices must be isomorphic to \( K_{n-\gamma+1} \), and so we have \( G_1 \cong K_{n-\gamma+1} \cup (\gamma - 1)K_1 \).

If \( u \) has one nonneighbor in \( G_1 \), then \( G_1 \) as a \( (n - \gamma) \)-regular graph on \( n - \gamma + 2 \) vertices must be isomorphic to \( \frac{n-\gamma+2}{2} K_2 \), and so we have \( G_1 \cong \frac{n-\gamma+2}{2} K_2 \cup (\gamma - 2)K_1 \).

Thus, suppose that \( u \) has at least two nonneighbors in \( G_1 \). If any two nonneighbors \( v, w \) of \( u \) have a common neighbor \( s \) in \( G_1 \), then we can again form a dominating set of \( G \) of size less than \( \gamma \) by taking \( u, s \) and the remaining nonneighbors of \( u \). Therefore, no two nonneighbors of \( u \) in \( G_1 \) have a common neighbor in \( G_1 \). This implies that the closed neighborhoods of nonneighbors of \( u \) are mutually disjoint. Each of these \( \gamma - t \) closed neighborhoods contains \( n - \gamma + 1 \) vertices, while none of them contains \( u \). Thus, we have the following inequality:

\[ (\gamma - t)(n - \gamma + 1) \leq n - t \]

from where it follows that

\[ (\gamma - t - 1)n \leq (\gamma - 1)(\gamma - t) - t \]

and so

\[ \frac{(\gamma - t)(\gamma - 1) - t}{\gamma - t - 1} = \frac{(\gamma - t - 1)(\gamma - 1) + \gamma - 1 - t}{\gamma - t - 1} = \gamma \leq n. \]

Therefore, \( n = \gamma \). In such case, \( G \) consists of isolated vertices only, i.e., \( G_1 \cong K_1 \), and we have a contradiction as \( u \) does not have two nonneighbors in \( G_1 \).

At last, it is trivial to check that \( K_{n-\gamma+1} \cup (\gamma - 1)K_1 \) and, for \( n - \gamma \) even, \( \frac{n-\gamma+2}{2} K_2 \cup (\gamma - 2)K_1 \) have spectral radius \( n - \gamma \) and domination number \( \gamma \). \( \square \)

**Proof of Theorem 2.** (i) For any graph \( G \) on \( n \) vertices, \( \rho(G) \leq \rho(K_n) \) (see [6]), with equality if and only if \( G \cong K_n \). Since the complete graph \( K_n \) has domination number 1, this completes the case \( \gamma = 1 \).

(ii, iii) Suppose that \( \gamma = 2 \) and let \( G^* \) be a graph having the maximum spectral radius among all graphs on \( n \) vertices with no isolated vertices and domination number 2.

If \( G^* \) is disconnected with components \( G_1, \ldots, G_t, t \geq 2 \), then each component has at least two vertices, and thus, the biggest component has at most \( n - 2 \) vertices. Consequently, as \( \rho(G^*) = \max_{1 \leq i \leq t} \rho(G_i) \), we have \( \rho(G^*) \leq \rho(K_{n-2}) = n - 3 \).

On the other hand, suppose that \( G^* \) is connected. Since the spectral radius of a connected graph strictly increases by adding an edge, we see that \( G^* \) has to be domination-critical: the graph \( G^* + e \) has domination number less than \( \gamma \) for every edge \( e \) that does not belong to \( G^* \).

The structure of the domination-critical graphs with domination number 2 has been determined
in [18]: a graph with domination number 2 is domination-critical if and only if it is isomorphic to \( \bigcup_{i=1}^{t} K_{1,n_i} \) for some \( n_1, n_2, \ldots, n_t \), where \( K_{1,n_i} \) denotes the star on \( n_i + 1 \) vertices.

Thus, \( G^* \), as a domination-critical graph with domination number 2, is a complement of union of stars, and consequently, it is also a radius-critical graph with radius 2 [11]. Therefore, \( G^* \), which has maximum spectral radius among all connected graphs on \( n \) vertices with domination number 2, also has maximum spectral radius among all connected graphs on \( n \) vertices with radius 2. However, two of present authors have already solved the latter problem in [10, Theorem 3.1], showing that \( G^* \sim n/2 K_2 \) for even \( n \) and \( G^* \sim (\lfloor n/2 \rfloor - 1)K_2 \cup P_3 \) for odd \( n \). Since both of these graphs have average vertex degree larger than \( n - 3 \), and the spectral radius of a graph is always at least its average vertex degree (see [6]), we conclude that \( G^* \) are indeed the graphs which maximize the spectral radius among graphs with no isolated vertices and domination number 2.

(iv) Let \( 3 \leq \gamma \leq n/2 \) and let \( G^* \) be a graph on \( n \) vertices with no isolated vertices, domination number \( \gamma \) and the maximum spectral radius \( \rho^*(G^*) = \rho(G^*) \). Let \( S^* \) be the surjective split graph \( \text{SSG}(n, \gamma; n - 2\gamma + 1, 1, \ldots, 1) \).

As the rest of the proof is somewhat involved, above the beginning of every new part of the proof we put its main theme in bold type. The bolded sentences then yield an overview of the proof.

**\( G^* \) has at least two vertices of degree \( n - \gamma \).**

From the well-known bound \( A(G^*) \geq \rho^* \) (see [6]), the fact that \( S^* \) has domination number \( \gamma \) and Lemma 3, we get

\[ A(G^*) \geq \rho^* \geq \rho(S^*) \geq n - \gamma - 1 + \frac{1}{n - \gamma}. \]

Together with (1), this yields

\[ A(G^*) = n - \gamma. \]

Lemma 4 now implies that \( G^* \) contains at least two vertices of degree \( n - \gamma \). Suppose that \( w' \) and \( w'' \) are vertices of degree \( n - \gamma \).

**\( G^* \) is connected.**

On the contrary, suppose that \( G^* \) has components \( G^*_1, G^*_2, \ldots, G^*_t, t \geq 2 \). Suppose that \( w' \) belongs to \( G^*_1 \). Vertex \( w' \) together with its \( \gamma - 1 \) nonneighbors forms a minimal dominating set \( D \) in \( G^* \). There exists no edge between any two nonneighbors of \( w' \), as the existence any such edge yields a smaller dominating set. In particular, it follows that the components \( G^*_2, \ldots, G^*_t \) consist of isolated vertices. However, this is in contradiction with our premise that \( G^* \) has no isolated vertices.

**\( G^* \) is domination-critical.**

If any edge \( e \) may be added to \( G^* \) without decreasing its domination number, then \( G^* + e \) has greater spectral radius than \( G^* \) and domination number \( \gamma \), which is a contradiction. Thus, no edge may be added to \( G^* \) without decreasing its domination number, and so, \( G^* \) is domination-critical.

**The local structure of \( G^* \) imposed by \( w' \).**

Let

\[ S_{w'} = \{s_1, s_2, \ldots, s_{\gamma - 1}\} \]

be the set of \( \gamma - 1 \) vertices that are not adjacent to \( w' \). The subgraph induced by \( S_{w'} \) contains no edges: any edge between vertices of \( S_{w'} \) leads to a dominating set of size \( \gamma - 1 \), a contradiction.
Similarly, no two vertices from $S_w'$ may have a common neighbor: for if $t$ is a vertex of $G^*$ adjacent to vertices $u$ and $v$ of $S_w'$, then $\{w', t\} \cup S_{w'} \setminus \{u, v\}$ would be again a dominating set of size $\gamma - 1$.

Let $Y_w'$ be the set of vertices that are adjacent both to $w'$ and to a vertex from $S_{w'}$. In particular, for each $u \in S_{w'}$, let $Y_{w', u}$ be the set of vertices that are adjacent to $w'$ and $u$. The set $Y_{w', u}$ is not empty, as $G^*$ does not contain isolated vertices, and from the previous paragraph it follows that each neighbor of $u$ must also be a neighbor of $w'$. Moreover, it also follows that the sets $Y_{w', u}$, $u \in S_{w'}$, are mutually distinct.

Finally, let $Z_{w'}$ be the set of remaining vertices of $G^*$, those which are adjacent to $w'$ and to no vertex of $S_{w'}$. The set $Z_{w'}$ is not empty: otherwise, a set $X$ obtained by choosing an arbitrary vertex from each $Y_{w', u}, u \in S_{w'}$, would be a dominating set of size $\gamma - 1$. Actually, for each such $X$ an even stronger statement holds:

There exists a vertex $z_X$ in $Z_{w'}$ that is not adjacent to any vertex in $X$. \hfill (3)

We may now see that for any $u \in S_{w'}$, every dominating set $X$ of $G^*$ must contain either the vertex $u$ or a vertex from $Y_{w', u}$. In particular, if $|X| = \gamma$, then $\gamma - 1$ vertices of $X$ belong to sets $\{u\} \cup Y_{w', u}, u \in S_{w'}$, and the remaining vertex belongs to $\{w'\} \cup Z_{w'}$.

Next, the subgraph of $G^*$ induced by $Y_{w'}$ is a clique: otherwise, if $uv$ is not an edge of $G^*$ for $u, v \in Y_w'$, then $G^* + uv$ also has domination number $\gamma$, but its spectral radius is larger than $\rho^*$, a contradiction. From a similar reason, the subgraph induced by $Z_{w'}$ is also a clique.

Where does $w''$ appear: in $S_{w'}, Y_{w'}$ or $Z_{w'}$? The only part of $G^*$ that we do not know anything about is the set of edges between vertices of $Y_{w'}$ and $Z_{w'}$. This is where the second vertex $w''$ of degree $n - \gamma$ helps us. Note that the sets $S_{w''}, Y_{w''}$ and $Z_{w''}$ may be defined in the same manner and share similar properties as their counterparts $S_{w'}, Y_{w'}$ and $Z_{w'}$. So, let us consider in which of the three sets $S_{w''}, Y_{w''}$ and $Z_{w''}$ may $w''$ appear?

The case $w'' \in S_{w''}$ is impossible. First, $w''$ may not belong to $S_{w'}$, as the degrees of vertices in $S_{w'}$ are too small. Namely, a vertex $u \in S_{w'}$ is not adjacent to any vertex from

$$\{w'\} \cup Z_{w'} \cup (S_{w'} \setminus \{u\}) \cup (Y_{w'} \setminus Y_{w', u}),$$

and its degree is, thus, at most $n - 1 - (1 + 1 + (\gamma - 2) + (\gamma - 2)) < n - \gamma$.

If $w'' \in Y_{w'}$, then $G^*$ is a surjective split graph. Next, suppose that $w'' \in Y_{w'}$ and let $s$ be the unique vertex of $S_{w'}$ adjacent to $w''$. Then $w''$ is adjacent to all vertices of $Z_{w'}$ but one, which we denote by $z$. It is easy to see that

$$S_{w''} = \{z\} \cup S_{w'} \setminus \{s\},$$
$$Y_{w''} \supseteq \{w'\} \cup (Y_{w'} \setminus Y_{w', s}) \cup (Z_{w'} \setminus \{z\}),$$
$$Z_{w''} \subseteq \{s\} \cup Y_{w', s} \setminus \{w''\}.$$ \hfill (4) \hfill (5)

We show that equality holds in (4) and (5). Suppose that $t \in Y_{w', s} \cap Y_{w''}$. Since the subgraph induced by $Y_{w''}$ is a clique, $t$ must be adjacent to all vertices of $Z_{w'} \setminus \{z\}$. Further, as an element of $Y_{w''}, t$ must be adjacent to a vertex of $S_{w''}$. Since it is not adjacent to any vertex of $S_{w'} \setminus \{s\}$, we conclude that $t$ is adjacent to $z$ as well. But then $t$ has degree $n - \gamma + 1$, a contradiction. Thus, it follows that $Y_{w', s} \cap Y_{w''} = \emptyset$ and then the equality holds in (4) and (5). Moreover, one has

$$Y_{w'', u} = Y_{w', u}, \quad u \in S_{w'} \setminus \{s\}$$
and
\[ Y_{w''} = \{w'\} \cup Z_{w'} \setminus \{z\}. \]
As a consequence, \( z \) is adjacent to vertices of \( Y_{w''} \), and so \( z \) is not adjacent to any vertex from \( Y_{w'} \). Then \( G^* \), as a domination-critical graph, must already contain all edges between a vertex of \( Y_{w'} \) and \( Z_{w'} \setminus \{z\} \). In such case, \( G^* \) is indeed a surjective split graph:
\[ G^* \cong SS(n, \gamma; |Z_{w'}|, |Y_{w',s_1}|, \ldots, |Y_{w',s_{\gamma-1}}|). \]

**If** \( w'' \in Z_{w'} \), **then** \( G^* \) **is a surjective split graph.**
The last option for \( w'' \) is that it belongs to \( Z_{w'} \). We may freely suppose then that no vertex of \( Y_{w'} \) has degree \( n - \gamma \) (otherwise, rename any such vertex to \( w'' \) and return to the previous paragraph).

Let \( U \) be the set of all vertices of \( G^* \) having degree \( n - \gamma \). Then \( U \subseteq \{w'\} \cup Z_{w'} \). The vertices of \( U \) imply the same local structure in \( G^* \) – for any \( w \in U \) one has
\[ S_w = S_{w'}, \quad Y_w = Y_{w'}, \quad Z_w = \{w'\} \cup Z_{w'} \setminus \{w\}. \]

Finally, let \( Z' = Z_{w'} \setminus U \). Any vertex \( z' \in Z' \) has degree less than \( n - \gamma \) and, thus, there exists a vertex \( y' \in Y_{w'} \) not adjacent to \( z' \). The graph \( G^* + y'z' \) has a dominating set \( X \) of cardinality \( \gamma - 1 \). Note that \( y' \in X \subseteq Y_{w'} \) and that \( X \) does not dominate \( z' \) in \( G^* \). Thus, \( z' \) is not adjacent to any vertex of \( X \) in \( G^* \). In other words, for any \( z' \in Z' \), \( G^* \) does not contain at least \( \gamma - 1 \) edges of the form \( z'v \). This can be used to give an upper bound on the number of edges \( m^* \) of \( G^* \):
\[ m^* \leq \left( \frac{n - \gamma + 1}{2} \right) - |Z'| (\gamma - 1) + (\gamma - 1). \] (6)

(The last term above counts the edges between \( S_{w'} \) and \( Y_{w'} \).)

We can pair this upper bound with a lower bound on \( m^* \) obtained from Lemma 3 and the bound of Hong [12]
\[ \rho(G^*) \leq \sqrt{2m^* - n + 1}. \]

Namely, we have
\[ 2m^* \geq \rho(G^*)^2 + n - 1 \]
\[ = \left( n - \gamma - \frac{(2n - 3\gamma)(\gamma - 1)}{n - \gamma} \right)^2 + n - 1 \]
\[ \geq (n - \gamma)^2 - \frac{2(2n - 3\gamma)(\gamma - 1)}{n - \gamma} + n - 1 \]
\[ > (n - \gamma)^2 - 4(\gamma - 1) + n - 1 = (n - \gamma + 1)(n - \gamma) - 3(\gamma - 1), \]
i.e.,
\[ m^* > \left( \frac{n - \gamma + 1}{2} \right) - \frac{3}{2}(\gamma - 1). \] (7)

Inequalities (6) and (7), taken together, yield
\[ |Z'| \leq \frac{5}{2}. \]
Note that the case $|Z'| = 2$ is impossible. Namely, since each vertex $y' \in Y_{w'}$ has degree less than $n - \gamma$, there are at least two vertices in $Z'$ not adjacent to $y'$. Thus, neither of two vertices of $Z'$ is adjacent to any vertex of $Y_{w'}$. However, we can then add to $G^*$ all edges between one vertex of $Z'$ and all vertices of $Y_{w'}$ without decreasing its domination number, which is a contradiction.

Thus, $|Z'| = 1$. Then $G^*$ is again a surjective split graph

$$G^* \cong SSG(n, \gamma'; |U|, |Y_{w',x_1}|, \ldots, |Y_{w',s_{\gamma'-1}}|).$$

$S^*$ has maximum spectral radius among surjective split graphs.

Finally, we may suppose that $G^* \cong SSG(n, \gamma; a_1, \ldots, a_{\gamma'})$ for $a_1 \geq \cdots \geq a_{\gamma'} \geq 1$. Our goal is to show that $a_1 = n - 2\gamma + 1$, while $a_2 = \cdots = a_{\gamma'} = 1$.

For this purpose we will use the concept of edge rotations introduced in [15]. Let $G = (V, E)$ be a connected graph with a Perron vector $x$. If, for vertices $r, s, t \in V$, it holds $rs \in E$, $rt \not\in E$ and $x_s \leq x_t$, then the rotation of an edge $rs$ into $rt$, meaning a deletion of an edge $rs$ followed by addition of an edge $rt$, strictly increases the index of $G$. We have

$$\rho(G - rs + rt) \geq \frac{x^T A(G - rs + rt) x}{x^T x} = \frac{x^T A(G) x + 2x_r (x_t - x_s)}{x^T x} \geq \rho(G).$$

However, the equality $\rho(G - rs + rt) = \rho(G)$ cannot hold. In such a case, one would have that $x_s = x_t$ and that $x$ is a Perron vector of $G - rs + rt$. The eigenvalue equations at $s$ in graphs $G$ and $G - rs + rt$ would then give

$$\rho(G)x_s = \sum_{u:us \in E} x_u,$$

$$\rho(G)x_s = \rho(G - rs + rt)x_s = -x_r + \sum_{u:us \in E} x_u$$

implying that $x_r = 0$, which is a contradiction. Thus, the strict inequality holds

$$\rho(G - rs + rt) > \rho(G).$$

Back to $G^*$, let $x^*$ be the Perron vector of $G^*$. Let $S = \{s_1, \ldots, s_{\gamma'}\}$ be the independent set of $G^*$ such that, for $1 \leq i \leq \gamma'$, the vertex $s_i$ has $a_i$ neighbors in the clique $K$ of $G^*$. Suppose that there exist vertices $s_i, s_j \in S$ such that $a_i, a_j \geq 2$, and without loss of generality, suppose that $x^*_{s_i} \leq x^*_{s_j}$. Let $y$ be an arbitrary vertex adjacent to $s_i$. By rotating the edge $ys_i$ to $ys_j$, we get that

$$\rho(G^* - ys_i + ys_j) > \rho(G^*).$$

However, this is contradiction, as the connected graph

$$G^* - ys_i + ys_j \cong SSG(n, \gamma; a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_{\gamma'})$$

also has domination number $\gamma$.

Thus, at most one number among $a_1, \ldots, a_{\gamma'}$ may be larger than one. This shows that $G^* \cong S^* \cong SSG(n, \gamma; n - 2\gamma + 1, 1, \ldots, 1)$. □

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References