On the independence polynomial of an antiregular graph

Vadim E. Levit  
Ariel University Center of Samaria, Israel  
levitv@ariel.ac.il

Eugen Mandrescu  
Holon Institute of Technology, Israel  
eugen.m@hit.ac.il

Abstract

A graph with at most two vertices of the same degree is called antiregular [25], maximally nonregular [32], or quasiperfect [2]. If $s_k$ is the number of independent sets of cardinality $k$ in a graph $G$, then

$$I(G; x) = s_0 + s_1 x + ... + s_\alpha x^\alpha$$

is the independence polynomial of $G$ [10], where $\alpha = \alpha(G)$ is the size of a maximum independent set.

In this paper we derive closed formulae for the independence polynomials of antiregular graphs. In particular, we deduce that every antiregular graph $A$ is uniquely defined by its independence polynomial $I(A; x)$, within the family of threshold graphs. Moreover, $I(A; x)$ is log-concave with at most two real roots, and $I(A; -1) \in \{ -1, 0 \}$.

Keywords: independent set, independence polynomial, antiregular graph, threshold graph.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The neighborhood of a vertex $v \in V$ is the set $N_G(v) = \{ w : w \in V \text{ and } vw \in E \}$, and $N_G[v] = N_G(v) \cup \{ v \}$.

if there is ambiguity on $G$, we use $N(v)$ and $N [v]$, respectively. If $|N(v)| = 1$, then $v$ is a pendant vertex of $G$ and $v$ is a simplicial vertex if $G[N[v]]$ is a complete graph. A maximal clique containing at least simplicial vertex is called a simplex. By simp($G$) we mean the set of all simplicial vertices of $G$. A graph $G$ is said to be simplicial if every vertex of $G$ is simplicial or is adjacent to a simplicial vertex [5].

$G$ stands for the complement of $G$. $K_n$, $K_{m,n}$, $P_n$ denote the complete graph on $n \geq 1$ vertices, the complete bipartite graph on $m, n \geq 1$ vertices, and the chordless path on $n \geq 1$ vertices, respectively.

The disjoint union of the graphs $G_1, G_2$ is the graph $G = G_1 \cup G_2$ having the disjoint union of $V(G_1), V(G_2)$ as a vertex set, and the disjoint union of $E(G_1), E(G_2)$ as an edge set. In particular, $nG$ denotes the disjoint union of $n > 1$ copies of the graph $G$. 

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If $G_1, G_2$ are disjoint graphs, then their Zykov sum is the graph $G_1 + G_2$ with $V(G_1) \cup V(G_2)$ as a vertex set and $E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$ as an edge set.

A set of pairwise non-adjacent vertices is called independent. If $S$ is an independent set, then we denote $N(S) = \{v : N(v) \cap S \neq \emptyset\}$ and $N[S] = N(S) \cup S$.

An independent set of maximum size will be referred to as a maximum independent set of $G$. The independence number of $G$, denoted by $\alpha(G)$, is the number of vertices of a maximum independent set in $G$.

A matching is a set of non-incident edges of $G$, and $\mu(G)$ is the maximum cardinality of a matching. $G$ is a König-Egerváry graph provided $\alpha(G) + \mu(G) = |V(G)|$, [7, 30].

Let $s_k$ be the number of independent sets in $G$ of cardinality $k \in \{0, 1, ..., \alpha(G)\}$. The polynomial

$$I(G; x) = s_0 + s_1x + s_2x^2 + ... + s_{\alpha}x^\alpha, \alpha = \alpha(G),$$

is called the independence polynomial of $G$ (Gutman and Harary [10]).

For a survey on independence polynomials the reader is referred to [17].

![Figure 1: $G_2$ is the line-graph of $G_1$.](image)

The independence polynomial is a generalization of the matching polynomial [10], because the matching polynomial of a graph and the independence polynomial of its line graph are identical. Recall that given a graph $G$, its line graph $L(G)$ is the graph whose vertex set is the edge set of $G$, and two vertices are adjacent if they share an end vertex in $G$. For instance, the graphs $G_1$ and $G_2$ depicted in Figure 1 satisfy $G_2 = L(G_1)$ and, hence,

$$I(G_2; x) = 1 + 6x + 7x^2 + x^3 = M(G_1; x),$$

where $M(G_1; x)$ is the matching polynomial of the graph $G_1$. In [10] a number of general properties of the independence polynomial of a graph are presented. As examples, we mention that:

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x), \quad I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$  

The following equality, due to Gutman and Harary [10], is very useful in calculating the independence polynomial for various families of graphs.

**Proposition 1.1** [10] If $w \in V(G)$, then $I(G; x) = I(G - w; x) + x \cdot I(G - N[w]; x)$.

A finite sequence of real numbers $(a_0, a_1, a_2, ..., a_n)$ is said to be:

- **unimodal** if there is some $k \in \{0, 1, ..., n\}$, called the mode of the sequence, such that $a_0 \leq ... \leq a_{k-1} \leq a_k \geq a_{k+1} \geq ... \geq a_n$;

- **log-concave** if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ for $i \in \{1, 2, ..., n - 1\}$.  

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It is known that every log-concave sequence of positive numbers is unimodal as well. A polynomial is called \emph{unimodal (log-concave)} if the sequence of its coefficients is unimodal (log-concave, respectively). The product of two unimodal polynomials is not always unimodal, even if they are independence polynomials; e.g.,

\[ I(K_{100} + 3K_7; x) = 1 + 121x + 147x^2 + 343x^3 \]

and

\[ I(K_{120} + 3K_7; x) = 1 + 141x + 147x^2 + 343x^3, \]

while their product is not unimodal:

\[ 1 + 262x + 17355x^2 + 39200x^3 + 111475x^4 + 100842x^5 + 117649x^6. \]

Theorem 1.2 \cite{12} If \( P, Q \) are polynomials, such that \( P \) is log-concave and \( Q \) is unimodal, then \( P \cdot Q \) is unimodal, while the product of two log-concave polynomials is log-concave.

Alavi, Malde, Schwenk and Erdős \cite{1} proved that for any permutation \( \pi \) of \( \{ 1, 2, ..., \alpha \} \) there is a graph \( G \) with \( \alpha(G) = \alpha \), such that the coefficients of \( I(G; x) \) satisfy

\[ s_{\pi(1)} < s_{\pi(2)} < s_{\pi(3)} < ... < s_{\pi(\alpha)}. \]

For instance, the independence polynomial:

- \( I(K_{42} + 3K_7; x) = 1 + 63x + 147x^2 + 343x^3 \) is log-concave;
- \( I(K_{43} + 3K_7; x) = 1 + 64x + 147x^2 + 343x^3 \) is unimodal, but non-log-concave, because \( 147 \cdot 147 - 64 \cdot 343 = -343 < 0 \);
- \( I(K_{127} + 3K_7; x) = 1 + 148x + 147x^2 + 343x^3 \) is non-unimodal.

Some more discussion on independence polynomials may be found in \cite{4, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23}.

Recall that \( G \) is a \emph{threshold} graph if it can be obtained from \( K_1 \) by iterating the operations of complementation and disjoint union with a new copy of \( K_1 \) in any order \cite{6}. In other words, \( G \) is a threshold graph if it can be obtained from \( K_1 \) by iterating the operations of adding in a new vertex which is connected to no other vertex (i.e., an \emph{isolated vertex}) or adding in a new vertex connected to every other vertex (i.e., a \emph{cone vertex}, or a \emph{universal vertex}, or a \emph{dominating vertex}). The sequence of operations which describes this process can be represented as a binary string, which we call the \emph{binary building string} of \( G \), where "0" means "adding an isolated vertex" and "1" corresponds to "adding a dominating vertex". Clearly, each such a string begins by a "0". For some examples see Figure 2, where the vertex \( v_i \) is dominating if and only if the \( i^{th} \) bit in the binary building string equals "1". Chvatal and Hammer \cite{6} showed that threshold graphs are exactly the graphs having no induced subgraph isomorphic to either a \( P_4 \), or a \( C_4 \), or a \( 2K_2 \).

Following Hoede and Li \cite{11}, \( G \) is called a \emph{clique-unique graph} if the relation \( I(G; x) = I(H; x) \) implies that \( G \) and \( H \) are isomorphic (or, equivalently, \( G \) and \( H \) are isomorphic). One of the problems they proposed was to determine clique-unique graphs (Problem 4.1, \cite{11}). In \cite{23} it is proved that spiders are \emph{independence-unique graphs} within the family of well-covered trees, i.e., they are uniquely defined by their independence polynomials in the context of well-covered trees. The following result, due to Stevanović, says that every threshold graph is completely determined by its independence polynomial within the class of threshold graphs.
Theorem 1.3 [31] Two threshold graphs have the same independence polynomial if and only if they are isomorphic.

It is well-known that every graph of order at least two has at least two vertices of the same degree. A graph having at most two vertices of the same degree is called antiregular [24, 25], maximally nonregular [32] or quasiperfect [2, 27, 29]. Some examples of antiregular graphs are presented in Figure 3.

Figure 3: Antiregular graphs: $A_1, A_2, A_3, A_4$ and $A_{n+2} = K_1 + (K_1 \cup A_n)$.

It is intuitively clear that the number of different antiregular graphs of the same order is quite small.

Theorem 1.4 [2] For every positive integer $n \geq 2$ there is a unique connected antiregular graph of order $n$, denoted by $A_n$, and a unique non-connected antiregular graph of order $n$, namely $\overline{A_n}$.

Moreover, antiregular graphs enjoy a very specific recursive structure.

Theorem 1.5 [25] The antiregular graphs can be defined by the following recurrences:

$$A_1 = K_1, A_{n+1} = K_1 + \overline{A_n}, n \geq 1, \text{ or}$$

$$A_1 = K_1, A_2 = K_2, A_{n+2} = K_1 + (K_1 \cup A_n), n \geq 1.$$

Characteristic, admittance and matching polynomials of antiregular graphs were studied in [26].

In this paper we present closed formulae for the independence polynomial of an antiregular graph. We also show that independence polynomials of antiregular graphs are log-concave. Moreover, it turns out that antiregular graphs are completely determined by their independence polynomials within the class of threshold graphs.
2 Results

Antiregular graphs have a number of nice properties. Some of them are presented in [2, 24, 25, 27, 29].

Theorem 2.1 Every antiregular graph is threshold, simplicial, and a König-Egerváry graph.

Proof.

• The building binary strings of the forms 001010101... and 01010101... give rise to antiregular graphs. Hence, any antiregular graph is threshold.

• We prove that every antiregular graph is simplicial, by induction on \( n \). Clearly, \( A_1 = K_1 \), \( A_2 = K_2 \) are simplicial graphs. Since, by Theorem 1.3, we have \( A_3 = K_1 + (K_1 \cup K_1) \), it is easy to see that \( A_3 = P_3 \) and, consequently, \( A_3 \) is a simplicial graph.

Assume that the assertion is true for \( 2 \leq m \leq n + 1 \).

Since \( A_{n+2} = K_1 + (K_1 \cup A_n), n \geq 1 \), it follows that \( A_{n+2} \) has a vertex of degree one, say \( v_{n+2} \) and a vertex of degree \( n + 1 \), say \( v_{n+1} \). Clearly, \( v_{n+2} \) is a simplicial vertex, as it is a pendant vertex, while \( v_{n+1} \) is not a simplicial one. On the other hand, if \( v \in \text{simp}(A_n) \), then \( N_{A_n}[v] \) induces a complete subgraph of \( A_n \). Hence, we infer that

\[
N_{A_{n+2}}[v] = N_{A_n}[v] \cup \{v_{n+1}\}
\]

induces a complete subgraph of \( A_{n+2} \), since \( N_{A_{n+2}}[v_{n+1}] = V(A_{n+2}) - \{v_{n+1}\} \). Consequently, it follows that

\[
\text{simp}(A_{n+2}) = \text{simp}(A_n) \cup \{v_{n+2}\}.
\]

In other words, each vertex of \( A_{n+2} \) is either a simplicial vertex or it is adjacent to a simplicial vertex. Therefore, \( A_n \) is a simplicial graph, for every \( n \geq 1 \). Since, clearly, \( A_n = K_1 + A_{n-1} = K_1 \cup A_{n-1} = K_1 \cup A_{n-1} \), we get that \( A_n, n \geq 1 \), is a simplicial graph, too.

• It is easy to see that the independence number \( \alpha(A_n) \) is equal to \( \left\lceil \frac{n}{2} \right\rceil \). On the other hand, using the fact that \( A_n = K_1 + (K_1 \cup A_{n-2}), n \geq 3 \), one can easily see that its matching number \( \mu(A_n) \) equals \( \left\lceil \frac{n}{2} \right\rceil \). Since \( \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n \), the graph \( A_n \) is a König-Egerváry graph.

Using the recursive structure of antiregular graphs we get the following.

Lemma 2.2 \( I(A_{n+2}; x) = (1 + x) \bullet (1 + I(A_n; x)) - 1 \) holds for every \( n \geq 1 \).

Proof. Clearly, \( I(A_1; x) = I(K_1; x) = 1 + x \), while \( I(A_2; x) = I(K_2; x) = 1 + 2x \). Further, according to Proposition 1.1 and Theorem 1.3, we infer that:

\[
I(A_{n+2}; x) = I(K_1 + (K_1 \cup A_{n}); x) = I(K_1; x) + I(K_1 \cup A_n; x) - 1 = I(K_1; x) + I(K_1; x) \bullet I(A_n; x) - 1 = (1 + x) \bullet (1 + I(A_n; x)) - 1,
\]

as required. \( \blacksquare \)
Using Lemma 2.2, one can easily compute the following independence polynomials:

\[ I(A_3; x) = 1 + 3x + x^2, \quad I(A_4; x) = 1 + 4x + 2x^2 \]
\[ I(A_5; x) = 1 + 5x + 4x^2 + x^3, \quad I(A_6; x) = 1 + 6x + 6x^2 + 2x^3 \]
\[ I(A_7; x) = 1 + 7x + 9x^2 + 5x^3 + x^4, \quad I(A_8; x) = 1 + 8x + 12x^2 + 8x^3 + 2x^4. \]

**Theorem 2.3** The independence polynomial of \( A_n \) is:

\[ I(A_{2k-1}; x) = (1 + x)^k + (1 + x)^{k-1} - 1, \quad k \geq 1, \]
\[ I(A_{2k}; x) = 2 \cdot (1 + x)^k - 1, \quad k \geq 1, \]

The independence polynomial of \( \overline{A_n} \) is

\[ I(\overline{A_n}; x) = (1 + x) \cdot I(\overline{A_{n-1}}; x), \quad n \geq 2, \]

and, consequently,

\[ I(\overline{A_{2k-1}}; x) = 2 \cdot (1 + x)^k - x - 1, \quad k \geq 1, \]
\[ I(\overline{A_{2k}}; x) = (1 + x)^{k+1} + (1 + x)^k - x - 1, \quad k \geq 1. \]

**Proof.** We prove the formulae for \( I(A_n; x) \) by induction on \( n \).
If \( n = 1 \) or \( 2 \), then

\[ I(A_1; x) = I(K_1; x) = 1 + x = (1 + x)^1 + (1 + x)^0 - 1 \]

and

\[ I(A_2; x) = I(K_2; x) = 1 + 2x = 2 \cdot (1 + x)^1 - 1. \]

Assume that the formulae are true for \( 2 \leq m \leq n \).

Let \( n + 1 = 2k + 1 \). Then, by Lemma 2.2 and induction hypothesis, we obtain:

\[ I(A_{n+1}; x) = I(A_{2k+1}; x) = (1 + x) \cdot (1 + I(A_{2k-1}; x)) - 1 = \]
\[ = (1 + x) \cdot (1 + (1 + x)^k + (1 + x)^{k-1} - 1) - 1 = \]
\[ = (1 + x)^{k+1} + (1 + x)^k - 1. \]

Let \( n + 1 = 2k \). Then again, using Lemma 2.2 and the induction hypothesis, we get:

\[ I(A_{n+1}; x) = I(A_{2k}; x) = (1 + x) \cdot (1 + I(A_{2k-2}; x)) - 1 = \]
\[ = (1 + x) \cdot (1 + 2 \cdot (1 + x)^{k-1} - 1) - 1 = \]
\[ = 2 \cdot (1 + x)^k - 1. \]

In conclusion, both formulae are true.

According to Theorem 1.3, we have \( A_n = K_1 + \overline{A_{n-1}} \), which implies

\[ \overline{A_n} = K_1 + \overline{A_{n-1}} = K_1 \cup \overline{A_{n-1}} = K_1 \cup A_{n-1}, \]

and hence, we deduce that \( I(\overline{A_n}; x) = (1 + x) \cdot I(A_{n-1}; x) \). Further, using the result obtained for \( I(A_{n-1}; x) \), we finally infer the closed formulae for \( I(\overline{A_n}; x) \), as claimed.
The Fibonacci number of a graph $G$ is the number of all its independent sets. Obviously, the Fibonacci number of $G$ is equal to

$$I(G; 1) = s_0 + s_1 + s_2 + \ldots + s_{\alpha(G)},$$

where

$$I(G; x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_{\alpha(G)} x^{\alpha(G)}$$

is the independence polynomial of $G$. Using Theorem 2.3, we immediately obtain the following.

**Corollary 2.4** The Fibonacci numbers of $A_n$ are:

$$I(A_{2k-1}; 1) = 3 \cdot 2^{k-1} - 1 \quad \text{and} \quad I(A_{2k}; 1) = 2^{k+1} - 1, \quad k \geq 1,$$

while the Fibonacci numbers of $\overline{A_n}$ are:

$$I(\overline{A_{2k-1}}; 1) = 2^{k+1} - 2 \quad \text{and} \quad I(\overline{A_{2k}}; 1) = 3 \cdot 2^k - 2, \quad k \geq 1.$$

If $G$ has $s_k$ independent sets of size $k$, then

$$s_0 - s_1 + s_2 - s_3 + \ldots + (-1)^{\alpha(G)} s_{\alpha(G)}$$

is called the alternating number of independent sets of $G$. Evidently,

$$I(G; -1) = s_0 - s_1 + s_2 - s_3 + s_4 - \ldots + (-1)^{\alpha(G)} s_{\alpha(G)}$$

$$= (s_0 + s_2 + s_4 + \ldots) - (s_1 + s_3 + s_5 + \ldots).$$

In addition, if we denote

$$\text{even}(G) = s_0 + s_2 + s_4 + \ldots,$$

$$\text{odd}(G) = s_1 + s_3 + s_5 + \ldots,$$

then we may conclude that:

"the alternating number of independent sets of $G" = I(G; -1) = \text{even}(G) - \text{odd}(G)."$

Let us notice that the difference $|\text{even}(G) - \text{odd}(G)|$ can be indefinitely large. For instance, $I(K_n; x) = 1 + nx$ and hence, $I(K_n; -1) = 1 - n \leq 0$. On the other hand, the graph $H = (K_m \cup K_n) + K_1$ has

$$I(H; x) = (1 + mx)(1 + nx) + x$$

and, consequently,

$$I(H; -1) = (1 - m)(1 - n) - 1 > 0, \quad m > 2, \quad n > 1.$$

**Corollary 2.5** For a connected antiregular graph the number of independent sets of odd size is greater by one than the number of independent sets of even size, while for a disconnected antiregular graph, the two numbers are equal.
Proof. Let denote by \( \alpha \) the independence number of \( A_n \), i.e., \( \alpha = \alpha(A_n) = \lceil n/2 \rceil \), and 
\[ I(A_n; x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_n x^n. \]

According to Theorem 2.3, it follows that \( I(A_n; -1) = -1 \), which clearly implies
\[ (s_0 + s_2 + s_4 + \ldots) + 1 = s_1 + s_3 + s_5 + \ldots, \]
i.e., even\((G) + 1 = \text{odd}(G)\), as required.

Similarly, in accordance with Theorem 2.3, we obtain that \( I(A_n; -1) = 0 \), which ensures that
\[ s_0 + s_2 + s_4 + \ldots = s_1 + s_3 + s_5 + \ldots, \]
i.e., even\((G) = \text{odd}(G)\), and this completes the proof.

Let us notice that there are threshold graphs, whose independence polynomials are

- non-unimodal, e.g., \( G = 6K_1 + K_{10} \), whose binary building string is \( 6[0]10[1] \) and independence polynomial is
  \[ I(G; x) = (1 + x)^6 + 10x = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 16x + 1; \]

- unimodal, but non-log-concave, e.g., \( G = 3K_1 + K_7 \), whose binary building string is \( 3[0]7[1] \) and independence polynomial is
  \[ I(G; x) = (1 + x)^3 + 7x = x^3 + 3x^2 + 10x + 1; \]

- log-concave, e.g., \( G = 7K_1 + K_5 \), whose binary building string is \( 7[0]5[1] \) and independence polynomial is
  \[ I(G; x) = (1 + x)^7 + 5x = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 12x + 1. \]

Corollary 2.6 The independence polynomials of \( A_n \) and \( \overline{A_n} \) are log-concave, for every integer \( n \geq 1 \).

Proof. According to Theorem 2.3, \( I(A_{2k}; x) = 2 \cdot (1 + x)^k - 1 \) and hence, \( I(A_{2k-1}; x) \) is
log-concave, because \((1 + x)^k\) is log-concave. The polynomial
\[ I(A_{2k-1}; x) = (1 + x)^k + (1 + x)^{k-1} - 1 = (1 + x)^{k-1} (2 + x) - 1 \]
is log-concave, since the product of two log-concave polynomials is again log-concave, by Theorem 1.2.

Similarly, \( I(\overline{A_n}; x) \) is log-concave, because \( I(\overline{A_n}; x) = (1 + x) \cdot I(A_{n-1}; x) \).

Figure 4: Non-isomorphic trees with the same independence polynomial.

Let us mention that there are non-isomorphic graphs with the same independence polynomial. For instance, Dohmen, Pönitz and Tittmann [8] have found two non-isomorphic trees (Figure 4) having the same independence polynomial, namely,
\[ I(T_1; x) = I(T_2; x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 12x^5 + x^6. \]
Let us notice that $I(A_{2k}; x) = I(K_{k,k}; x)$ and $I(A_{2k-1}; x) = I(K_{k,k-1}; x)$. For $k \geq 3$ neither $K_{k,k}$ nor $K_{k,k-1}$ is a threshold graph, because they contain an induced subgraph isomorphic to $C_4$.

Figure 5: Non-isomorphic graphs having $I(K_{2,3}; x) = I(A_5; x) = 1 + 5x + 4x^2 + x^3$.

Using Theorems 1.3 and 2.1, we infer the following.

**Corollary 2.7** Every antiregular graph is a unique-independence graph within the family of threshold graphs, i.e., if a threshold graph $G$ has $I(G; x) = I(A_n; x)$ or $I(G; x) = I(A_n^{-}; x)$, then $G$ is isomorphic to $A_n$ or $A_n^{-}$, respectively.

It is known that the independence polynomial has at least one real root [9].

**Corollary 2.8** The polynomial $I(A_{2k}; x)$ has only one real root for every odd $k$ and exactly two real roots for each even $k$. If $k$ is odd, then the only real root $x_0 = -1 + \frac{1}{\sqrt{2}}$ belongs to $(-1,0)$. If $k$ is even, then the only real roots $x_{1,2} = -1 \pm \frac{1}{\sqrt{2}}$, while $x_1 \in (-2,-1)$ and $x_2 \in (-1,0)$.

**Corollary 2.9** The polynomial $I(A_{2k-1}; x)$ has only one real root for every odd $k$, and exactly two real roots for each even $k$. If $k$ is odd, then the only one real root belongs to $(-1,0)$. If $k$ is even, then one root belongs to $(-3,-2)$, and the other belongs to $(-1,0)$.

Theorem 2.3 claims that

\[ I(A_n^{-}; x) = (1 + x) \cdot I(A_{n-1}; x), n \geq 2. \]

Therefore, the set of roots of $I(A_n^{-}; x)$ is the union of the set of roots of $I(A_{n-1}; x)$ and $\{-1\}$.

**3 Conclusions**

It is amusing, but antiregular graphs are, actually, very “regular”. One can easily see their pattern, when antiregular graphs are considered in the context of threshold graphs. Namely, their building binary strings are of the forms $001010101\ldots$ and $01010101\ldots$. Let us define a $(a,b)$-pattern graph as a graph with the building binary string of the form $a_1a_2\ldots a_q (b_1b_2\ldots b_p)$, where all $a_i, b_j \in \{0,1\}$, and the sequence $b_1b_2\ldots b_p$ is a periodic part of the string. For instance, connected antiregular graphs may be described as 0(01)-pattern or (01)-pattern graphs. This definition opens an interesting research program. Its main goal is to recognize such patterns $(a,b)$ that ensure for the independence polynomial of the corresponding $(a,b)$-pattern graphs to be unimodal, log-concave, or even to have only real roots.
References


