On the Laplacian Spectral Radius of Trees

TAN Shang-wang

(Department of Applied Mathematics, China University of Petroleum, Dongying 257061, China)

Abstract: Some sharp upper bounds of Laplacian spectral radius of trees in terms of order, diameter, pendant vertex number, covering number, edge covering number or total independence number are given. And the ninth to thirteenth largest values of Laplacian spectral radius over the class of trees on a given order are also given.

Key words: Laplacian spectral radius; tree; diameter

2000 MR Subject Classification: 05C50

CLC number: O157.5 Document code: A


§1. Introduction

Let $G$ be a simple graph on order $n$. Its adjacency matrix and diagonal matrix of vertex degrees are denoted by $A(G)$ and $D(G)$, respectively. Then the Laplacian matrix $L(G)$ of $G$ is defined to be $L(G) = D(G) - A(G)$. The Laplacian spectral radius of $G$ is the largest eigenvalue of $L(G)$, denoted by $\mu(G)$.

Let $\text{dia}(G)$ be the diameter of $G$ and $P_k$ be a path with order $k$. A pendant vertex of $G$ is a vertex of degree 1. A path $wv_1v_2\cdots v_k$ of $G$ is called a pendant path at the vertex $w$ with length $k$ if $d_G(w) \geq 3, d_G(v_k) = 1$ and $d_G(v_i) = 2$ for $1 \leq i \leq k - 1$. If graphs $G$ and $H$ are isomorphic, we write $G \cong H$, and $G \not\cong H$ otherwise.

Many authors have done a lot of work on the Laplacian spectral radius of graphs (see [1]). For some special types of graphs, the sharp upper bounds of Laplacian spectral radius of graphs have been obtained. For example, the first eight largest values of Laplacian spectral radius of trees in terms of order were obtained in [2-4]; the sharp upper bounds of Laplacian spectral radius of trees in terms of order and independence number were given in [5]; the sharp upper bounds of Laplacian spectral radius of trees and unicyclic graphs in terms of order and edge...
independence number were given in [3, 6]; the sharp upper bounds of Laplacian spectral radius of trees in terms of order and pendant vertex number were obtained in [7-8].

This paper has two purposes. In Section 2 we will present some sharp upper bounds of Laplacian spectral radius of trees in terms of order, diameter, pendant vertex number, covering number, edge covering number or total independence number. In Section 3 we will give the ninth to thirteenth largest values of Laplacian spectral radius over the class of trees on a given order. Now we give the following two results used in this paper.

**Lemma 1.1** Let $w$ be a vertex in a non-trivial connected bipartite graph $G$. Suppose that two paths with lengths $k$ and $s$ are attached to $G$ at $v$, respectively, to form $G_{k,s}$. Then

$$\mu(\text{G}_{k+1,s-1}) < \mu(G_{k,s})$$

if $k \geq s \geq 1$.

**Lemma 1.2** Let $u, v$ be two vertices of a connected bipartite graph $G$. Suppose

$$\{v_1, v_2, \cdots, v_p\} \subseteq N_G(v) \setminus (N_G(u) \bigcup \{u\}), \text{ where } p \geq 1.$$ 

Let $x$ be a unit eigenvector of $L(G)$ corresponding to $\mu(G)$ and $x_z$ be the coordinate corresponding to the vertex $z$ in $x$. Then $\mu(G) < \mu(G(v \rightarrow u))$ if $G(v \rightarrow u)$ is also a bipartite graph and $|x_u| \geq |x_v|$.

The procedure from $G_{k+1,s-1}$ to $G_{k,s}$ is called the first edge transformation, for short 1.e.t. The producer from $G$ to $G(v \rightarrow u)$ is called the second edge transformation, for short 2.e.t.

### §2. On the Laplacian Spectral Radius of Trees with Some Given Invariants

In this section, we investigate some sharp upper bounds of Laplacian spectral radius of trees in terms of order, diameter, pendant vertex number, covering number, edge covering number or total independence number.

For the positive integers $n, d$ and $s$ such that $3 \leq s \leq n - d + 1$, it is obvious that there are two non-negative integers $l$ and $k$ such that $n - d - 1 = l(s - 2) + k$, where $0 \leq k \leq s - 3$. Let $T_{n,d,s}$ be the tree with order $n$, diameter $d$ and pendant vertex number $s$ shown in Fig. 1 and $S(m,k;t,l)$ be the tree shown in Fig. 2.
Let $v$ be a vertex in a tree $T$ such that $m = d_T(v) \geq 3$ and $N_T(v) = \{v_1, v_2, \cdots, v_m\}$. Denote the component containing $v_i$ in $T - v$ by $T_i$. $v$ is called an end branch vertex of $T$ if at most one among $T_1 + v_1v$, $T_2 + v_2v$, \cdots, $T_m + v_mv$ is not a path.

**Theorem 2.1** Let $T$ be a tree with order $n$, diameter $d$ and pendant vertex number $s$. Then $\mu(T) \leq \mu(T_{n,d,s})$, with equality if and only if $T \cong T_{n,d,s}$.

**Proof** Let $t$ be the number of vertices with degree no less than 3 in $T$.

**Case 1** $t = 0$. Obviously, $T \cong P_n$, i.e., $T \cong T_{n,n-1,2}$. So $\mu(T) = \mu(T_{n,n-1,2})$.

**Case 2** $t = 1$. Obviously, the vertex with degree no less than 3 is unique. To $T$ repeat using l.e.t, $T$ can be transformed into $T_{n,d,s}$. By Lemma 1.1, we have $\mu(T) \leq \mu(T_{n,d,s})$, and with equality if and only if $T \cong T_{n,d,s}$.

**Case 3** $t \geq 2$. It is obvious that $T$ has at least two end branch vertices, so $T \not\cong T_{n,d,s}$. Let $v$ and $u$ be two end branch vertices of $T$. Since $T$ is a tree, there is the unique path $P$ between $v$ and $u$. Denote the unique adjacent vertex of $v$ by $v'$ and the unique adjacent vertex of $u$ by $u'$ in $P$. Let $T_v$ be the component containing $v$ in $T - vv'$, $v''$ the adjacent vertex of $v$ in the longest path with an end vertex $v$ in $T_v$ and $N_T(v) = \{v_1, v_2, \cdots, v_{d_T(v)-2}, v', v''\}$. Let $T_u$ be the component containing $u$ in $T - uu'$, $uu''$ the adjacent vertex of $u$ in the longest path with an end vertex $u$ in $T_u$ and $N_T(u) = \{u_1, u_2, \cdots, u_{d_T(u)-2}, u', u''\}$. Set

$$T(v \rightarrow u) = T - vv_1 - vv_2 - \cdots - vv_{d_T(v)-2} + uv_1 + uv_2 + \cdots + uv_{d_T(v)-2},$$

$$T(u \rightarrow v) = T - uu_1 - uu_2 - \cdots - uu_{d_T(u)-2} + vu_1 + vu_2 + \cdots + vu_{d_T(u)-2}.$$  

Then $T(v \rightarrow u)$ and $T(u \rightarrow v)$ are two new trees with pendant vertex number $s$ and diameter no less dia($T$). By Lemma 1.2, we have $\mu(T) < \mu(T(v \rightarrow u))$ or $\mu(T) < \mu(T(u \rightarrow v))$.

We write $T_1^* = T(v \rightarrow u)$ if $\mu(T) < \mu(T(v \rightarrow u))$, $T_1^* = T(u \rightarrow v)$ otherwise. Then $T_1^*$ contains $t - 1$ vertices with degree no less than 3. If $t - 1 \geq 2$, then to $T_1^*$ repeat the above steps until the number is only one. So we get trees $T_1^*, T_2^*, \cdots, T_{t-1}^*$ such that

$$\mu(T) < \mu(T_1^*) < \mu(T_2^*) < \cdots < \mu(T_{t-1}^*),$$

$$\text{dia}(T) \leq \text{dia}(T_1^*) \leq \text{dia}(T_2^*) \leq \cdots \leq \text{dia}(T_{t-1}^*).$$

Moreover, each $T_i^*$ has pendant vertex number $s(1 \leq i \leq t - 1)$. For $T_{t-1}^*$, repeat using l.e.t., it can be transformed into a tree $T_t^*$ such that it has diameter $d$, pendant vertex number $s$ and the unique vertex with degree no less than 3. By Lemma 1.1, we have $\mu(T_{t-1}^*) \leq \mu(T_t^*)$. Referring to Case 2, we have $\mu(T_t^*) \leq \mu(T_{n,d,s})$. Therefore, $\mu(T) < \mu(T_{n,d,s})$.

By the above three cases, the proof is completed.

**Corollary 2.2** Let $T$ be a tree with order $n$ and $\text{dia}(T) \geq d$. Then $\mu(T) \leq \mu(T_{n,d,n-d+1})$.
with equality if and only if \( T \cong T_{n,d,n-d+1} \).

**Proof** Using 1.e.t, \( T_{n,d,s} \) can be transformed into \( T_{n,d-1,s} \) if \( d \geq 3 \) and \( T_{n,d,s} \) can be transformed into \( T_{n,d,s+1} \) if \( s \leq n-d \). So by Lemma 1.1, we get that

\[
\mu(T_{n,d,s}) < \mu(T_{n,d-1,s}), \quad \mu(T_{n,d,s}) < \mu(T_{n,d,s+1}).
\]

(1)

Now suppose \( T \not\cong T_{n,d,n-d+1} \), \( \text{dia}(T) = m \) and \( s \) is the number of pendant vertices of \( T \). Then \( m \geq d \) and \( s \leq n-d+1 \). From Theorem 2.1 and Eq. (1), we have

\[
\mu(T) \leq \mu(T_{n,m,s}) \leq \mu(T_{n,d,s}) \leq \mu(T_{n,d,n-d+1}),
\]

and at least one equality does not hold by \( T \not\cong T_{n,d,n-d+1} \). The proof is completed.

From Corollary 2.2, we immediately obtain the following corollary.

**Corollary 2.3** Let \( T \) be a tree with order \( n \) and diameter \( d \). Then

\[
\mu(T) \leq \mu(T_{n,d,n-d+1})
\]

with equality if and only if \( T \cong T_{n,d,n-d+1} \).

Let \( n \) and \( s \) be two fixed positive integers. Then there are two non-negative integers \( p \) and \( q \) such that

\[
n - 1 = ps + q, \quad \text{where} \quad 2 \leq s \leq n - 1, 0 \leq q \leq s - 1.
\]

For \( q = 0, q = 1, q \geq 2 \), let \( \delta(q) \) be \( 0, 1, 2 \), respectively. Then

\[
S(s-q,p;q,p+1) = T_{n,2p+\delta(q),s}.
\]

Therefore, by Theorem 2.1 and Lemma 1.1, we easily obtain the following corollary.

**Corollary 2.4**[7-8] Let \( T \) be a tree with order \( n \) and pendant vertex number \( s \). Then

\[
\mu(T) \leq \mu(S(s-q,p;q,p+1))
\]

with equality if and only if \( T \cong S(s-q,p;q,p+1) \).

**Lemma 2.5**[7-8] \( \mu(S(s-q,p;q,p+1)) \) is strictly increasing in \( s(2 \leq s \leq n - 2) \).

Let \( \alpha, \alpha', \beta, \beta' \) denote independence number, edge independence number, covering number, edge covering number of a bipartite graph, respectively. Then(see [10])

\[
\alpha + \beta = \alpha' + \beta' = n, \quad \alpha' = \beta, \quad \beta' = \alpha.
\]

(2)

**Lemma 2.6**[5] Let \( T \) be a tree with order \( n \) and independence number \( \alpha \). Then

\[
\mu(T) \leq \mu(S(2\alpha - n + 1,1;n - \alpha - 1,2))
\]

with equality if and only if \( T \cong S(2\alpha - n + 1,1;n - \alpha - 1,2) \).

By \( \beta' = \alpha \) and Lemma 2.6, we immediately obtain the following corollary.
Corollary 2.7 Let $T$ be a tree with order $n$ and edge covering number $\beta'$. Then
\[ \mu(T) \leq \mu(S(2\beta' - n + 1; n - \beta' - 1, 2)) \]
with equality if and only if $T \cong S(2\beta' - n + 1; n - \beta' - 1, 2)$.

Lemma 2.8\([3,6]\) Let $T$ be a tree with order $n$ and edge independence number $\alpha'$. Then
\[ \mu(T) \leq \mu(S(n - 2\alpha' + 1, 1; \alpha' - 1, 2)) \]
with equality if and only if $T \cong S(n - 2\alpha' + 1, 1; \alpha' - 1, 2)$.

By $\beta = \alpha'$ and Lemma 2.8, we immediately obtain the following corollary.

Corollary 2.9 Let $T$ be a tree with order $n$ and covering number $\beta$. Then
\[ \mu(T) \leq \mu(S(n - 2\beta + 1, 1; \beta - 1, 2)) \]
with equality if and only if $T \cong S(n - 2\beta + 1, 1; \beta - 1, 2)$.

Let $G = (V, E)$ be a simple graph and $A_t$ a subset of $V \cup E$. $A_t$ is called a total independent set of $G$ if its two arbitrary elements are neither adjacent nor incident. The number of elements in a total independent set of maximum cardinality is called the total independence number of $G$.

Corollary 2.10 Let $T$ be a tree with order $n$ and total independence number $\gamma$. Then
\[ \mu(T) \leq \mu(S(2\gamma - n + 1, 1; n - \gamma - 1, 2)) \]
with equality if and only if $T \cong S(2\gamma - n + 1, 1; n - \gamma - 1, 2)$.

Proof Suppose that $T$ has $s$ pendant vertices. Then $s \leq \alpha \leq \gamma$ by the definitions of independence number and total independence number. There are two non-negative integers $p$ and $q$ such that $n - 1 = ps + q$, where $0 \leq q \leq s - 1$. It is obvious that $\alpha' \leq \frac{n}{2}$ by the definition of edge independence number. Therefore, by Eq. (2), we have
\[ \gamma \geq \alpha = n - \beta = n - \alpha' \geq \frac{n}{2}. \]
So $\gamma = s = \gamma$, we have $p = 1$ and $q = n - \gamma - 1$. From Corollary 2.4 and Lemma 2.5, we get
\[ \mu(T) \leq \mu(S(s - q; p; q, p + 1)) \leq \mu(S(2\gamma - n + 1, 1; n - \gamma - 1, 2)). \quad (3) \]

Suppose that $\mu(T) = \mu(S(2\gamma - n + 1, 1; n - \gamma - 1, 2))$. By Eq. (3), we have
\[ \mu(T) = \mu(S(s - q; p; q, p + 1)) = \mu(S(2\gamma - n + 1, 1; n - \gamma - 1, 2)). \]
So by Corollary 2.4 and Lemma 2.5, we have $T \cong S(s - q; p; q, p + 1)$, and $s = \gamma, p = 1, q = n - \gamma - 1$. Hence, $T \cong S(2\gamma - n + 1, 1; n - \gamma - 1, 2)$. This completes the proof.
§3. On the Five New Upper Bounds of Laplacian Spectral Radius of Trees

In this section, we will investigate the ninth to thirteenth largest values of Laplacian spectral radius over the class of trees on a given order.

**Lemma 3.1**\(^1\) Let \(T\) be a tree on the largest degree \(\Delta(T)\). Then \(\mu(T) \geq \Delta(T) + 1\).

**Lemma 3.2** Let \(D_{s,t}\) be the tree obtained by joining two centers of \(K_{1,s}\) and \(K_{1,t}\) with an edge, where \(s + t = n - 2\). If \(n \geq 8\) and \(\frac{n-2}{2} \leq s \leq n - 3\), then
\[
\mu(D_{n-3,1}) > \mu(D_{n-4,2}) > \mu(D_{n-5,3}) > \cdots > \mu(D_{n,n-s-2}).
\]

**Proof** By an elementary calculation, we have
\[
\phi(D_{s,t}, \lambda) = \lambda(\lambda - 1)^{n-4}[\lambda^3 - (n + 2)\lambda^2 + (2n + 1 - s^2 + sn - 2s)\lambda - n].
\]
\[
\phi(D_{s,t}, \lambda) - \phi(D_{s+1,t-1}, \lambda) = (2s + 3 - n)\lambda^2(\lambda - 1)^{n-4}.
\]
So \(\phi(D_{s,t}, \lambda) > \phi(D_{s+1,t-1}, \lambda)\) for \(\frac{n-2}{2} \leq s \leq n - 3\) and \(\lambda \geq \mu(D_{s+1,t-1})\). This implies that \(\mu(D_{s,t}) < \mu(D_{s+1,t-1})\) for \(\frac{n-2}{2} \leq s \leq n - 3\). Hence this completes the proof.

**Lemma 3.3**\(^8\) Let \(u_0\) be a vertex in a non-trivial connected bipartite graph \(G\) and let \(G_{u_0}\) be the graph obtained from \(G\) and a path \(u_1u_2\cdots u_n\) by joining \(u_0\) and \(u_1\) with an edge. Suppose that \(x\) is a unit eigenvector of \(L(G_{u_0})\) corresponding to \(\mu(G_{u_0}) = \mu\) and \(x_{u_i}\) the coordinate corresponding to \(u_i\) in \(x\). Let
\[
a_1(y) = 1 - y, a_i(y) = 2 - y - \frac{1}{a_{i-1}(y)}, \quad i \geq 2.
\]
Then \(x_{u_i} = a_{s-i}(\mu)x_{u_{i+1}} \neq 0, \quad i = 0, 1, \ldots, s - 1\).

\[
H(m,t,n_1,\ldots,n_t,k)
\]

\[
G_{s,t,l,r}
\]

**Fig. 3**

**Fig. 4**

**Lemma 3.4** The tree \(H(m,t,n_1,\ldots,n_t,k)\) with order \(n\) and diameter 4 is shown in Fig. 3, where \(m \geq k \geq n_1 \geq \cdots \geq n_t \geq 1\). The tree \(G_{s,t,l,r}\) is shown in Fig. 4, where \(s \geq r\) and \(s + t + 2l + r = n - 5\). Let \(n \geq 10\) and \(H(m,t,n_1,\ldots,n_t,k) \notin J\), where
\[
J = \{G_{0,n-5,0,0}, G_{n-5,0,0,0}, G_{1,n-6,0,0}, G_{0,n-7,1,0}, G_{n-6,1,0,0}, G_{n-6,0,0,1}, G_{2,n-7,0,0}, G_{n-7,2,0,0}, G_{1,n-7,0,1}\}.
\]
Then $\mu(H(m, t, n_1, \cdots, n_l, k)) < \mu(G_{n-7,2,0,0})$.

**Proof** Using 2.e.t, $H(m, t, n_1, \cdots, n_l, k)$ can be transformed into

$$H(m + \sum_{i=1}^{l} (n_i - 1), t, 1, \cdots, 1, k) = G_{s,t,l,r},$$

where $s = -1 + m + \sum_{i=1}^{l} (n_i - 1)$ and $r = k - 1$. By Lemma 1.2, we have

$$\mu(H(m, t, n_1, \cdots, n_l, k)) \leq \mu(G_{s,t,l,r}).$$

(4)

Let $f(s) = \lambda^2 - (s + 3)\lambda + 1$. By an elementary calculation, we have

$$\phi(G_{s,t,l,r}, \lambda) = (\lambda - 1)^s + t + r - (\lambda^2 - 3\lambda + 1)^{t-1} \psi,$$

where

$$\psi = f(s)f(r) \left( [f(0)f(l + t) + 1] - l\lambda \right) - (\lambda - 1)^2 f(0)[f(r) + f(s)].$$

(5)

From $H(m, t, n_1, \cdots, n_l, k) \not\in J$, we have $G_{s,t,l,r} \not\in J$. Let $x$ be a unit eigenvector of $L(G_{s,t,l,r})$ corresponding to $\mu(G_{s,t,l,r}) = \mu$ and $x_w$ be the coordinate corresponding to the vertex $w$ in $x$. By Lemma 3.3, we have

$$x_{w_i} = \frac{1}{a_2(\mu)} x_v = \frac{1 - \mu}{\mu^2 - 3\mu + 1} x_v, \quad i = 1, 2, \cdots, l.$$

$$x_{u_i} = \frac{1}{a_1(\mu)} x_u = \frac{1 - \mu}{1 - \mu} x_u, \quad i = 0, 1, 2, \cdots, s.$$  (6)

By $L(G_{s,t,l,r})x = \mu x$, we have

$$\mu x_u = (s + 2)x_u - x_v - \sum_{i=0}^{s} x_{u_i}.$$  

So by Eq. (6), $x_u = \frac{1 - \mu}{\mu^2 - (s + 3)\mu + 1} x_v$. In the similar way, we have $x_z = \frac{1 - \mu}{\mu_2 - (r + 3)\mu + 1} x_v$. It is clear that $|x_u| \geq |x_z| \geq |x_{w_i}|$.

**Case 1** $l = 0$.

**Case 1.1** $t \neq 0$.

**Case 1.1.1** $r = 0$. By $G_{s,t,l,r} \not\in J$, we have $3 \leq s \leq n - 8$. Since $\Delta(G_{n-7,2,0,0}) \geq 5$, by Lemma 3.1, $\mu(G_{n-7,2,0,0}) \geq 6$. By Eq. (5), we have

$$\phi(G_{s,t,l,r}, \lambda) - \phi(G_{n-7,2,0,0}, \lambda) = (n - s - 7)\lambda^2 (\lambda - 1)^{n-6} [s - 2)(\lambda^2 - 3\lambda + 1) - 1].$$

So for $\lambda \geq \rho(G_{n-7,2,0,0})$, $\phi(G_{s,t,l,r}, \lambda) > \phi(G_{n-7,2,0,0}, \lambda)$, i.e., $\mu(G_{s,t,l,r}) < \mu(G_{n-7,2,0,0})$.

**Case 1.1.2** $r \neq 0$. From $G_{s,t,l,r} \not\in G_{1,n-7,0,1}$, we know $3 \leq s + r = n - 5 - t$.

(a) Let $t \geq 3$. By $|x_u| \geq |x_z|$, Lemma 1.2 and Case 1.1.1, we have

$$\mu(G_{s,t,l,r}) < \mu(G_{s,t,0,r} - zz_1 - \cdots - zz_r + uz_1 + \cdots + uz_r) = \mu(G_{s+r,0,0})$$
Lemma 1.2 and Case 1.1.2, we have

\[ \mu(G_{s,t,l,r}) < \mu(G_{s,2,0,r} - zz_1 - \cdots - zz_r + uz_1 + \cdots + uz_r) = \mu(G_{n-7,2,0,0}). \]

(c) Let \( t = 1 \) and \( s > r \). By Lemma 3.1, \( \mu(G_{s,t,l,r}) \geq s + 3 \geq r + 4 \), so \( |x_o| \geq |x_z| \). Recall \( |x_u| \geq |x_z| \), by Lemma 1.2 and (c), we have

\[ \mu(G_{s,t,l,r}) < \mu(G_{s,1,0,r} - zz_1 + vz_1) = \mu(G_{s,2,0,r-1}) \leq \mu(G_{n-7,2,0,0}). \]

(d) Let \( t = 1 \) and \( s = r \). Then \( r \geq 2 \) by \( s + r = n - 6 \). By \( |x_u| \geq |x_z| \), Lemma 1.2 and (c), we have

\[ \mu(G_{s,t,l,r}) < \mu(G_{r,1,0,r} - zz_1 + uz_1) = \mu(G_{r+1,1,0,r-1}) < \mu(G_{n-7,2,0,0}). \]

Case 1.2 \( t = 0 \). Since \( G_{s,t,l,r} \notin \{G_{n-5,0,0,0,0}, G_{n-6,0,0,0,0}\} \), we have \( s \geq r \geq 2 \).

If \( s > r \), then by Lemma 3.1, we have \( \mu(G_{s,t,l,r}) \geq s + 3 \geq r + 4 \), so \( |x_o| \geq |x_z| \). Hence by Lemma 1.2 and (c) of Case 1.1.2, we have

\[ \mu(G_{s,t,l,r}) < \mu(G_{s,0,0,r} - zz_1 + vz_1) = \mu(G_{s,1,0,r-1}) < \mu(G_{n-7,2,0,0}). \]

If \( s = r \), then \( r \geq 3 \) by \( s + r = n \geq 5 \). By \( |x_u| \geq |x_z| \), Lemma 1.2 and the above result,

\[ \mu(G_{s,t,l,r}) < \mu(G_{r,0,0,r} - zz_1 + uz_1) = \mu(G_{r+1,0,0,r-1}) < \mu(G_{n-7,2,0,0}). \]

Case 2 \( l \neq 0 \). Let \( r \neq 0 \). By \( |x_u| \geq |x_w| \), Lemma 1.2 and Case 1.1.2, we have

\[ \mu(G_{s,t,l,r}) < \mu(G_{s,t,l,r} - w_1b_1 - \cdots - w_i b_i + ub_1 + \cdots + ub_1) = \mu(G_{s+l,t+l,0,0}). \]

Let \( r = 0 \). Then \( l \geq 2 \) when \( s = 0 \) from \( G_{s,t,l,r} \notin G_{0,n-7,1,0} \). So by \( |x_u| \geq |x_z| \geq |x_w| \), Lemma 1.2 and Case 1.1.2, we have

\[ \mu(G_{s,t,l,r}) < \mu(G_{0,t,l,0} - w_1 b_1 - \cdots - w_i b_i + ub_1 + \cdots + ub_{i-1} + zb_i) = \mu(G_{s+l-1,l+l,0,1}) < \mu(G_{n-7,2,0,0}). \]

By Case 1, Case 2 and Eq. (4), we complete the proof.

Let \( P_0 = u_1u_2 \cdots u_6 \) and \( T_{n,5}^4 \) be the tree obtained from \( P_0 \) by attaching \( n - 6 \) new pendant edges at \( u_i (i = 2, 3) \). Let \( T_{n,5}^4 \) be the tree obtained from \( P_0 \) by attaching \( n - 7 \) new pendant edges at \( u_3 \) and a new pendant edge at \( u_4 \). It is easy to see \( T_{n,5}^4 = T_{n,5,n-4} \).

Lemma 3.5 Let \( n \geq 10 \). Then

\[ \mu(G_{n-6,0,0,0,0}) > \mu(T_{n,5}^3) > \mu(G_{n-6,0,0,0,1}) > \mu(T_{n,5}^2) > \mu(D_{n-6,4}) \]
Lemma 1.2, we have

Above discussion, we have

But for \( \phi \), \( g \), \( \lambda \), we have

Proof

Applying 2.e.t between \( u_2 \) and \( u_4 \), \( T_{n,5}^3 \) can be transformed into \( G_{n-6,1,0,0} \). So by Lemma 1.2, we have

By an elementary calculation, we have

Let \( g(\lambda) = \lambda^3 - (n - 1)\lambda^2 + (n + 1)\lambda + n - 9 \) and \( \lambda_0 \) be the largest root of \( g(\lambda) = 0 \). From \( g(n - 3) < 0 \), we have \( \lambda_0 > n - 3 \). By Eq. (5), it is easy to see that

Since \( g'(\lambda) > 0 \) for \( \lambda \geq n - 3 \), \( g(\lambda) \) is strict monotone increasing on \( \lambda \geq n - 3 \). Hence for \( \lambda \geq \lambda_0 \), by \( g(\lambda) > g(\lambda_0) = 0 \), we have

Combing \( \phi(G_{n-6,0,0,1}, n-3) < 0 \) and \( \phi(T_{n,5}^3, n-3) < 0 \), we have

But for \( \lambda \in (n-3, \lambda_0) \), by \( g(\lambda) < g(\lambda_0) = 0 \), we have \( \phi(T_{n,5}^3, \lambda) < \phi(G_{n-6,0,0,1}, \lambda) \). So from the above discussion, we have

Since \( T_{n,5}^2 = u_5u_6 + u_4u_6 \cong G_{n-6,0,0,1} \), by Lemma 1.1, we have

From \( \phi(D_{s,6}, \lambda) \), we have \( \phi(D_{s,6}, \lambda) = \lambda(\lambda - 1)^{-4}h(\lambda) \), where

Since \( h'(\lambda) > 0 \) for \( \lambda \geq n - 3 \), we have \( h(\lambda) > h(n - 3) > 0 \) for \( \lambda \geq n - 3 \). This implies that \( \mu(D_{n-6,4}) < n - 3 \). But by Lemma 3.1, \( \mu(T_{n,5}^2) \geq \Delta(T_{n,5}^2) + 1 = n - 3 \), so

Using 2.e.t between \( u_2 \) and \( u_4 \), \( T_{n,5}^* \) can be transformed into \( G_{n-7,2,0,0} \). Using 1.e.t, \( G_{n-7,2,0,0} \) can be transformed into \( D_{n-6,4} \). Using 2.e.t, \( G_{1,n-7,0,1} \) and \( G_{2,n-7,0,0} \) can be transformed into \( D_{n-6,4} \). So by Lemma 1.1 and Lemma 1.2, we get

\[
\mu(D_{n-6,4}) > \max\{\mu(G_{1,n-7,0,1}), \mu(G_{2,n-7,0,0}), \mu(G_{n-7,2,0,0}), \mu(T_{n,5}^*)\}. \tag{11}
\]
Combining Eqs. (7)∼(11), we complete the proof.

**Theorem 3.6** Let $T$ be a tree with order $n(n \geq 10)$ and $T \notin S$, where

$$S = \{ K_{1,n-1}, D_{n-3,1}, D_{n-4,2}, G_{0,n-5,0,0}, G_{n-5,0,0,0}, D_{n-5,3}, G_{1,n-6,0,0}, G_{0,n-7,1,0},$$

$$G_{n-6,1,0,0}, T_{n,5}^3, G_{n-6,0,0,1}, T_{n,5}^2, D_{n-6,4} \}.$$

Then $\mu(T) < \mu(D_{n-6,4})$.

**Proof** From $n \geq 10$ and $T \neq K_{1,n-1}$, we have $\text{dia}(T) \geq 3$.

Let $\text{dia}(T) \geq 6$. Using 2.e.t, $T_{n,6,n-5}$ can be transformed into $G_{n-7,0,1,0}$. Using 1.e.t, $G_{n-7,0,1,0}$ can be transformed into $D_{n-6,4}$. By Corollary 2.2, Lemma 1.2 and Lemma 1.1, we get

$$\mu(T) \leq \mu(T_{n,6,n-5}) < \mu(G_{n-7,0,1,0}) < \mu(D_{n-6,4}).$$

Let $\text{dia}(T) = 5$. Since $T \notin S$, we have $T \neq T_{n,5}^3, T_{n,5}^2$. By Theorem 3.1 in [9] and Lemma 3.5, we get

$$\mu(T) \leq \mu(T_{n,5}^*) < \mu(D_{n-6,4}).$$

Let $\text{dia}(T) = 4$. Then there are non-negative integers $m, t, n_1, \cdots, n_t, k$ such that $T = H(m, t, n_1, \cdots, n_t, k)$. By Lemma 3.5, now assume $T \notin \{ G_{2,n-7,0,0}, G_{n-7,2,0,0}, G_{1,n-7,0,1} \}$. From $T \notin S$, we have $T \notin J$, where $J$ is the set defined in Lemma 3.4. By Lemma 3.4 and Lemma 3.5, we get

$$\mu(T) < \mu(G_{n-7,2,0,0}) < \mu(D_{n-6,4}).$$

Let $\text{dia}(T) = 3$. Then there is a positive integer $s$ such that $\frac{n-2}{2} \leq s \leq n-3$ and $T = D_{s,t}$. From $T \notin S$, we have $s \leq n - 6$. By Lemma 3.2, we get

$$\mu(T) = \mu(D_{s,t}) < \mu(D_{n-6,4}).$$

This completes the proof.

**Remark** By [2–4], Lemma 3.5 and Theorem 3.6, for the proper positive integer $n$, the first thirteenth largest values of Laplacian spectral radius of trees on order $n$ are ordered as follows.

$$\mu(K_{1,n-1}) > \mu(D_{n-3,1}) > \mu(D_{n-4,2}) > \mu(G_{0,n-5,0,0}) > \mu(G_{n-5,0,0,0}) > \mu(D_{n-5,3}) > \mu(G_{1,n-6,0,0}) > \mu(G_{0,n-7,1,0}) > \mu(G_{n-6,1,0,0}) > \mu(T_{n,5}^3) > \mu(G_{n-6,0,0,1}) > \mu(T_{n,5}^2) > \mu(D_{n-6,4}).$$
[References]


