On upper bounds for Laplacian graph eigenvalues

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ABSTRACT

In this paper, we obtain the following upper bound for the largest Laplacian graph eigenvalue \( \lambda(G) \):

\[
\lambda(G) \leq \max \left\{ \frac{d(u) (d(u) + m(u)) + d(v) (d(v) + m(v))}{d(u) + d(v)} \right. \\
- \left. 2 \sum_{w \in N(u) \cap N(v)} d(w) \right\},
\]

where the maximum is taken over all pairs \( (u, v) \in E(G) \). This is an improvement on Li and Zhang’s result with \(-2 \sum_{w \in N(u) \cap N(v)} d(w) / d(u) + d(v)\) omitted. We also present another new upper bound for \( \lambda(G) \):

\[
\lambda(G) \leq \max \left\{ \frac{d(u)}{d(v)} m(u) + \frac{d(v)}{d(u)} m(v) : (u, v) \in E(G) \right\}.
\]

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1. Introduction

Let \( G = (V, E) \) be a simple graph on vertex set \( V = \{v_1, \ldots, v_n\} \) and edge set \( E \). All graphs in this paper have at least one edge. We use \( d(u) \) to denote the degree of vertex \( u \). Let \( D = D(G) = \text{diag}(d(v_1), \ldots, d(v_n)) \) be the diagonal matrix of vertex degrees, and let \( A = A(G) \) be the adjacency matrix of \( G \). Then the Laplacian matrix of \( G \) is \( L = L(G) = D(G) - A(G) \). The eigenvalues of \( L(G) \) are called Laplacian eigenvalues of \( G \). Denote by \( \lambda_1, \lambda_2, \ldots, \lambda_n \) the Laplacian eigenvalues of \( G \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). It is known that \( |V| \geq \lambda_1 \geq \lambda_n = 0 \). The largest eigenvalue of \( L(G) \) is denoted by \( \lambda(G) \). There are some known results for upper bounds of \( \lambda(G) \).

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In 1985, Anderson and Morley [1] proved:
\[ \lambda(G) \leq \max \{d(u) + d(v) : (u, v) \in E(G)\}. \] (1.1)

In 1997, Li and Zhang [7] improved (1.1) by establishing:
\[ \lambda(G) \leq 2 + \sqrt{(r - 2)(s - 2)}, \] (1.2)
where \( r = \max \{d(u) + d(v) : (u, v) \in E(G)\} \), and \( s = \max \{d(u) + d(v) : (u, v) \in E(G) - (x, y)\} \) with \((x, y) \in E(G)\) and \( d(x) + d(y) = r \).

In 1998, Merris [9] showed that:
\[ \lambda(G) \leq \max \{d(u) + m(u) : u \in V(G)\}, \] (1.3)
where \( m(u) = \frac{1}{d(u)} \sum_{v \in N(u)} d(v) \), i.e. \( m(u) \) is the average degree of the adjacent vertices of \( u \).

In 1998, Li and Zhang [8] improved (1.3) as follows:
\[ \lambda(G) \leq \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} : (u, v) \in E(G) \right\}. \] (1.4)

In 2000, Rojo et al. [11] obtained an always-nontrivial bound:
\[ \lambda(G) \leq \max \{d(u) + d(v) - |N(u) \cap N(v)| : u \neq v, u, v \in V(G)\}, \] (1.5)
where \( N(w) = \{u \in V(G) : (u, w) \in E(G)\} \).

In 2002, Pan [10] improved (1.4):
\[ \lambda(G) \leq 2 + \sqrt{(t - 2)(b - 2)}, \] (1.6)
where \( t = \max \{h(u, v) : (u, v) \in E(G)\} \), \( s = \max \{h(u, v) : (u, v) \in E(G) - (x, y)\} \) with \((x, y) \in E(G)\) such that \( d(x) + d(y) = r \), and
\[ h(u, v) = \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)}. \]

In 2003, Das [2] improved the bound of (1.3):
\[ \lambda(G) \leq \max \{d(u) + d(v) - |N(u) \cap N(v)| : (u, v) \in V(G)\}. \] (1.7)

In this paper, we first improve (1.4) as follows:

**Theorem 1.1.** If \( G \) is a simple graph, then
\[ \lambda(G) \leq \max \left\{ \frac{d(u)t(u) + d(v)t(v)}{d(u) + d(v)} - 2 \frac{\sum_{w \in N(u) \cap N(v)} d(w)}{d(u) + d(v)} : (u, v) \in E \right\}, \] (1.8)
where \( t(w) = d(w) + m(w) \) for \( w \in V(G) \).

We also have the following new upper bounds:

**Theorem 1.2.** Let \( G = (V, E) \) be a simple graph. Denote by \( T(u, v) = \frac{d(u)}{d(v)} m(u) + \frac{d(v)}{d(u)} m(v) \). Then
\[ \lambda(G) \leq \max \left\{ 2 + \sqrt{(T(u, v) - 2)(T(u, w) - 2)} : (u, v), (u, w) \in E, v \neq w \right\}. \] (1.9)

**Corollary 1.1.** If \( G \) is a simple graph, then
\[ \lambda(G) \leq \max \left\{ \frac{d(u)}{d(v)} m(u) + \frac{d(v)}{d(u)} m(v) : (u, v) \in E(G) \right\}. \] (1.10)
In the next section, we will prove Theorems 1.1 and 1.2. And in Section 3, we shall discuss some further upper bounds.

2. Proof of Theorems 1.1 and 1.2

We will prove two general results which imply Theorems 1.1 and 1.2. For graph \( G = (V, E) \), let \( f : V \times V \to \mathbb{R} \) be a function. If \( f(u, v) > 0 \) for all \((u, v) \in E\), we say that \( f \) is positive on edges.

**Theorem 2.3.** Let \( G = (V, E) \) be a simple graph. Let \( f : V \times V \to \mathbb{R}_+ \cup \{0\} \) be a nonnegative function which is positive on edges. Then \( \lambda(G) \) is less than or equal to

\[
\max \left\{ \left| N(u) \cap N(v) \right| + \frac{\sum_{w \in N(u) \setminus N(v)} f(u, w) + \sum_{w \in N(v) \setminus N(u)} f(v, w)}{f(u, v)} : (u, v) \in E \right\}. \tag{2.1}
\]

**Remark 2.1.**

1. Set \( f(u, v) = 1 \) for all \( u, v \in V \). Clearly (2.1) equals \( d(u) + d(v) - \left| N(u) \cap N(v) \right| \). So Theorem 2.3 gives (1.7).
2. Setting \( f(u, v) = d(u) + d(v) \) for \( u, v \in V \), it is not difficult to see by some calculations that (2.1) coincides with the right side of (1.8):

\[
\begin{align*}
\sum_{w \in N(u) \setminus N(v)} (d(u) + d(w)) + \sum_{w \in N(v) \setminus N(u)} (d(v) + d(w)) \\
= \sum_{w \in N(u)} (d(u) + d(w)) + \sum_{w \in N(v)} (d(v) + d(w)) \\
- \sum_{w \in N(u) \cap N(v)} (d(u) + d(v) + 2d(w)) \\
= d(u)(d(u) + m(u)) + d(v)(d(v) + m(v)) \\
- \left| N(u) \cap N(v) \right|(d(u) + d(v)) - 2 \sum_{w \in N(u) \cap N(v)} d(w).
\end{align*}
\]

Therefore this proves Theorem 1.1.

**Proof of Theorem 2.3.** Let \( X = (x_1, \ldots, x_n) \) be an eigenvector corresponding to the eigenvalue \( \lambda = \lambda(G) \) of the Laplacian matrix \( L = L(G) \). Then

\[
LX = \lambda X. \tag{2.2}
\]

The \( j \)th equation of (2.2) gives

\[
\lambda x_j = d(v_j)x_j - \sum_{k=1}^{n} a_{jk}x_k = d(v_j)x_j - \sum_{v_k \in N(j)} x_k.
\]

Here for convenience, we use \( N(j) \) to denote \( N(v_j) \). And also we use \( N(i) \cap N(j) \) to denote \( N(v_i) \cap N(v_j) \). Then

\[
\lambda x_j = \sum_{v_k \in N(j)} (x_j - x_k). \tag{2.3}
\]

For \( 1 \leq i, j \leq n \), from (2.3)

\[
\lambda (x_j - x_i) = \sum_{v_k \in N(j)} (x_j - x_k) - \sum_{v_k \in N(i)} (x_i - x_k)
\]
Lemma 2.1

Theorem 2.4.

satisfying

Then

\[ \lambda(x_j - x_i) = \sum_{v_k \in N(i,j)} (x_j - x_k) + \sum_{v_k \in N(j) \setminus N(i,j)} (x_j - x_k) - \sum_{v_k \in N(i) \setminus N(j)} (x_i - x_k). \]

i.e.,

\[ \lambda(x_j - x_i) = |N(i,j)| (x_j - x_i) + \sum_{v_k \in N(j) \setminus N(i,j)} (x_j - x_k) - \sum_{v_k \in N(i) \setminus N(j)} (x_i - x_k). \tag{2.4} \]

For convenience, we use \( f(j, i) \) to denote \( f(v_j, v_i) \). Set \( g(j, i) = \frac{x_j - x_i}{f(j, i)}. \) For \( (v_i, v_j) \in E \), by (2.4) we get

\[ (\lambda - |N(i,j)|) f(j, i) g(j, i) = \sum_{v_k \in N(j) \setminus N(i,j)} f(j, k) g(j, k) - \sum_{v_k \in N(i) \setminus N(j)} f(i, k) g(i, k). \]

Now choose \( i', j' \) such that \( (v_{i'}, v_{j'}) \in E \) and \( |g(j', i')| = \max \{|g(j, i)| : (v_j, v_i) \in E(G)\} \). Clearly \( |g(j', i')| \neq 0 \). Otherwise \( x_j - x_i = 0 \) for all edges \( (v_j, v_i) \in E(G) \). By (2.3), \( \lambda(G) = 0 \) which is impossible.

Then

\[ (\lambda - |N(i', j')|) f(j', i') \leq \sum_{v_k \in N(j') \setminus N(i', j')} f(j', k) \left| g(j', k) \right| g(j', i') + \sum_{v_k \in N(i') \setminus N(i', j')} f(i', k) \left| g(i', k) \right| g(j', i'). \]

\[ \leq \sum_{v_k \in N(j') \setminus N(i', j')} f(j', k) + \sum_{v_k \in N(i') \setminus N(i', j')} f(i', k). \]

Therefore, we obtain

\[ \lambda \leq |N(i', j')| + \frac{1}{f(j', i')} \sum_{v_k \in N(j') \setminus N(i', j')} f(j', k) + \frac{1}{f(i', i')} \sum_{v_k \in N(i') \setminus N(i', j')} f(i', k). \]

This proves the desired result. \( \square \)

The matrix \( K(G) = D(G) + A(G) \) is called the un-oriented Laplacian matrix of \( G \). It is not difficult to prove that all the eigenvalues of \( L(G) \) and \( K(G) \) are nonnegative. For a matrix \( M \), denote by \( \rho(M) \) the spectral radius of \( M \). We need a result from [10]:

Lemma 2.1 ([10]). Let \( G = (V, E) \) be a simple graph. Then

\[ \lambda(G) \leq \rho(K(G)). \]

For any \( f : V \times V \to \mathbb{R}^+ \cup \{0\} \) if \( f(u, v) \neq 0 \) we define

\[ \tilde{f}(u, v) = \frac{\sum_{w \in N(u) \setminus \{v\}} f(u, w) + \sum_{w \in N(v) \setminus \{u\}} f(v, w)}{f(u, v)}, \]

for \( u, v \in V \).

Theorem 2.4. Let \( G = (V, E) \) be a simple graph and \( f : V \times V \to \mathbb{R}^+ \cup \{0\} \) be positive on edges. Denote by \( R = \max \{ \tilde{f}(u, v) : (u, v) \in E(G) \} \) and \( S = \max \{ \tilde{f}(u, v) : (u, v) \in E(G) - (x, y) \} \) with \( (x, y) \in E(G) \) satisfying \( \tilde{f}(x, y) = R \). Then
Remark 2.2.

Proof of Theorem 2.4.

By (2.8),
\[
\lambda(G) \leq \rho(K(G)) \leq 2 + \sqrt{RS}.
\]

Furthermore, if \( f(u,v) = f(v,u) \) for any \( u,v \in V \), then
\[
\lambda(G) \leq \rho(K(G)) \leq \max \left\{ 2 + \sqrt{\tilde{f}(u,v)\tilde{f}(u,w)} : (u,v),(u,w) \in E, v \neq w \right\}.
\]

Proof of Theorem 2.4. In light of Lemma 2.1, we only need to prove the second inequalities of (2.5) and (2.6). Let \( X = (x_1, \ldots, x_n) \) be an eigenvector corresponding to the eigenvalue \( \rho = \rho(K(G)) \) of the matrix \( K = K(G) \). Then
\[
KK = \rho X.
\]

Then for \( 1 \leq j \leq n \),
\[
\rho x_j = d(v_j)x_j + \sum_{k=1}^{n} a_{jk}x_k = d(v_j)x_j + \sum_{v_k \in N(j)} x_k.
\]

As in the proof of Theorem 2.3, we still use \( N(j) \) to denote \( N(v_j), N(i,j) = N(i) \cap N(j) = N(v_i) \cap N(v_j) \). Then
\[
\rho x_j = \sum_{v_k \in N(j)} (x_j + x_k).
\]

By (2.8),
\[
(\rho - 2)(x_j + x_i) = \sum_{v_k \in N(j) \setminus \{v_i\}} (x_j + x_k) + \sum_{v_k \in N(i) \setminus \{v_j\}} (x_i + x_k).
\]

i.e.,
\[
(\rho - 2)(x_j + x_i) = \sum_{v_k \in N(j) \setminus \{v_i\}} (x_j + x_k) + \sum_{v_k \in N(i) \setminus \{v_j\}} (x_i + x_k).
\]

Set \( g(j,i) = \frac{x_j + x_i}{f(j,i)} \). If \( (v_i, v_j) \in E \), then
\[
(\rho - 2)f(j,i)g(j,i) = \sum_{v_k \in N(j) \setminus \{v_i\}} f(j,k)g(j,k) + \sum_{v_k \in N(i) \setminus \{v_j\}} f(i,k)g(i,k).
\]

Choose \( j_1, i_1 \) such that \( |g(j_1,i_1)| = M_1 := \max \{|g(j,i)| : (v_j, v_i) \in E(G) \} \) and \( (v_{j_1}, v_{i_1}) \in E \). Then choose \( j_2, i_2 \) such that \( |g(j_2,i_2)| = M_2 := \max \{|g(j,i)| : (v_j, v_i) \in E(G) - (v_{j_1}, v_{i_1}) \} \) and \( (v_{j_2}, v_{i_2}) \in E(G) - (v_{j_1}, v_{i_1}) \). If \( M_1 = 0 \), then \( x_j + x_i = 0 \) for all edges \( (v_j, v_i) \). So by (2.8) \( \rho = 0 \) which is impossible. If \( M_2 = 0 \) then \( x_j + x_i = 0 \) for all edges \( (v_j, v_i) \). Then by (2.9), we have \( \rho = 2 \) and (2.5) holds trivially. So we assume \( M_1 \geq M_2 > 0 \) below. Now clearly we have
\[
|\rho - 2|f(j_2,i_2)g(j_2,i_2)| \leq \sum_{v_k \in N(j_2) \setminus \{v_{i_2}\}} f(j_2,k)M_1 + \sum_{v_k \in N(i_2) \setminus \{v_{j_2}\}} f(i_2,k)M_1.
\]

Choose \( j_1, i_1 \) such that \( |g(j_1,i_1)| = M_1 := \max \{|g(j,i)| : (v_j, v_i) \in E(G) \} \) and \( (v_{j_1}, v_{i_1}) \in E \). Then choose \( j_2, i_2 \) such that \( |g(j_2,i_2)| = M_2 := \max \{|g(j,i)| : (v_j, v_i) \in E(G) - (v_{j_1}, v_{i_1}) \} \) and \( (v_{j_2}, v_{i_2}) \in E(G) - (v_{j_1}, v_{i_1}) \). If \( M_1 = 0 \), then \( x_j + x_i = 0 \) for all edges \( (v_j, v_i) \). So by (2.8) \( \rho = 0 \) which is impossible. If \( M_2 = 0 \) then \( x_j + x_i = 0 \) for all edges \( (v_j, v_i) \). Then by (2.9), we have \( \rho = 2 \) and (2.5) holds trivially. So we assume \( M_1 \geq M_2 > 0 \) below. Now clearly we have
\[
|\rho - 2|f(j_2,i_2)g(j_2,i_2)| \leq \sum_{v_k \in N(j_2) \setminus \{v_{i_2}\}} f(j_2,k)M_1 + \sum_{v_k \in N(i_2) \setminus \{v_{j_2}\}} f(i_2,k)M_1.
\]
and

\[ |\rho - 2|f(j_1, i_1)g(j_1, i_1)| \leq \sum_{v_k \in N(j_1) \setminus \{v_{j_1}\}} f(j_1, k)M_2 + \sum_{v_k \in N(i_1) \setminus \{v_{i_1}\}} f(i_1, k)M_2. \] (2.11)

Combining (2.10) and (2.11), we have

\[ (\rho - 2)^2 \leq \frac{\sum_{v_k \in N(j_1) \setminus \{v_{j_1}\}} f(j_1, k) + \sum_{v_k \in N(i_1) \setminus \{v_{i_1}\}} f(i_1, k)}{f(j_1, i_1)} \times \frac{\sum_{v_k \in N(j_2) \setminus \{v_{j_2}\}} f(j_2, k) + \sum_{v_k \in N(i_2) \setminus \{v_{i_2}\}} f(i_2, k)}{f(j_2, i_2)}. \] (2.12)

Since \((v_{j_1}, v_{i_1})\) and \((v_{j_2}, v_{i_2})\) are different edges of \(G\), we have \(\rho(K(G)) \leq 2 + \sqrt{\rho S}\).

In order to prove (2.6), we restart from the choice of \(j_2\) and \(i_2\). Now choose \(j_2, i_2\) such that \((v_{j_2}, v_{i_2}) \in E(G) - \{(v_{j_1}, v_{i_1})\}, \{v_{j_1}, v_{i_1}\} \cap \{v_{j_2}, v_{i_2}\} = \emptyset\) and \(|g(j_2, i_2)| = M_3 := \max\{|g(k_1, k_2)| : (v_{k_1}, v_{k_2}) \in E(G), \{v_{j_1}, v_{i_1}\} \cap \{v_{k_1}, v_{k_2}\} = 1\}. Without loss of generality, we assume that \(v_{j_2} = v_{j_1}\), since otherwise \(v_{i_1} = v_{i_2}\), and note that \(g(i_1, j_1) = g(j_1, i_1) = M_1, g(i_1, j_2) = g(j_2, i_1) = g(j_2, i_2) = M_3\). Then similar to the above proof of (2.5), we conclude that (2.12) still holds with \(v_{j_1} = v_{j_2}\) now, i.e.

\[ (\rho - 2)^2 \leq \tilde{f}(v_{j_1}, v_{i_1})f(v_{j_1}, v_{i_2}). \]

The proof of (2.6) is completed. \(\Box\)

### 3. Further discussions of upper bounds

In [6] Li and Pan showed the following result based on the relationship between eigenvalues and eigenvectors:

\[ \lambda(G) \leq \max \left\{ \sqrt{2d(u)(d(u) + m(u))} : u \in V(G) \right\}. \] (3.1)

Zhang [13] followed the method in [6] to improve the above bound:

\[ \lambda(G) \leq \max \left\{ d(u) + \sqrt{d(u)m(u)} : u \in V(G) \right\}. \] (3.2)

And Shi [12] got that

\[ \lambda(G) \leq \sqrt{2} \max \left\{ \sqrt{d(u)^2 + d(u)m(u)} : u \in V(G) \right\}. \] (3.3)

The following result was obtained by Zhang and Li [14], and Das [3]:

\[ \lambda(G) \leq \max \left\{ \frac{d(u) + d(v) + \sqrt{(d(u) - d(v))^2 + 4m(u)m(v)}}{2} : (u, v) \in E \right\}. \] (3.4)

Note that (3.1–3.3) all involve square root. We apply Theorem 2.4 with different choices of function \(f\). First we observe that if \(f(u, v) = f(v, u)\) for all \(u, v \in V\), then (2.5) or (2.6) implies:

\[ \lambda(G) \leq \max \left\{ \frac{\sum_{(w, u) \in E} f(u, w) + \sum_{(w, v) \in E} f(v, w)}{f(u, v)} : (u, v) \in E \right\}. \] (3.5)

Setting \(f(u, v) = \sqrt{d(u)d(v)}\), we obtain

\[ \lambda(G) \leq \max \left\{ \frac{m(u)}{d(u)} + d(v) \sqrt{\frac{m(v)}{d(u)}} : (u, v) \in E \right\}. \]
Now applying (3.5) with \( f(u, v) = \sqrt{d(u)} + \sqrt{d(v)} \) we have

**Theorem 3.5.**

\[
\lambda(G) \leq \max \left\{ \frac{d(u)}{\sqrt{d(u) + d(v)}} : (u, v) \in E(G) \right\}, 
\]

(3.6)

Since \( \sum_{(w, u) \in E} \sqrt{d(u)} + d(w) \leq \sqrt{d(u) \sum_{(w, u) \in E} (d(u) + d(w))} = d(u) \sqrt{d(u) + m(u)} \) by Cauchy-Schwarz inequality.

If we apply (3.5) with \( f(u, v) = \sqrt{d(u)} + \sqrt{d(v)} \), then we can get:

**Theorem 3.6.** For a simple graph \( G \),

\[
\lambda(G) \leq \max \left\{ \frac{d(u)}{\sqrt{d(u) + \sqrt{d(v)} + m(v)}} : (u, v) \in E \right\}. 
\]

(3.7)

Follows from \( \sum_{(w, u) \in E} \left( \sqrt{d(u)} + \sqrt{d(w)} \right) = d(u) \sqrt{d(u)} + \sum_{(w, u) \in E} \sqrt{d(w)} \leq d(u)^{1/2} + \sqrt{d(u) \sum_{(w, u) \in E} d(w)} = d(u) \left( \sqrt{d(u)} + \sqrt{m(u)} \right) \) by Cauchy-Schwarz inequality.

Notice that (3.6) and (3.7) can be viewed as adding square roots to (1.4) at different places. Meanwhile, Theorems 3.5 and 3.6 are also improvements on Guo’s inequality (3.3) and Zhang’s inequality (3.2) respectively.

We mention that there are some other upper bounds involving square roots. Liu et al. [5] proved that

\[
\lambda(G) \leq \frac{\left( \Delta + \delta - 1 \right) + \sqrt{\left( \Delta + \delta - 1 \right)^2 + 4(4|E| - 2\delta(|V| - 1))}}{2},
\]

where \( \Delta \) and \( \delta \) are the minimum degree and maximal degree of vertices of \( G \) respectively. Guo [4] obtained

\[
\lambda(G) \leq \max \left\{ \frac{d(v) + \sqrt{d(v)^2 + 8d(v)m(v) - 8\sum_{(u, v) \in E} |N(u) \cap N(v)|}}{2} : v \in V \right\}. 
\]

To end this paper, we follow the method in [6,13] to prove a bound which is very similar to (3.4).

**Theorem 3.7.** Let \( G \) be a simple graph. Then

\[
\lambda(G) \leq \max \left\{ \frac{d(u) + d(v) + \sqrt{(d(u) - d(v))^2 + 4\sqrt{d(u)d(v)m(u)m(v)}}}{2} : (u, v) \in E \right\}. 
\]

(3.8)

**Proof.** Let \( X = (x_1, \ldots, x_n) \) be an eigenvector corresponding to \( \lambda(G) \). Then

\[
(\lambda(G) - d(v_j))x_j = \sum_{(v_k, v_j) \in E} x_k \leq \sqrt{d(v_j) \sum_{(v_k, v_j) \in E} x_k^2}. 
\]

i.e.,

\[
(\lambda(G) - d(v_j))^2 x_j^2 \leq d(v_j) \sum_{(v_k, v_j) \in E} x_k^2. 
\]

So for any \( (v_j, v_j) \in E \), we have
(\lambda(G) - d(v_j))^2 (\lambda(G) - d(v_i))^2 x_i^2 x_j^2 \leq d(v_j) d(v_i) \left( \sum_{(v_k,v_i) \in E} x_k^2 \right) \left( \sum_{(v_k,v_j) \in E} x_k^2 \right).

i.e.,

\begin{align*}
(\lambda(G) - d(v_j))^2 (\lambda(G) - d(v_i))^2 & \leq \left( \sum_{(v_k,v_i) \in E} d(v_k) x_k^2 \right) \left( \sum_{(v_k,v_j) \in E} d(v_k) x_k^2 \right).
\end{align*}

Choose \( v_i, v_j \) such that \( \frac{x_i^2}{d(v_i)} \frac{x_j^2}{d(v_j)} = M := \max \left\{ \frac{x_{i'}^2}{d(v_{i'})} : (v_{i'}, v_{j'}) \in E \right\} \) and \((v_i, v_j) \in E\). If \( M > 0 \), then

\begin{align*}
(\lambda(G) - d(v_j))^2 (\lambda(G) - d(v_i))^2 M & \leq \sum_{(v_k,v_i),(v_m,v_j) \in E} d(v_k) d(v_m) x_k^2 x_m^2 d(v_k) d(v_j) d(v_i) d(v_m) \\
& = \frac{1}{M} \sum_{(v_k,v_i),(v_m,v_j) \in E} d(v_k) d(v_m) x_k^2 x_m^2 x_j^2 x_i^2 d(v_k) d(v_j) d(v_i) d(v_m) \\
& \leq \frac{1}{M} \sum_{(v_k,v_i),(v_m,v_j) \in E} d(v_k) d(v_m) M^2 \\
& = M \sum_{(v_k,v_i),(v_m,v_j) \in E} d(v_k) d(v_m) = M d(v_j) m(v_j) d(v_i) m(v_i).
\end{align*}

So

\begin{equation}
(\lambda(G) - d(v_j))(\lambda(G) - d(v_i)) \leq \sqrt{d(v_j) d(v_i) m(v_j) m(v_i)}. \tag{3.9}
\end{equation}

If \( M = 0 \), then for any \( x_j \neq 0 \) we have \( (\lambda(G) - d(v_j)) x_j^2 = \sum_{(v_k,v_j) \in E} x_k x_j = 0 \). Therefore \( \lambda(G) = d(v_j) \) and \( \text{(3.9)} \) holds.

It is clear that \( \text{(3.9)} \) implies \( \text{(3.8)} \). \( \square \)

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\section*{References}