A note on upper bounds for the spectral radius of weighted graphs

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Abstract

Let \( G = (V, E) \) be a simple connected weighted graph on \( n \) vertices, in which the edge weights are positive definite matrices. The eigenvalues of \( G \) are the eigenvalues of its adjacency matrix. In this note, we present a correction in equality part in Theorem 2 [S. Sorgun, S. Büyükköse, The new upper bounds on the spectral radius of weighted graphs, Appl. Math. Comput. 218 (2012) 5231–5238]. In addition, some related results are also provided.

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1. Introduction

We only consider undirected graphs which have no loops or multiple edges. Let \( G = (V, E) \) be a connected graph with vertex set \( V = \{1, 2, \ldots, n\} \) and edge set \( E \). A weighted graph is a graph, each edge of which has been assigned a square matrix, called the weight of the edge. All the weight matrices will be assumed to be of the same order and will be assumed to be positive definite. In particular, if the weight matrix of each of its edges is a positive number, then \( G \) is usual weighted graph.

An unweighted graph is thus a weighted graph with each of the edges bearing weight 1.

Let \( G \) be a connected weighted graph on \( n \) vertices. Denote by \( w_i \), the positive definite weight matrix of order \( p \) of the edge \( i,j \), and assume that \( w_{i,j} = w_{j,i} \). We write \( i \sim j \) if the vertices \( i \) and \( j \) are adjacent. Let \( w_i = \sum_j w_{i,j} \), where \( N_i \) stands for the neighbor set of vertex \( i \).

The adjacency matrix of a weighted graph \( G \) is denoted by \( A(G) \) and is defined as \( A(G) = (a_{i,j}) \), where

\[
a_{i,j} = \begin{cases} w_{i,j} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}
\]

Notice that in the definition above, the zero denotes the \( p \)-by-\( p \) zero matrix. Thus \( A(G) \) is a square matrix of order \( np \).

For any symmetric matrix \( B \), denote by \( \rho_1 (B) \) the largest eigenvalue (in modulus) of \( B \). For a connected weighted graph \( G \), let \( i = \rho_1 (w_i) \) and \( \bar{i} = \frac{\sum_{j} w_{i,j} \rho_1 (w_{j})}{\rho_1 (w_{i})} \) for all \( i \in V \). For all \( i,j \in V \), \( w_{i,j} \) are positive definite weight matrices, which implies \( i > 0 \) and \( \bar{i} > 0 \) for all \( i \in V \). If \( G \) is usual weighted graph, that is, edge weights are positive numbers, then \( i = w_i \) and \( \bar{i} = w_i = \frac{\sum_{j} w_{i,j} \rho_1 (w_{j})}{\rho_1 (w_{i})} \) for all \( i \in V \); If \( G \) is an unweighted graph, then \( i = \bar{i} = m_i = \frac{\sum_{j} d_j}{V} \) is called the average of degrees of the vertices adjacent to \( i \). The following definitions are introduced for the sake of convenience.

Definition 1.1. A graph \( G = (V, E) \) is called bipartite if \( G \) has no cycles of odd length; the vertex set \( V \) can be partitioned into two sets \( V_1 \) and \( V_2 \) in such a way that every edge in \( E \) connects a vertex in \( V_1 \) with a vertex in \( V_2 \). For a weighted bipartite

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graph with a bipartition $V_1$, $V_2$ of $V$, if every vertex $i$ in $V_1$ has the same $\gamma_i$ and every vertex $j$ in $V_2$ has the same $\gamma_j$, then $G$ is called a weighted pseudo-semiregular bipartite graph. If every vertex $i$ in $V$ of weighted graph $G$ has the same $\gamma_i$, then $G$ is called a weighted pseudo-regular graph.

If $G$ is an unweighted graph, then weighted pseudo-semiregular bipartite graph and weighted pseudo-regular graph are ordinary unweighted pseudo-semiregular bipartite graph and unweighted pseudo-regular graph, respectively (see [7,13]).

Upper and lower bounds for the spectral radius of unweighted graphs have been extensively investigated for a long time, reader may refer to [1,2,5–7,9,12,13] and the references therein. Recently, Upper and lower bounds for the spectral radius of weighted graphs have been studied in [3,4,10,11]. In 2012, Sorgun and Büyükköse [10] obtained

\[ |\rho_i| \leq \max_{j \neq i} \left\{ \sqrt{\gamma_i \gamma_j} \right\}. \tag{1} \]

Moreover, they claimed that the equality in (1) holds if and only if

(i) $G$ is either a weighted regular graph or a weighted semiregular bipartite graph;
(ii) $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_i(w_{ij})$ for all $i,j \in V$.

However, it is not true! For example,

**Example 2.1.** Let $G = (V,E)$ be a weighted graph with vertex set $V = \{1,2,\ldots,9\}$ and edge set $E = \{12,14,23,25,36,45,47,56,58,69,78,89\}$. Let us take the weights of the edges as follows:

\[ w_{12} = w_{14} = w_{23} = w_{36} = w_{47} = w_{69} = w_{78} = w_{89} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } w_{25} = w_{45} = w_{56} = w_{58} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}. \]

Applying the inequality (1), one has $|\rho_1| \leq 6\sqrt{10}$. By a direct calculation, $|\rho_1| = 6\sqrt{10}$, which shows the equality in (1) holds for the weighted graph $G$. However, $G$ is neither a weighted regular graph nor a weighted semiregular bipartite graph as $\rho_1(w_{12}) = 6$, $\rho_1(w_{25}) = 15$, $\rho_1(w_{58}) = 36$. On the other hand, by a direct calculation, one has

\[ \gamma_i = \begin{cases} 15, & i = 1, 3, 5, 7, 9; \\ 24, & i = 2, 4, 6, 8. \end{cases} \]

Hence, $G$ is a weighted pseudo-semiregular bipartite graph. It is also easy to verify that $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_i(w_{ij})$ for all $i,j \in V$.

In this note, we present a correction in the equality part in (1). It is proved that the equality in (1) holds if and only if:

(i) $G$ is either a weighted pseudo-regular graph or a weighted pseudo-semiregular bipartite graph;
(ii) $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_i(w_{ij})$ for all $i,j \in V$.

Finally, some related results are also provided, which show that this result generalizes some known results for unweighted graphs.

**2. Main results**

The following Lemmas 2.1 and 2.2 are directly consequences of the Cauchy–Schwarz inequality and Lemma 2.3 in [4], respectively.

**Lemma 2.1 ([4,8]).** Let $B$ be a Hermitian $n$-by-$n$ matrix with $\rho_1$ as its largest eigenvalue, in modulus. Then for any $x \in \mathbb{R}^n (x \neq 0)$ and $y \in \mathbb{R}^n (y \neq 0)$, the spectral radius $|\rho_1|$ satisfies

\[ |x^T B y| \leq |\rho_1| \sqrt{x^T x} \sqrt{y^T y}. \tag{2} \]

Equality holds if and only if $x$ is an eigenvector of $B$ corresponding to the largest eigenvalue $\rho_1$ and $y = z x$ for some $z \in \mathbb{R}$.

**Lemma 2.2 ([4]).** Let $G$ be a weighted graph, and let $w_{ij}$ be the positive definite weight matrix of order $p$ of the edge $ij$. Then $\gamma_i = \rho_1(w_{ij}) > 0$, where $w_i = \sum_{j \in N_i} w_{ij}$. Moreover, let $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all $i,j \in V$. Then

\[ \gamma_i = \rho_1(w_{ij}) = \sum_{j \in N_i} \rho_1(w_{ij}). \tag{3} \]

**Theorem 2.3.** Let $G$ be a simple connected weighted graph and let $w_{ij}$ be the positive definite weight matrix of order $p$ of the edge $ij$. Also let $\rho_1$ be the largest eigenvalue, in modulus, so that $|\rho_1|$ is the spectral radius of $G$. Then the inequality (1) holds. Moreover, the equality in (1) holds if and only if
(i) $G$ is either a weighted pseudo-regular graph or a weighted pseudo-semiregular bipartite graph;
(ii) $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_i(w_{ij})$ for all $i,j \in V$.

**Proof.** Remark that the inequality (1) has been proved in [10]. For need of the proof in equality part, the proof of the inequality (1) is also provided here.

Let $A(G)$ be the adjacency matrix of weighted graph $G$ and let $M(G) = \text{diag}(\gamma_1 I_{pp}, \gamma_2 I_{pp}, \ldots, \gamma_n I_{pp})$, where $\gamma_i = \rho_i(w_i)$, $i = 1,2,\ldots,n$ and $I_{pp}$ is the identity matrix of order $p$. Lemma 2.2 implies $\gamma_i > 0$ for all $i \in V$. We have

The $(i,j)$th block of $(M(G))^{-1}A(G)M(G)$ is

$$a'_{i,j} = \begin{cases} \frac{\gamma_j}{\gamma_i} w_{i,j} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = (x_1^T, x_2^T, \ldots, x_n^T)^T$ be an eigenvector corresponding to the largest eigenvalue, in modulus, $\rho_1$ of $(M(G))^{-1}A(G)M(G)$, that is,

$$M(G)^{-1}A(G)M(G)X = \rho_1 X.$$  \hspace{1cm} (4)

Let $x_k$ be the vector component of $X$ such that $x_k^T x_0 = \max_{k \in V} \{x_k^T x_k\}$ and $x_k^T x_0 = \max_{k \in N_0} \{x_k^T x_k\}$. Thus $x_k^T x_0 \geq x_0^T x_k$ for all $k \in N_0$. Since $X$ is nonzero, so is $x_0$.

From the $j$th equation of (4), we have

$$\rho_1 x_0 = \sum_{k \in N_0} \frac{\gamma_k}{\gamma_j} w_{i,k} x_k.$$  \hspace{1cm} (5)

Multiplying both sides by $x_j^T$ to the left of the above equation, we get

$$\rho_1 x_j^T x_0 = \sum_{k \in N_0} x_j^T \frac{\gamma_k}{\gamma_j} w_{i,k} x_k.$$  \hspace{1cm} (6)

Taking modulus on both sides of the above equation, we have

$$|\rho_1| |x_j^T x_0| \leq \sum_{k \in N_0} \left| x_j^T \frac{\gamma_k}{\gamma_j} w_{i,k} x_k \right|,$$

$$\leq \sum_{k \in N_0} \frac{\gamma_k}{\gamma_j} \sqrt{x_j^T x_0} \sqrt{x_k^T x_k} \rho_1(w_{i,k}),$$

$$\leq \sum_{k \in N_0} \frac{\gamma_k}{\gamma_j} \sqrt{x_j^T x_0} \sqrt{x_k^T x_k} \rho_1(w_{i,k}),$$

where the second inequality follows from (2) and the last inequality holds since $x_j^T x_0 \geq x_k^T x_k$ for all $k \in N_0$.

Since $|\rho_1|$ is the spectral radius of a nonzero matrix, $\rho_1 \neq 0$. It follows from (7) and $x_0 \neq 0$ that $x_k \neq 0$.

Similarly, from the $j$th equation of (4), we get

$$|\rho_1| |x_j^T x_0| \leq \sum_{k \in N_0} \frac{\gamma_k}{\gamma_j} \sqrt{x_j^T x_0} \sqrt{x_k^T x_k} \rho_1(w_{i,k}),$$

Multiplying (7) and (8), we get

$$|\rho_1|^2 x_j^T x_0 \cdot x_k^T x_0 \leq \sum_{k \in N_0} \frac{\gamma_k}{\gamma_j} \rho_1(w_{i,k}) \sum_{k \in N_0} \frac{\gamma_k}{\gamma_j} \rho_1(w_{i,k}) x_j^T x_k \cdot x_k^T x_j,$$

which implies that the inequality (1) holds.

Now suppose that the equality in (1) holds. Then all inequalities in the above argument must be equalities. From (7), we have, for all $k \in N_0$, $x_j^T x_k = x_k^T x_j$, which implies $x_k = x_j$ for all $k \in N_0$. By Lemma 2.1, we get that the equality in 6 holds if and only if both $x_k$ and $x_0$ are eigenvectors of $\frac{\gamma}{\gamma_j} w_{i,k}$ corresponding to eigenvalue $\frac{\gamma}{\gamma_j} \rho_1(w_{i,k})$ for all $k \in N_0$, that is, both $x_0$ and $x_k$ are eigenvectors of $w_{i,k}$ corresponding to largest eigenvalue $\rho_1(w_{i,k})$ for all $k \in N_0$. Thus there exits some nonzero $t_{i,j} \in \mathbb{R}$ such that $x_k = t_{i,j} x_0$ for all $k \in N_0$.

Since the equality in (5) holds, we have

$$\sum_{k \in N_0} \frac{\gamma_k}{\gamma_j} w_{i,k} x_k = \sum_{k \in N_0} x_j^T \frac{\gamma_k}{\gamma_j} w_{i,k} x_k,$$
that is,
\[ \left| \sum_{k \in N_b} t_{b,k} \frac{\gamma_k}{\gamma} X^T W_{b,k} X \right| = \sum_{k \in N_b} |t_{b,k}| \frac{\gamma_k}{\gamma} |X^T W_{b,k} X|. \]

Notice that \( \bar{x}_b \) is an eigenvector of positive definite matrix \( W_{b,k} \) corresponding to the largest eigenvalue \( \rho_1(W_{b,k}) \). Thus \( t_{b,k} \) must be the same sign for all \( k \in N_b \).

Now let \( X_p = t_{i,p} x_{i,p} \) and \( X_q = t_{i,q} x_{i,q} \) for \( p, q \in N_b \). It follows from \( \bar{x}_k^T X_p = \bar{x}_k^T x_{i,p} \) for each \( k \in N_b \) that \( t_{i,p}^2 = t_{i,q}^2 \). Thus \( t_{i,p} = t_{i,q} = t \). Hence \( \bar{x}_k = t x_{i,p} \) for all \( k \in N_b \), where \( t \) is a nonzero constant.

Similarly, we may get that both \( \bar{x}_b \) and \( \bar{x}_c \) are eigenvectors of positive definite matrix \( W_{b,k} \) corresponding to the largest eigenvalue \( \rho_1(W_{b,k}) \) for all \( k \in N_b \). Moreover, \( \bar{x}_k = s x_{i,s} \) for all \( k \in N_b \), where \( s \) is a nonzero constant.

It follows from \( t_0 = t_j \) that \( \bar{x}_b = s \bar{x}_j \) and \( \bar{x}_c = t \bar{x}_j \), which implies \( st = 1 \). Thus we have \( \bar{x}_k = t x_{i,p} \) for all \( k \in N_b \) and \( \bar{x}_k = \bar{x}_b \) for all \( k \in N_b \).

Firstly, assume that \( t = 1 \). Let \( V_1 = \{ x : \bar{x}_b = \bar{x}_a \} \) be the following vector: \( \bar{x}_b = x \bar{x}_b \). We shall show that \( V_1 = V \). From the above argument, we have \( \bar{x}_b = \bar{x}_a \) for all \( k \in N_b \) (notice that \( t = 1 \)), that is, \( N_b \subseteq V_1 \). Moreover, for any \( r \in N_{b_0} \) (where \( N_{b_0} \) denotes the second neighbor set of vertex \( i_0 \), that is, \( N_{b_0} = \{ i_0 \} \) such that \( i_0 \sim p \) and \( p \sim r \). Since \( N_{b_0} \subseteq V_1 \), \( r \) becomes an eigenvector corresponding to the largest eigenvalue \( \rho_1(W_{b,k}) \) for all \( i_0 \in V_2 \), that is, \( \gamma_i \) is constant for all \( i \in V_2 \). Hence \( \bar{x}_b = \bar{x}_a \) for all \( k \in N_b \) that \( r \in V_1 \). Hence \( N_{b_0} \subseteq V_1 \). Continuing the same procedure, it is easy to see, since \( G \) is connected, that \( V_1 = V \). This implies \( \bar{x}_b \) is a common eigenvector of positive definite matrix \( W_{b,k} \) corresponding to the largest eigenvalue \( \rho_1(W_{b,k}) \) for all \( i_0 \in V_2 \), that is, \( \gamma_i \) is constant for all \( i \in V_2 \). Hence \( \bar{x}_b = \bar{x}_a \) for all \( k \in N_b \) that \( r \in V_1 \). Hence \( G \) is a weighted pseudo-semi-regular bipartite graph.

Conversely, suppose that both (i) and (ii) hold for weighted graph \( G \). Let \( \bar{x} \) be a common eigenvector of positive definite matrix \( W_{b,k} \) corresponding to the largest eigenvalue \( \rho_1(W_{b,k}) \) for all \( i_0 \in V_2 \). It follows from Lemma 2.2 that \( x \) is an eigenvector of \( W_{b,k} \) corresponding to the largest eigenvalue \( \rho_1(W_{b,k}) \) for all \( i_0 \in V_2 \).

Firstly, assume that \( G \) is a weighted pseudo-semi-regular bipartite graph with bipartition \( V_1, V_2 \). Let \( \bar{\gamma}_i = p_1 \) for all \( i \in V_1 \) and \( \bar{\gamma}_j = p_2 \) for all \( j \in V_2 \). Without loss generality, we also assume that \( V_1 = \{ 1, 2, \ldots, k \} \) and \( V_2 = \{ k+1, k+2, \ldots, n \} \). Let \( X \) be the following vector:
\[ X = \left( \begin{array}{c}
(w_1 x)^T, (w_2 x)^T, \ldots, (w_k x)^T, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_1}
\end{array} \right)^T.
\]

It is easy to verify that \( A X = \sqrt{p_1} p_2 X \). Thus \( p_1 p_2 \leq |\rho_1| \). From (1), we have \( |\rho_1| \leq \max_{i-j} \{ \sqrt{|\gamma_i \gamma_j|} \} = \sqrt{p_1 p_2} \), which implies \( |\rho_1| = \sqrt{p_1 p_2} \).

Now assume that \( G \) is a weighted pseudo-regular graph. Let \( \bar{\gamma}_i = p \) for all \( i \in V \). Also let \( X = \left( \begin{array}{c}
(w_1 x)^T, (w_2 x)^T, \ldots, (w_k x)^T, (w_{k+1} x)^T, \ldots, (w_n x)^T
\end{array} \right)^T.
\]

Similarly, we get \( |\rho_1| = p \). This completes the proof. □

Corollary 2.4.

(1) Suppose that \( G \) is a pseudo-regular graph.
(i) Let G be a weighted graph, in which the edge weights $w_{ij}$ are positive definite matrices. Also let $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all $i, j \in V$. If $\bar{\gamma}_i = p$ for all $i \in V$, then $|\rho_1| = p$.

(ii) Let G be a weighted graph, in which the edge weights are positive numbers. If $w_i = p$ for all $i \in V$, then $\rho_1 = p$.

(iii) Suppose that G is a weighted pseudo-semiregular bipartite graph with the bipartition $V_1, V_2$ of V.

(1) Let G be a weighted graph, in which the edge weights $w_{ij}$ are positive definite matrices. Also let $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all $i, j \in V$. If $\bar{\gamma}_i = p_i$ for all $i \in V_1$ and $\bar{\gamma}_i = p_2$ for all $i \in V_2$, then $|\rho_1| = \sqrt{p_1p_2}$.

(ii) Let G be a weighted graph, in which the edge weights are positive numbers. If $w_i = p_i$ for all $i \in V_1$ and $w_i = p_2$ for all $i \in V_2$, then $\rho_1 = \sqrt{p_1p_2}$.

(iii) Suppose that G is an unweighted graph. If $m_i = p_i$ for all $i \in V_1$ and $m_i = p_2$ for all $i \in V_2$, then $\rho_1 = \sqrt{p_1p_2}$.

**Proof.** The proof follows directly from the proof of Theorem 2.3. \qed

**Corollary 2.5.** Suppose that G is a simple connected graph.

(i) If G is a weighted graph, in which the edge weights are positive numbers, then

$$\rho_1 \leq \max_{i,j} \sqrt{w_iw_j}.$$  \hfill (9)

Moreover, the equality in (9) holds if and only if G is either a weighted pseudo-regular graph or a weighted pseudo-semiregular bipartite graph.

(ii) Suppose that G is an unweighted graph, then

$$\rho_1 \leq \max_{i,j} \sqrt{m_im_j}.$$  \hfill (10)

Moreover, the equality in (10) holds if and only if G is either an unweighted pseudo-regular graph or an unweighted pseudo-semiregular bipartite graph.

*Note: Corollary 2.5 corrects the mistakes of Corollaries 2 and 3 in [10].*

**Proof.** These are direct consequences of Theorem 2.3. \qed

**Corollary 2.6.** Let G be a simple connected weighted graph, and let $\rho_1$ be the largest eigenvalue, in modulus, of G, so that $|\rho_1|$ is the spectral radius of G. Then

$$|\rho_1| \leq \max_{i \in V} \bar{\gamma}_i.$$  \hfill (11)

Moreover, the equality in (11) holds if and only if G is a weighted pseudo-regular graph and $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all $i, j \in V$.

**Proof.** It follows from Theorem 2.3 that the inequality (11) holds.

Now suppose that equality in (11) holds. Also applying Theorem 2.3, we get (i) G is either a weighted pseudo-regular graph or a weighted pseudo-semiregular bipartite graph; (ii) $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all $i, j \in V$.

Now assume that G is a weighted pseudo-semiregular bipartite graph, but not a weighted pseudo-regular graph. Let $V_1, V_2$ be the bipartition of V. Also let $\bar{\gamma}_i = p_1$ for all $i \in V_1$ and $\bar{\gamma}_i = p_2$ for all $i \in V_2$. From our assumption, $p_1 \neq p_2$. Thus $|\rho_1| \leq \sqrt{p_1p_2} < \max\{p_1, p_2\}$, which contradicts our assumption.

Conversely, Suppose that G is a weighted pseudo-regular graph and $w_{ij}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all $i, j \in V$. From Corollary 2.4, we get the required result. \qed

**Corollary 2.7.** Suppose that G is a simple connected graph.

(i) If G is a weighted graph, in which the edge weights are positive numbers, then

$$\rho_1 \leq \max_{i \in V} w_i$$  \hfill (12)

with equality holds if and only if G is a weighted pseudo-regular graph.

(ii) Suppose that G is an unweighted graph, then
\[ \rho_1 \leq \max_{i \in V} m_i \]  
with equality holds if and only if \( G \) is an unweighted pseudo-regular graph.

**Proof.** Applying Corollary 2.6, we get easily the required results. \( \Box \)

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