Sharp bounds for the signless Laplacian spectral radius of digraphs

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ARTICLE INFO

Keywords:
Digraph
Spectral radius
Signless Laplacian

ABSTRACT

Let \( G = (V, E) \) be a digraph with \( n \) vertices and \( m \) arcs without loops and multiarcs, and vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \). Denote the outdegree and average 2-outdegree of the vertex \( v_i \) by \( d_i^+ \) and \( m_i^+ \), respectively. Let \( A(G) \) be the adjacency matrix and \( D(G) = \text{diag}(d_1^+, d_2^+, \ldots, d_n^+) \) be the diagonal matrix with outdegree of the vertices of the digraph \( G \). Then we call \( Q(G) = D(G) + A(G) \) the signless Laplacian matrix of \( G \). Let \( q(G) \) denote the signless Laplacian spectral radius of the digraph \( G \). In this paper, we present several improved bounds in terms of outdegree and average 2-outdegree for the signless Laplacian spectral radius of digraphs. Then we give an example to compare the bounds.

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1. Introduction

Let \( G = (V, E) \) be a digraph with \( n \) vertices and \( m \) arcs without loops and multiarcs, and vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \). Let \((v_i, v_j)\) be an arc of \( G \). Then \( v_i \) is called the initial vertex and \( v_j \) is called the terminal vertex of this arc. The outdegree \( d_i^+ \) of a vertex \( v_i \) in the digraph \( G \) is defined to be the number of arcs in \( G \) with initial vertex \( v_i \). Let \( t_i^+ \) be the sum of the outdegrees of all vertices in \( N^+(v_i) = \{ v_j : (v_i, v_j) \in E \} \) and call it 2-outdegree. Moreover, we call \( m_i^+ = \frac{t_i^+}{n} \) average 2-outdegree, \( 1 \leq i \leq n \). If the average 2-outdegrees of vertices in \( V \) are the same, we call \( G \) average 2-outdegree regular digraph. If \( V = V_1 \cup V_2 \) and the average 2-outdegrees of the vertices in \( V_1 \) and \( V_2 \) are \( m_1^+ \) and \( m_2^+ \), respectively, we call \( G \) average 2-outdegree semiregular digraph. \( \delta^+(G) \) is defined to be the minimum outdegree among all the vertices of \( G \). For convenience, we sometimes abbreviate \( \delta^+(G) \) to \( \delta^+ \).

Let \( A(G) \) be the adjacency matrix and \( D(G) = \text{diag}(d_1^+, d_2^+, \ldots, d_n^+) \) be the diagonal matrix with outdegree of the vertices of the digraph \( G \). Then we call \( Q(G) = D(G) + A(G) \) the signless Laplacian matrix of \( G \). In general, \( Q(G) \) is not symmetric and so its eigenvalues can be complex numbers. Let \( q(G) \) denote the signless Laplacian spectral radius of digraph \( G \). Since \( Q(G) \) is a nonnegative matrix, it follows from Perron Frobenius theory that \( q = q(G) \) is a real number.

For the signless Laplacian spectral radius of a simple, undirected graph, this is a classical problem with numerous results pertaining to it (see [1,3,4,7,9]). There are some papers that give the bounds for the spectral radius of digraphs (see [5,8,10]). Now we consider the signless Laplacian spectral radius of a digraph. For applications it is crucial to be able to compute or at least estimate \( q(G) \) for a given digraph \( G \). Let \( \mathbb{R} \) be the set of real numbers and \( \mathbb{R}^+ = \{ x : x \in \mathbb{R}, x > 0 \} \). Now we present the

* This work was supported by the National Natural Science Foundation of China (No. 11171273).

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http://dx.doi.org/10.1016/j.amc.2014.04.001
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main results of this paper. In [2], Bozkurt et al. obtained the following bounds for signless Laplacian spectral radius of a digraph.

**Lemma 1.1** [2]. Let $G = (V, E)$ be a digraph on $n$ vertices and $b^+_i \in \mathbb{R}^+ (1 \leq i \leq n)$. Then

$$
\min \{r^+_i : v_i \in V\} \leq q(G) \leq \max \{r^+_i : v_i \in V\}
$$

(1)

where $r^+_i = d^+_i + \frac{1}{n} \sum_{(v_i, v_j) \in E} b^+_j (1 \leq i \leq n)$. Moreover, if $G$ is a strongly connected digraph, the equality holds on both sides of (1) if and only if $r^+_1 = r^+_2 = \cdots = r^+_n$.

**Lemma 1.2** [2]. Let $G = (V, E)$ be a strongly connected digraph on $n$ vertices and $b^+_i \in \mathbb{R}^+ (1 \leq i \leq n)$. Then

$$
q(G) \leq \max \left\{ \frac{d^+_i + d^+_j + \sqrt{(d^+_i - d^+_j)^2 + 4c^+_i c^+_j}}{2} : (v_i, v_j) \in E \right\}
$$

(2)

and if $q(G) > \max \left\{ \frac{d^+_i + d^+_j - \sqrt{(d^+_i - d^+_j)^2 + 4c^+_i c^+_j}}{2} : (v_i, v_j) \in E \right\}$, then

$$
q(G) \geq \min \left\{ \frac{d^+_i + d^+_j + \sqrt{(d^+_i - d^+_j)^2 + 4c^+_i c^+_j}}{2} : (v_i, v_j) \in E \right\}
$$

(3)

where $c^+_i = \frac{1}{n} \sum_{(v_i, v_j) \in E} b^+_j (1 \leq i \leq n)$.

**Lemma 1.3** [2]. Let $G = (V, E)$ be a strongly connected digraph on $n$ vertices and $b^+_i \in \mathbb{R}^+ (1 \leq i \leq n)$. Then

$$
q(G) \leq \max \left\{ \frac{d^+_i + \sqrt{\sum_{(v_i, v_j) \in E} s^+_j}}{2} : v_i \in V \right\}
$$

(4)

where $s^+_i = \frac{1}{n^2} \sum_{(v_i, v_j) \in E} b^+_j \cdot b^+_j$ and if the equality holds then $d^+_i + \sqrt{\sum_{(v_i, v_j) \in E} s^+_j} (1 \leq i \leq n)$ is a constant.

From Lemmas 1.1, 1.2, and 1.3, the following bounds are obtained in [2].

1. Taking $b^+_i = d^+_i$ in (1), we have the bounds

$$
\min \{d^+_i + m^+_i : v_i \in V\} \leq q(G) \leq \max \{d^+_i + m^+_i : v_i \in V\}.
$$

(5)

2. Taking $b^+_i = 1$ in (2) and (3), we have the upper and lower bounds

$$
\min \{d^+_i + d^+_j : (v_i, v_j) \in E\} \leq q(G) \leq \max \{d^+_i + d^+_j : (v_i, v_j) \in E\}
$$

(6)

3. Taking $b^+_i = d^+_i$ in (2) and (3), we have the upper bound

$$
q(G) \leq \max \left\{ \frac{d^+_i + d^+_j + \sqrt{(d^+_i - d^+_j)^2 + 4m^+_i m^+_j}}{2} : (v_i, v_j) \in E \right\}
$$

(7)

and the lower bound

$$
q(G) \geq \min \left\{ \frac{d^+_i + d^+_j + \sqrt{(d^+_i - d^+_j)^2 + 4m^+_i m^+_j}}{2} : (v_i, v_j) \in E \right\}
$$

(8)

In this paper, we present several bounds in terms of outdegree and average 2-outdegree for the signless Laplacian spectral radius of digraphs. Then we give an example to compare the bounds for signless Laplacian spectral radius of digraphs.

2. Some bounds for the signless Laplacian spectral radius of digraphs

In this section, we first present some known lemmas and results that will be used in the following study.
Lemma 2.1 [6]. Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix and let $r_i(M)$ be the $i$-th row sum of $M$, i.e., $r_i(M) = \sum_{j=1}^{n} m_{ij}(1 \leq i \leq n)$. Then
\[
\min\{r_i(M) : 1 \leq i \leq n\} \leq \rho(M) \leq \max\{r_i(M) : 1 \leq i \leq n\}.
\]
If $M$ is irreducible, then each equality holds if and only if $r_1(M) = r_2(M) = \cdots = r_n(M)$.

Lemma 2.2 [6]. Let $M$ be an irreducible nonnegative matrix. Then $\rho(M)$ is an eigenvalue of $M$ and there is a positive vector $X$ such that $MX = \rho(M)X$.

Now we give some bounds for the signless Laplacian spectral radius of digraphs.

Theorem 2.3. Let $G$ be a digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and arc set $E$. Then
\[
q(G) \leq \max \left\{ \frac{d_i^- + \sqrt{d_i^-^2 + 4m_i^+ \left(d_i^- + m_i^+\right)}}{2} : (v_i, v_j) \in E \right\}.
\]
(9)
Moreover, if $G$ is a strongly connected digraph, the equality holds if and only if $G$ is average 2-outdegree regular.

Proof. Let $G$ be a digraph and let $D(G) = \text{diag}(d_1^-, d_2^-, \ldots, d_n^-)$ be the diagonal matrix with outdegree of the vertices of the digraph $G$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $D(G)^{-1}Q(G)D(G)$ corresponding to the spectral radius $q(G)$. We assume that one of the eigencomponents, say $x_i$, is equal to 1, and the other eigencomponents are less than or equal to 1, i.e., $x_i = 1$ and $0 < x_k \leq 1$ for all $k \neq i$. Let $x_j = \max\{x_k : (v_i, v_k) \in E\}$. Since

\[
D(G)^{-1}Q(G)D(G)X = q(G)X,
\]
(10)
From the equation, we have
\[
q(G)x_i = d_i^+x_i + \sum_{(v_i, v_j) \in E} \frac{d_j^- x_j}{d_i^-}.
\]
And $x_i = 1$ and $x_j = \max\{x_k : (v_i, v_k) \in E\} \leq 1$, we have
\[
q(G) \leq d_i^+ + \sum_{(v_i, v_j) \in E} \frac{d_j^- x_j}{d_i^-} = d_i^+ + m_i^+ x_j
\]
(11)
Similarly, from (10) we have
\[
q(G)x_j = d_j^+ x_j + \sum_{(v_j, v_i) \in E} \frac{d_i^- x_i}{d_j^-}.
\]
i.e.
\[
q(G)x_j \leq d_j^+ x_j + m_j^+ \leq d_j^+ + m_j^+.
\]
(12)
From (11) and (12), we get
\[
q(G)^2 \leq (d_i^+ + m_i^+ x_i)q(G) = d_i^+ q(G) + m_i^+ x_i q(G) \leq d_i^+ q(G) + m_i^+ (d_j^+ + m_j^+).
\]
Therefore
\[
q(G) \leq \frac{d_i^+ + \sqrt{d_i^+^2 + 4m_i^+(d_j^+ + m_j^+)}}{2}.
\]
Then (9) follows.

Now suppose that the equality in (9) holds. Then all inequalities in the above argument must be equalities. In particular, from (11), we get that $x_j = x_k$ for all $k$ such that $(v_i, v_k) \in E$. Also from (12), we get that $x_k = x_i = 1$ and $x_j = 1$ for all $k$ such that $(v_j, v_k) \in E$.

Assume that $G$ is a strongly connected digraph such that the equality in (9) holds. Let $V_1 = \{v_k : x_k = 1\}$, if $V_1 \neq V$, there exist vertices $v_r, v_p \in V_1$ and $v_q \notin V_1$ such that $(v_r, v_p) \in E$ and $(v_p, v_q) \in E$. Therefore, from $q(G)x_r = d_i^+ x_r + \sum_{(v_r, v_j) \in E} \frac{d_j^- x_j}{d_i^-} \leq d_i^+ + m_i^+ x_p$ and $q(G)x_p = d_i^+ x_p + \sum_{(v_p, v_j) \in E} \frac{d_j^- x_j}{d_i^-} \leq d_i^+ + m_i^+$. Then we have
\[
q(G) < \frac{d_i^+ + \sqrt{d_i^+^2 + 4m_i^+(d_j^+ + m_j^+)}}{2},
\]
which contradicts the hypothesis that the equality holds in (9). Thus $V_1 = V$ and $G$ is a digraph with all the vertices have equal average 2-outdegree. So $G$ is average 2-outdegree regular. \qed
Similarly, we can obtain a sharp lower bound on the signless Laplacian spectral radius \( q(G) \).

**Theorem 2.4.** Let \( G \) be a digraph with vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \) and arc set \( E \). If
\[
q(G) > \max \left\{ \frac{d_i^+ - \sqrt{d_i^+ - 2 + 4m_i^+ (d_i^+ + m_i^+)}}{2} : (v_i, v_j) \in E \right\},
\]
then
\[
q(G) \geq \min \left\{ \frac{d_i^+ + \sqrt{d_i^+ - 2 + 4m_i^+ (d_i^+ + m_i^+)}}{2} : (v_i, v_j) \in E \right\}.
\]

Moreover, if \( G \) is a strongly connected digraph, the equality holds if and only if \( G \) is average 2-outdegree regular.

**Proof.** Let \( G \) be a digraph and let \( D(G) = \text{diag}(d_1^+, d_2^+, \ldots, d_n^+) \) be the diagonal matrix with outdegree of the vertices of the digraph \( G \). Let \( X = (x_1, x_2, \ldots, x_n)^T \) be an eigenvector of \( D(G)^{-1}Q(G)D(G) \) corresponding to the spectral radius \( q(G) \).

We assume that one of the eigencomponents, say \( x_i \), is equal to 1, and the other eigencomponents are greater than or equal to 1, i.e., \( x_i = 1 \) and \( x_k \geq 1 \) for all \( k \neq i \). Let \( x_j = \min\{x_k : (v_i, v_k) \in E\} \).

Since
\[
D(G)^{-1}Q(G)D(G)X = q(G)X,
\]
From the equation, we have
\[
q(G)x_i = d_i^+x_i + \sum_{(v_i, v_k) \in E} \frac{d_k^+x_k}{d_i^+}.
\]
And \( x_i = 1 \) and \( x_j = \min\{x_k : (v_i, v_k) \in E\} \geq 1 \), we have
\[
q(G) \geq d_i^+ + \sum_{(v_i, v_k) \in E} \frac{d_k^+x_k}{d_i^+} = d_i^+ + m_i^+x_j
\]
Similarly, from (10) we have
\[
q(G)x_j = d_j^+x_j + \sum_{(v_j, v_k) \in E} \frac{d_k^+x_k}{d_j^+},
\]
i.e.
\[
q(G)x_j \geq d_j^+x_j + m_j^+ \geq d_j^+ + m_j^+.
\]
From (14) and (15), we get
\[
q(G) \geq (d_i^+ + m_i^+x_j)q(G) = d_i^+q(G) + m_i^+x_jq(G) \geq d_i^+q(G) + m_i^+(d_i^+ + m_i^+).
\]
By solving the above inequality with the respect to the condition
\[
q(G) > \max \left\{ \frac{d_i^+ - \sqrt{d_i^+ - 2 + 4m_i^+ (d_i^+ + m_i^+)}}{2} : (v_i, v_j) \in E \right\},
\]
we arrive at
\[
q(G) \geq \min \left\{ \frac{d_i^+ + \sqrt{d_i^+ - 2 + 4m_i^+ (d_i^+ + m_i^+)}}{2} : (v_i, v_j) \in E \right\}.
\]
Then (13) follows.

Similarly as the proof of Theorem 2.4, we can show that the equality holds if and only if \( G \) is average 2-outdegree regular. \( \square \)

**Theorem 2.5.** Let \( G \) be a strongly connected digraph with vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \) and \( d_1^+ \geq d_2^+ \geq \ldots \geq d_n^+ \). Then
\[ q(G) \leq \min_{i \in [n]} \left\{ d_i^+ + 2d_i^+ - 1 + \sqrt{\left(2d_i^+ - d_i^+ + 1\right)^2 + 8(i-1)(d_i^+ - d_i^-)} \right\}. \]  

Moreover, if \( i = 1 \), the equality holds if and only if \( G \) is a regular digraph. If \( 2 \leq i \leq n \), the equality holds if and only if \( G \) is a regular digraph or a bidigreed digraph in which \( d_i^+ = d_i^- = \cdots = d_{i-1}^+ = n-1 \) and \( d_i^+ = d_i^- = \cdots = d_n^+ = \delta^+ \).

**Proof.** When \( i = 1 \) or \( d_i^+ = d_i^- \), by (6), it is clearly that the inequality (16) is true and the equality holds if and only if \( G \) is a regular digraph.

Now we suppose \( d_i^+ > d_i^- \), clearly, \( 2 \leq i \leq n \). The signless Laplacian matrix of \( G \) can be written as

\[ Q(G) = (q_{ij})_{n \times n} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \]

where \( Q_{11} \) is an \((i-1) \times (i-1)\) matrix and \( Q_{22} \) is an \((n-i+1) \times (n-i+1)\) matrix. Let

\[ U = \begin{pmatrix} x & I_{i-1} \\ 0 & I_{n-i+1} \end{pmatrix}, \]

\[ U^{-1} = \begin{pmatrix} \frac{1}{x}I_{i-1} & 0 \\ 0 & I_{n-i+1} \end{pmatrix}, \]

where \( x > 1 \), \( I_{i-1} \) is an \((i-1) \times (i-1)\) unit matrix, and \( I_{n-i+1} \) is an \((n-i+1) \times (n-i+1)\) unit matrix. Then

\[ B = U^{-1}Q(G)U = \begin{pmatrix} Q_{11} & \frac{1}{x}Q_{12} \\ xQ_{21} & Q_{22} \end{pmatrix}. \]

Obviously, \( Q(G) \) and \( B \) are similar matrices, they have the same characteristic roots. So, \( q(G) = q(B) \), where \( q(B) \) is the largest eigenvalue of the matrix \( B \). Now we consider the row sums \( r_1(B), r_2(B), \ldots, r_n(B) \) of the matrix \( B \).

\[ r_l(B) = \sum_{j=1}^{l-1} q_{lj} + \frac{1}{x} \sum_{j=1}^{l-1} q_{lj} = \frac{1}{x} \sum_{j=1}^{n} q_{lj} + \left(1 - \frac{1}{x}\right) \sum_{j=1}^{l-1} q_{lj} = \frac{2}{x} d_i^+ + \left(1 - \frac{1}{x}\right) \sum_{j=1}^{l-1} q_{lj}, \]

where \( 1 \leq l \leq i-1 \), and

\[ r_k(B) = \sum_{j=1}^{k-1} q_{kj} + \sum_{j=1}^{n} q_{kj} = \sum_{j=1}^{n} q_{kj} + (x-1) \sum_{j=1}^{l-1} q_{kj} = 2d_i^+ + (x-1) \sum_{j=1}^{l-1} q_{kj}, \]

where \( i \leq k \leq n \). In the matrix \( Q(G) \), \( q_{ii} = d_i^+, q_{ij} = 0 \) or 1 if \( i \neq j \), where \( i, j = 1, 2, \ldots, n \). Hence \( \sum_{j=1}^{l-1} q_{lj} \leq d_i^+ + i - 2 \), where \( 1 \leq l \leq i-1 \) and \( \sum_{j=1}^{l-1} q_{kj} \leq i - 1 \), where \( i \leq k \leq n \). As \( x > 1 \), \( d_i^+ > d_i^- \Rightarrow \cdots \Rightarrow d_n^+ \), then

\[ r_l(B) \leq \frac{2}{x} d_i^+ + \left(1 - \frac{1}{x}\right) (d_i^+ + i - 2) = \left(1 + \frac{1}{x}\right) d_i^+ + \left(1 - \frac{1}{x}\right) (i - 2), \quad 1 \leq l \leq i-1. \]  

(17)

and

\[ r_k(B) \leq 2d_i^+ + (x-1)(i-1), \quad i \leq k \leq n. \]  

(18)

From (17) and (18), we get

\[ \max(r_1(B), \ldots, r_n(B)) \leq \max \left\{ \left(1 + \frac{1}{x}\right) d_i^+ + \left(1 - \frac{1}{x}\right) (i - 2), 2d_i^+ + (x-1)(i-1) \right\}. \]

Let

\[ \left(1 + \frac{1}{x}\right) d_i^+ + \left(1 - \frac{1}{x}\right) (i - 2) = 2d_i^+ + (x-1)(i-1). \]

Solving the equality, we get

\[ x = \frac{2i + d_i^+ - 2d_i^+ - 3 + \sqrt{(2d_i^+ - d_i^- + 1)^2 + 8(i-1)(d_i^+ - d_i^-)}}{2(i-1)}. \]

Since \( i \geq 2 \), \( d_i^+ > d_i^- \), we can easily obtain that \( x > 1 \). By Lemma 2.1, we get

\[ q(G) \leq \max\{r_i(B) : 1 \leq i \leq n\} \leq 2d_i^+ + (x-1)(i-1) = \frac{d_i^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_i^- + 1)^2 + 8(i-1)(d_i^+ - d_i^-)}}{2}. \]
Then (16) follows.

For equality to hold in (16), all inequalities in the above argument must be equalities. Let \( Q(G) = (q_{ij})_{n \times n} \). In particular, from (17) we have that \( d_i^+ = d_i^- \) when \( 1 \leq l \leq i - 1 \), and \( q_{ij} = 1 \) when \( 1 \leq l \leq i - 1 \) and \( 1 \leq j \leq i - 1 \). From (18) we get that \( q_{ij} = 1 \) and \( d_i^+ = d_i^- \) where \( i \leq k \leq n \), \( 1 \leq j \leq i - 1 \). So \( q_{ij} = 1 \) when \( i \leq k \leq n \), \( 1 \leq j \leq i - 1 \). Thus \( d_i^+ = n - 1 \). This means that the degree sequence of \( G \) satisfying \( n - 1 = d_i^+ = \cdots = d_{i-1}^+ > d_i^- = \cdots = d_n^+ \). Hence \( G \) is a bidegreed digraph in which \( d_i^+ = \cdots = d_{i-1}^+ = n - 1 \) and \( d_i^- = \cdots = d_n^- = \delta^+ \). Conversely, it is easy to prove that the equality holds when \( G \) is a bidegreed digraph in which \( d_1^+ = \cdots = d_{n-1}^+ = n - 1 \) and \( d_n^- = \cdots = d_n^+ = \delta^+ \).

This completes the proof. \( \Box \)

**Theorem 2.6.** Let \( G \) be a digraph with vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \) and let \( d_i^+ \) be the outdegree of vertex \( v_i \), \( i = 1, 2, \ldots, n \). Then

\[
q(G) \leq \max \left\{ d_i^+ + \frac{d_i^+ (m_i^+ + \sqrt{m_i^-})}{d_i^- + \sqrt{d_i^+}} : v_i \in V \right\}.
\]  

**Proof.** Define a diagonal matrix \( P = \text{diag} \left( d_i^+ + \sqrt{d_i^+} : v_i \in V \right) \). Now we consider the matrix \( M = P^{-1} Q(G) P \), clearly \( M \) is similar to \( Q(G) \). The \((i,j)\)-th entry of \( M \) is equal to

\[
m_{ij} = \begin{cases} 
 d_i^+, & \text{if } i = j; \\
 \frac{d_i^+ + \sqrt{d_j^+}}{d_i^+ + \sqrt{d_i^-}}, & \text{if } i \sim j; \\
 0, & \text{otherwise.}
\end{cases}
\]

where \( i \sim j \) means that there exist an arc from \( i \) to \( j \). Obviously, the \( i \)-th row sum of \( M \) is

\[
r_i(M) = d_i^+ + \sum_{i \sim j} \left( \frac{d_i^+ + \sqrt{d_j^+}}{d_i^+ + \sqrt{d_i^-}} \right) = d_i^+ + \frac{d_i^+ m_i^+}{d_i^- + \sqrt{d_i^+}} + \sum_{i \sim j} \sqrt{d_j^+}.
\]

By the Cauchy–Schwarz inequality, we have

\[
\left( \sum_{i \sim j} \sqrt{d_j^+} \right)^2 \leq d_i^+ \sum_{i \sim j} d_j^+ = (d_i^+) m_i^+.
\]

Substituting (21) into (20) and simplifying the inequality, then we obtain

\[
r_i(M) \leq d_i^+ + \frac{d_i^+ (m_i^+ + \sqrt{m_i^-})}{d_i^- + \sqrt{d_i^+}}.
\]

By Lemma 2.1, we have

\[
q(G) \leq \max \{ r_i(M) : v_i \in V \}.
\]

Then (19) follows. \( \Box \)

3. Example

**Example 1.** Let \( G_1, G_2, G_3 \) be the digraphs of order 4, 5, 10, respectively, as shown in Fig. 1.

We summarize all bounds for the largest signless Laplacian eigenvalue of \( G_1, G_2, G_3 \) as follows:

![Fig. 1. Three digraphs G1, G2, and G3.](image-url)
Table 1
The value of some known upper bounds for Example 1.

<table>
<thead>
<tr>
<th></th>
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<th>(5)</th>
<th>(7)</th>
<th>(9)</th>
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<td>4.0000</td>
<td>3.5000</td>
<td>3.3028</td>
<td>3.3452</td>
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</tr>
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<td>(G_2)</td>
<td>3.8450</td>
<td>5.0000</td>
<td>4.3333</td>
<td>4.2078</td>
<td>4.1300</td>
<td>4.7321</td>
</tr>
<tr>
<td>(G_3)</td>
<td>2.8389</td>
<td>4.0000</td>
<td>3.5000</td>
<td>3.3028</td>
<td>3.3452</td>
<td>4.0000</td>
</tr>
</tbody>
</table>

Table 2
The value of some known lower bounds for Example 1.

<table>
<thead>
<tr>
<th></th>
<th>(6)</th>
<th>(5)</th>
<th>(8)</th>
<th>(13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_1)</td>
<td>3.0000</td>
<td>2.0000</td>
<td>2.0000</td>
<td>2.4142</td>
</tr>
<tr>
<td>(G_2)</td>
<td>3.8450</td>
<td>3.0000</td>
<td>3.0000</td>
<td>3.3028</td>
</tr>
<tr>
<td>(G_3)</td>
<td>2.8389</td>
<td>2.0000</td>
<td>2.0000</td>
<td>2.4142</td>
</tr>
</tbody>
</table>

Remark 1. Obviously, from Table 1, bound (9) is the best in all known upper bounds for \(G_2\). Bound (9) is the second-best bound for \(G_1\) and \(G_3\), respectively. Bound (5) is better than bounds (16) and (19) for \(G_1\), \(G_2\) and \(G_3\), respectively. In general, these bounds are incomparable.

Remark 2. From Table 2, bounds (8) and (13) are the best and the second-best bounds in all the lower bounds for digraphs \(G_1\), \(G_2\), \(G_3\), respectively. In general, these bounds are incomparable.

Acknowledgements

The authors would like to express their sincere gratitude to the anonymous referees for their comments and remarks, which improved the presentation of this paper.

References