REMARKS ON SPECTRAL RADIUS AND LAPLACIAN EIGENVALUES OF A GRAPH

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Abstract. Let $G$ be a graph with $n$ vertices, $m$ edges and a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \ldots \geq d_n$. The spectral radius and the largest Laplacian eigenvalue are denoted by $\rho(G)$ and $\mu(G)$, respectively. We determine the graphs with

$$\rho(G) = \frac{d_n - 1}{2} + \sqrt{2m - nd_n + \frac{(d_n + 1)^2}{4}}$$

and the graphs with $d_n \geq 1$ and

$$\mu(G) = d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i (d_i - d_n) + \left(d_n - \frac{1}{2}\right)^2}.$$ 

We also present some sharp lower bounds for the Laplacian eigenvalues of a connected graph.

Keywords: spectral radius, Laplacian eigenvalue, strongly regular graph

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1. Introduction

Let $G = (V, E)$ be a simple finite undirected graph with a vertex set $V$ and an edge set $E$. Let $\delta(G) = \delta$ be the minimal degree of vertices of $G$. Let $A(G)$ be the $(0, 1)$ adjacency matrix of $G$ and $D(G)$ the diagonal matrix of vertex degrees. An eigenvalue of $G$ is an eigenvalue of $A(G)$. The spectral radius $\rho(G)$ of $G$ is the largest

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eigenvalue of $G$. It turns out that the Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$ and $L(G)$ is positive semidefinite and singular. A Laplacian eigenvalue of $G$ is an eigenvalue of $L(G)$. Denote the Laplacian eigenvalues of $G$ by $\mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$. We also write $\mu(G)$ for $\mu_1(G)$. It is well known [3] that $\mu_{n-1}(G) > 0$ if and only if $G$ is connected.

Let $G$ be a graph with $n$ vertices, $m$ edges and a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \ldots \geq d_n$. We determine the graphs with

$$g(G) = \frac{d_n - 1}{2} + \sqrt{2m - nd_n + \frac{(d_n + 1)^2}{4}}$$

and the graphs with $d_n \geq 1$ and

$$\mu(G) = d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \left( d_n - \frac{1}{2} \right)^2}.$$  

We also present some lower bounds for the Laplacian eigenvalues of a connected graph.

2. Spectral radius

Recall that a bidegreed graph is a graph with two different vertex degrees. Hong, Shu and Fang [6] proved

**Theorem 1** [6]. Let $G$ be a connected graph with $n$ vertices and $m$ edges and let $\delta$ be the minimal degree of vertices of $G$. Then

$$g(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}},$$

and equality holds if and only if $G$ is either a regular graph or a bidegreed graph in which each vertex is of degree either $\delta$ or $n - 1$.

Recently, Nikiforov [10] proved the above inequality independently by a quite different method for a (not necessarily connected) graph, and mentioned that equality holds for regular graphs, the maximally irregular $n$-vertex graph which is the complement of $K_{n-k}$, and the disjoint union of $K_p$ and $K_{n-p}$. Based on Theorem 1, we can characterize the extreme case for not necessarily connected graphs.

**Lemma 1** [6, 10]. For nonnegative integers $p$ and $q$ with $2q \leq p(p - 1)$ and $0 \leq x \leq p - 1$, the function $f(x) = (x - 1)/2 + \sqrt{2q - px + (1 + x)^2}/4$ is decreasing with respect to $x$.
Theorem 2. Let $G$ be a graph with $n$ vertices and $m$ edges and let $\delta$ be the minimal degree of vertices of $G$. Then

\[ \varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}, \]

and equality holds if and only if in one component of $G$ each vertex is either of degree $\delta$ or adjacent to all other vertices, and all other components are regular with degree $\delta$.

**Proof.** The case that $G$ is connected is proved in [6]. Suppose $G$ is not connected. Then there is a component $G_1$ of $G$ such that $\varrho(G) = \varrho(G_1)$. Suppose $G_1$ has $n_1$ vertices, $m_1$ edges and a minimal vertex degree $\delta_1$. Let $G_2$ be the graph obtained from $G$ by deleting the component $G_1$. Suppose $G_2$ has $n_2$ vertices and $m_2$ edges. Then by Theorem 1,

\[ \varrho(G) = \varrho(G_1) \leq \frac{\delta_1 - 1}{2} + \sqrt{2m_1 - n_1\delta_1 + \frac{(\delta_1 + 1)^2}{4}}. \]

Note that $\delta_1 \geq \delta$ and $2m - n\delta = (2m_1 - n_1\delta) + (2m_2 - n_2\delta) \geq 2m_1 - n_1\delta$. By Lemma 1, we have

\[ \varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m_1 - n_1\delta + \frac{(\delta + 1)^2}{4}} \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}. \]

Suppose the equality holds in (1). Then all inequalities in the above argument are equalities. In particular, $2m_2 - n_2\delta = 0$, which implies that $G_2$ is regular with vertex degree $\delta$. We also have $\delta_1 = \delta$ and

\[ \varrho(G_1) = \frac{\delta_1 - 1}{2} + \sqrt{2m_1 - n_1\delta_1 + \frac{(\delta_1 + 1)^2}{4}}, \]

and hence by Theorem 1 we know that $G_1$ is either a regular graph with a vertex degree $\delta$ or $n_1 - 1$ or a bidegreed graph in which each vertex is of a degree either $\delta$ or $n_1 - 1$. So if the equality holds in (1), then $G_1$ is either a regular graph with a vertex degree $\delta$ or $n_1 - 1$ or a bidegreed graph in which each vertex is of a degree either $\delta$ or $n_1 - 1$ and $G_2$ is a regular graph with the vertex degree $\delta$.

Conversely, suppose one component of $G$, say, $G_1$ is a graph with $n_1$ vertices and $m_1$ edges, in which each vertex is either of a degree $\delta$ or $n_1 - 1$, and all other components are regular with the vertex degree $\delta$. Since $\varrho(G) \geq \delta$ and $2(m - m_1) - (n - n_1)\delta = 0$, we have by Theorem 1

\[ \varrho(G) = \varrho(G_1) = \frac{\delta - 1}{2} + \sqrt{2m_1 - n_1\delta + \frac{(\delta + 1)^2}{4}} \]

\[ = \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}. \]
Remark. Equality in (1) holds if and only if $G$ is a graph of one of the following four types:

(i) a regular graph with the vertex degree $\delta$;
(ii) the disjoint union of a complete graph with at least $\delta + 2$ vertices and a regular graph with the vertex degree $\delta$;
(iii) a bidegreed graph in which every vertex is either of a degree $\delta$ or $n-1$ ($\delta < n-1$);
(iv) the disjoint union of a connected bidegreed graph in which every vertex is either of the degree $\delta$ or adjacent to all other vertices, and a regular graph with the vertex degree $\delta$.

Let $G$ be a graph with $n$ vertices, $m$ edges and let $\delta$ be the minimal degree of vertices of $G$. Clearly $\delta \geq 0$. If $G$ has no isolated vertices, then $\delta \geq 1$. By Lemma 1 and Theorem 1 we have

**Corollary 1** [8]. Let $G$ be a graph with $n$ vertices and $m$ edges. Then

$$\varrho(G) \leq -\frac{1}{2} + \sqrt{2m + \frac{1}{4}},$$

and equality holds if and only if one component of $G$ is a complete graph with $m$ edges, and all other components are isolated vertices.

**Corollary 2** [5]. Let $G$ be a graph with $n$ vertices and $m$ edges. If $G$ has no isolated vertices, then

$$\varrho(G) \leq \sqrt{2m - n + 1},$$

and equality holds if and only if one component of $G$ is a star or a complete graph with at least 2 vertices, and all other components are $K_2$'s.

3. LARGEST LAPLACIAN EIGENVALUE

Recently Shu, Hong and Kai [9], using Theorem 1, provided an upper bound for the largest Laplacian eigenvalue of a connected graph in terms of the vertex degree sequence: Let $G$ be a connected graph with a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \ldots \geq d_n$. Then

$$\mu(G) \leq d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i (d_i - d_n) + \left( d_n - \frac{1}{2} \right)^2}.$$
They also pointed out that the equality holds if $G$ is a regular bipartite graph. It is mentioned in [1, p. 283] that the equality also holds if $G$ is the star graph. Let $L_G$ be the line graph of a graph $G$.

**Lemma 2** [7, 9]. If $G$ is a connected graph, then $\mu(G) \leq 2 + g(L_G)$, and equality holds if and only if $G$ is a bipartite graph.

For (not necessarily connected) graphs, we have

**Theorem 3.** Let $G$ be a graph with a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$. Then

\begin{equation}
\mu(G) \leq d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \left(\frac{d_n - 1}{2}\right)^2},
\end{equation}

and equality holds if and only if $G$ is a regular graph with at least one bipartite component, or $G$ is the disjoint union of a star graph and (possibly) $K_2$'s.

**Proof.** First suppose that $d_n \geq 2$. If $G$ is connected, then by the proof in [9, p. 128], we have (2), and equality holds in and only if $G$ is a regular gianti graph. Suppose $G$ is not connected. Then there is a component $G_1$ of $G$ such that $\mu(G) = \mu(G_1)$. Suppose $G_1$ has $n_1$ vertices, $m_1$ edges and a minimal vertex degree $\delta_1$. Suppose $L_G$ has $n'$ vertices, $m'$ edges and a minimal vertex degree $\delta'$, and $L_{G_1}$ has $n'_1$ vertices, $m'_1$ edges and a minimal vertex degree $\delta'_1$. We have

\[ n' = m = \frac{1}{2} \sum_{i=1}^{n} d_i, \quad 2m' = \sum_{i=1}^{n} d_i(d_i - 1) \quad \text{and} \quad \delta' \geq 2d_n - 2. \]

Note that $\delta'_1 \geq \delta' \geq 2d_n - 2$ and $2m' - n'\delta' \geq 2m'_1 - n'_1\delta'$. By Theorem 1 and Lemmas 1 and 2,

\[
\mu(G) = \mu(G_1) \leq 2 + g(L_{G_1}) = 2 + \frac{\delta'_1 - 1}{2} + \sqrt{\frac{(\delta'_1 + 1)^2}{4}}
\]
\[
\leq 2 + \frac{\delta' - 1}{2} + \sqrt{\frac{2m'_1 - n'_1\delta' + (\delta' + 1)^2}{4}}
\]
\[
\leq 2 + \frac{\delta' - 1}{2} + \sqrt{\frac{2m' - n'\delta' + (\delta' + 1)^2}{4}}
\]
\[
\leq d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \left(\frac{d_n - 1}{2}\right)^2}.
\]
This proves (2).

Suppose the equality holds in (2). Then all inequalities in the above argument are equalities. In particular, we have

\[ \delta' = 2d_n - 2 \quad \text{and} \quad 2(m' - m'_1) - (n' - n'_1)\delta' = 0. \]

So any component of \( L_G \) except \( L_{G_1} \) is regular with the vertex degree \( \delta' \), and hence any component \( H \) of \( G \) except \( G_1 \) is either a regular graph with the vertex degree \( d_n \) or a semi-regular bipartite graph. If \( H \) is a semi-regular bipartite graph with \( p_1 \) independent vertices of degree \( r_1 \) and \( p_2 \) independent vertices of degree \( r_2 \), then \( r_1 + r_2 = 2d_n \), which implies \( r_1 = r_2 = d_n \) since \( r_1, r_2 \geq d_n \). Hence any component \( H \) of \( G \) except \( G_1 \) is a regular graph with the vertex degree \( d_n \). Note that \( \mu(G) = \mu(G_1) \leq 2 + \varrho(L_{G_1}) \). We also have \( \delta'_1 = \delta' = 2d_n - 2 \), which implies the minimal vertex degree of \( G_1 \) is \( d_n \). Let \( (d_{11}, d_{12}, \ldots, d_{1n_1}) \) be the vertex degree sequence of \( G_1 \) with \( d_{11} \geq d_{12} \geq \ldots \geq d_{1n_1} = d_n \). Then

\[
\mu(G_1) = d_{1n_1} + \frac{1}{2} + \sqrt{\sum_{i=1}^{n_1} d_{1i}(d_{1i} - d_{1n_1}) + \left(d_{1n_1} - \frac{1}{2}\right)^2}.
\]

It follows that \( G_1 \) is a regular bipartite graph with the vertex degree \( d_n \). Hence \( G \) is regular with the vertex degree \( d_n \) and a bipartite component \( G_1 \).

Conversely, suppose \( G \) is a regular graph with at least one bipartite component. Then \( d_i = d_n = r \) for all \( i \). For any non-bipartite component \( H \) of \( G \), the smallest eigenvalue of \( H \) is \( > -r \), and hence \( \mu(H) < 2r \). For any bipartite component \( G_1 \) of \( G \), \( \mu(G) = 2r \). Hence

\[
\mu(G) = 2r = d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \left(d_n - \frac{1}{2}\right)^2}.
\]

Now suppose that \( d_n = 1 \) and \( \mu(G) = \mu(G_1) \) where \( G_1 \) is a component of \( G \) with vertex degree sequence \((d_{11}, d_{12}, \ldots, d_{1n_1})\), where \( d_{11} \geq d_{12} \geq \ldots \geq d_{1n_1} \). Let \( m'_1 \) be the number of edges of \( L_{G_1} \). Then by Corollary 1,

\[
\varrho(L_{G_1}) \geq -\frac{1}{2} + \sqrt{2m'_1 + \frac{1}{4}}.
\]
and equality holds if and only if one component of $L_{G_1}$ is a complete graph. Note that $2m'_1 = \sum_{j=1}^{n_1} d_{1j}(d_{1j} - 1)$, $(d_n = 1)$. Then by Lemma 2,

$$
\mu(G) = \mu(G_1) \leq 2 + \varrho(L_{G_1})
$$

$$
\leq \frac{3}{2} + \sqrt{2m'_1 + \frac{1}{4}}
$$

$$
= \frac{3}{2} + \sqrt{\sum_{j=1}^{n_1} d_{1j}(d_{1j} - 1) + \frac{1}{4}}
$$

$$
\leq \frac{3}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - 1) + \frac{1}{4}}
$$

$$
= d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + (d_n - \frac{1}{2})^2}.
$$

This proves (2) if $d_n = 1$.

Suppose the equality holds in (2) and $d_n = 1$. Then all inequalities in (3) and (4) are equalities. Hence $G_1$ is bipartite, $L_{G_1}$ is a complete graph, and the minimal vertex degree of any component of $G$ is 1. If $G$ is connected, then clearly $G$ is the star graph. If $G$ is not connected, then one component of $G$ is a star graph, and all other components are $K_2$’s.

Conversely, it can be easily checked that if one component of $G$ is a star graph, and all other components (if exist) are $K_2$’s, then the equality holds in (2).

$$\square$$

4. The $k$-th Laplacian eigenvalues with $k \geq 2$

Various lower bounds for $\mu_k$ ($1 \leq k \leq n - 1$) of a graph $G$ have been established, some in terms of the order, the degree sequence or the number of spanning trees of $G$ (see [4], [11]). In the following we suppose that $G$ is a connected graph with $n$ vertices and $m$ edges. Zhang and Li [11] have recently obtained a lower bound for $\mu_1(G)$ in terms of $n$ and $m$ in the form

$$
\mu_1(G) \geq \frac{1}{n - 1} \left( 2m + \sqrt{\frac{2(n^2 - n - 2m)m}{n(n - 2)}} \right),
$$

where equality holds if and only if $G = K_n$.

We present lower bounds for $\mu_k(G)$ ($2 \leq k \leq n - 1$) in terms of $n$ and $m$. 

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Lemma 3 [11]. Let $G$ be a graph with $n$ vertices, $m$ edges and a vertex degree sequence $(d_1, d_2, \ldots, d_n)$. Then
\[
\sum_{i=1}^{n} d_i^2 \leq \frac{nm^2}{n-1},
\]
where equality holds if and only if $G = K_{1,n-1}$.

The following lemma is well known [2].

Lemma 4. A connected graph with two distinct eigenvalues is complete, a regular connected graph with three distinct eigenvalues is strongly regular.

Theorem 4. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Write $M(G) = \min\{(nm - 4m + 2n - 2)m, 2(n^2 - n - 2m)m\}$. Then for $2 \leq k \leq n - 1$ we have
\[
\mu_k(G) \geq \frac{1}{n-1} \left(2m - \sqrt{\frac{k-1}{n-k} M(G)}\right),
\]
and equality holds for some $k$ with $2 \leq k \leq n - 1$ if and only if $G = K_n$.

Proof. Write $\mu_k$ for $\mu_k(G)$ and $L$ for $L(G)$. Let $\text{Tr}(B)$ be the trace of a square matrix $B$. Denote $N_k = \sum_{i=1}^{n-1} \mu_i$. Note that $\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n-1} d_i = 2m$. We have
\[
\text{Tr}(L^2) = \sum_{i=1}^{k-1} \mu_i^2 + \sum_{i=k}^{n-1} \mu_i^2 \geq \frac{1}{k-1} \left(\sum_{i=1}^{k-1} \mu_i\right)^2 + \frac{1}{n-k} \left(\sum_{i=k}^{n-1} \mu_i\right)^2
\]
\[
= \frac{(2m - N_k)^2}{k-1} + \frac{N_k^2}{n-k}.
\]
Hence
\[
N_k \geq \frac{1}{n-1} \left(2m(n-k) - \sqrt{(n-k)(k-1)((n-1) \text{Tr}(L^2) - 4m^2)}\right).
\]
Since $N_k \leq (n-k)\mu_k$, we have
\[
\mu_k \geq \frac{1}{n-1} \left(2m - \sqrt{\frac{k-1}{n-k} ((n-1) \text{Tr}(L^2) - 4m^2)}\right).
\]
By Lemma 3,
\[
(n-1) \text{Tr}(L^2) - 4m^2 = (n-1) \sum_{i=1}^{n} d_i(d_i + 1) - 4m^2 \leq (nm - 4m + 2n - 2)m.
\]
By virtue of the inequality $d_i \leq n - 1$ for $1 \leq i \leq n$ we obtain

$$(n - 1) \text{Tr}(L^2) - 4m^2 \leq (n - 1) \sum_{i=1}^{n} d_i n - 4m^2 = 2(n^2 - n - 2m)m.$$ 

Hence

$$(n - 1) \text{Tr}(L^2) - 4m^2 \leq M(G)$$ 

and (5) follows from (7).

Suppose that the equality in (5) holds for some $k_0$ with $2 \leq k_0 \leq n - 1$. Then

$$(n - 1) \text{Tr}(L^2) - 4m^2 = M(G),$$

and hence $\sum_{i=1}^{n} d_i^2 = nm^2/(n - 1)$ or $d_i = n - 1$ for $1 \leq i \leq n$. In the former case, we have $G = K_{1,n-1}$ by Lemma 3, and hence $\mu_{k_0} = 1$, which is impossible. In the latter case, we have $G = K_n$.

If $G = K_n$, then $M(G) = 0$ and hence the equality in (5) holds for each $k$ with $2 \leq k \leq n - 1$.

Note that the bound in (6) is trivial if $2m \leq \sqrt{(k - 1)/(n - k)M(G)}$. For a regular graph we give a finer lower bound for $\mu_k(G)$.

**Theorem 5.** Let $G$ be a connected regular graph with $n$ vertices and a vertex degree $\delta$. Then for $2 \leq k \leq n - 1$ we have

$$(8) \quad \mu_k(G) \geq \frac{1}{n - 1} \left( n\delta - \sqrt{\frac{k - 1}{n - k} n\delta(n - \delta - 1)} \right),$$

where equality holds for some $k$ with $2 \leq k \leq n - 1$ if and only if $G$ is $K_n$ or a strongly regular graph.

**Proof.** Note that $(n - 1) \text{Tr}(L(G)^2) - 4m^2 = (n - 1)n\delta(\delta + 1) - n^2\delta^2 = nd(n - \delta - 1)$. (8) follows from (7).

Suppose that equality in (8) holds for some $k_0$ with $2 \leq k_0 \leq n - 1$. Then the equality in (6) holds for $k = k_0$. It follows that $G$ has only two or three distinct Laplacian eigenvalues and hence has only two or three distinct eigenvalues. By Lemma 4, $G$ is $K_n$ or a strongly regular graph.

Conversely, if $G = K_n$, then equality in (8) holds for each $k$ with $2 \leq k \leq n - 1$; if $G$ is a strongly regular graph, then $G$ has three distinct eigenvalues $\delta$, $\varrho$ and $\sigma$ ($\delta > \varrho > \sigma$) with multiplicities 1, $r$ and $s$, and hence $G$ has three distinct Laplacian eigenvalues $\delta - \sigma$, $\delta - \varrho$ and 0 with multiplicities $s$, $r$ and 1, which implies that (6), (7) and hence (8) become equalities for $k = s + 1$. 

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