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Shang-Wang Tan; Jing-Jing Jiang
* Department of Mathematics, China University of Petroleum, Dongying 257061, China

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On the Laplacian spectral radius of weighted trees with fixed diameter and weight set

Shang-Wang Tan* and Jing-Jing Jiang

Department of Mathematics, China University of Petroleum, Dongying 257061, China

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The spectrum of weighted graphs is often used to solve the problems in the design of networks and electronic circuits. We first give some perturbational results on the (signless) Laplacian spectral radius of weighted graphs when some weights of edges are modified, then we determine the weighted tree with the largest Laplacian spectral radius in the set of all weighted trees with fixed diameter and positive weight set.

Keywords: weighted graph; weighted tree; Laplacian spectral radius; Perron vector

AMS Subject Classification: 05C50

1. Introduction

In this article, we only consider simple weighted graphs with positive weight set. Let \( G \) be a weighted graph with vertex set \( \{v_1, v_2, \ldots, v_n\} \), edge set \( E(G) \neq \emptyset \) and weight set \( W(G) = \{w_j > 0 : j = 1, 2, \ldots, |E(G)|\} \). The function \( w_G: E(G) \to W(G) \) is called a weight function of \( G \). It is obvious that each weighted graph corresponds to a weight function. For convenience, define \( w_G(uv) = 0 \) if \( uv \notin E(G) \). So \( G \) may be regarded as a weighted graph on a nonnegative weight set, where \( uv \in E(G) \) if and only if \( w_G(uv) > 0 \). Thus the adjacency matrix of \( G \) is defined to be the \( n \times n \) matrix \( A(G) = (w_G(v_i, v_j)) \). The weight of vertex \( v_i \), denoted by \( w_G(v_i) \), is the sum of weights of all edges incident to \( v_i \) in \( G \). Let \( W(G) = \text{diag}(w_G(v_1), w_G(v_2), \ldots, w_G(v_n)) \) be the diagonal matrix of vertex weights of \( G \). Then \( L(G) = W(G) - A(G) \) is called the Laplacian matrix of \( G \). From this fact and Geršgorin’s theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to 0, then 0 is its smallest eigenvalue. Thus the eigenvalues of \( L(G) \) are denoted by

\[
\lambda_1(L(G)) \geq \lambda_2(L(G)) \geq \cdots \geq \lambda_{n-1}(L(G)) \geq \lambda_n(L(G)) = 0,
\]

where \( \lambda_1(L(G)) \), denoted by \( \rho(G) \), is called the Laplacian spectral radius of \( G \) and \( \lambda_{n-1}(L(G)) \) is called the algebraic connectivity of \( G \).

The matrix \( R(G) = W(G) + A(G) \) is called the signless Laplacian matrix of \( G \). Note that \( R(G) \) is a nonnegative symmetric matrix, its eigenvalues are all real numbers.

*Corresponding author. Email: upctansw@yahoo.cn
numbers and its largest eigenvalue is a positive number. The largest eigenvalue \( \lambda_1(R(G)) \) of \( R(G) \) is called the signless Laplacian spectral radius of \( G \), denoted by \( \mu(G) \). Since \( R(G) \) is nonnegative, there is a nonnegative eigenvector corresponding to \( \mu(G) \). In particular, when \( G \) is connected, \( R(G) \) is irreducible and by the Perron–Frobenius theorem (see [8], for example), \( \mu(G) \) is simple and there is a unique positive unit eigenvector. We shall refer to such an eigenvector as the Perron vector of \( R(G) \).

Let \( H \) and \( G \) be two weighted graphs. \( G \) and \( H \) are called isomorphic, denoted by \( G \cong H \), if there is a bijection \( f \) from \( V(G) \) to \( V(H) \) such that \( ab \in E(G) \) if and only if \( f(a)f(b) \in E(H) \), and \( w_G(ab) = w_H(f(a)f(b)) \) for each \( ab \in E(G) \). \( H \) is called a weighted subgraph of \( G \) if \( H \) is a subgraph of \( G \) and \( w_H(e) = w_G(e) \) for each \( e \in E(H) \).

The Laplacian eigenvalues of weighted graphs have many important applications in combinatorial optimization, the design of networks, the design of electronic circuits and so on. On the other hand, unweighted graphs may be regarded as weighted graphs whose edges have weight 1. Therefore, it is significant and necessary to investigate the Laplacian eigenvalues of weighted graphs. Recently, this topic was mostly investigated in the literature. Das and Bapat [4], Rojo [10] obtained two upper bounds on the Laplacian spectral radius of weighted graphs, respectively. Rojo and Robbiano [11] gave an upper bound on the Laplacian spectral radius of weighted trees. Fernandes et al. [5] derived an upper bound on the Laplacian spectral radius of a weighted graph defined by a weighted tree and a weighted triangle attached, by one of its vertices, to a pendant vertex of the tree. Berman and Zhang [2] gave a lower bound on the second smallest Laplacian eigenvalue of weighted graphs. Kirkland and Neumann [9] researched the algebraic connectivity of weighted trees under perturbation.

The article studies the Laplacian spectral radius of weighted graphs. Throughout, let \( N_G(u) \) be the adjacent vertex set of a vertex \( u \) in \( G \), \( d_G(u) \) be the degree of \( u \) in \( G \) and \( \phi(M) = \phi(M, x) = \det(xI - M) \) be the characteristic polynomial of a matrix \( M \). In particular, let \( \phi(G) = \phi(R(G)) \). All other notations and definitions not given in the article are standard terminology of graph theory (see [3], for example).

The article is organized as follows: in Section 2 we give some perturbational results on the (signless) Laplacian spectral radius of weighted graphs when some weights of edges are modified. Furthermore, we also present some properties of characteristic polynomials of the (signless) Laplacian matrices of weighted graphs. In Section 3, we determine the weighted tree with the largest (signless) Laplacian spectral radius in the set of all weighted trees with fixed diameter and positive weight set (see [13] for similar results on the spectral radius of adjacency matrices).

2. Some perturbational results

Any modification of a weighted graph gives rise to perturbations of eigenvalues of its matrix. In the literature, this topic is extensively investigated for unweighted graphs. In this section, we mainly present some perturbational results on the signless Laplacian spectral radius of weighted graphs, which are useful and more ordinary than those of unweighted graphs.
**THEOREM 2.3**

Let $a, b, u, v$ be four vertices of a weighted graph $G$ and let $X = (x_1, x_2, \ldots, x_n)^t$ be a nonnegative unit eigenvector corresponding to $\mu(G)$, where $x_i$ corresponds to the vertex $v_i$ of $G$. For $0 < \delta \leq w_G(uv)$, let $G^1$ be the weighted graph obtained from $G$ such that

$$w_{G^1}(ab) = w_G(ab) + \delta, \quad w_{G^1}(uv) = w_G(uv) - \delta, \quad w_{G^1}(e) = w_G(e), \quad e \in E(G) \setminus \{ab, uv\}.$$  

If $x_u + x_v < x_a + x_b$, then $\mu(G) < \mu(G^1)$. In addition, if $x_u + x_v \leq x_a + x_b$ or $x_u + x_v \leq x_a + x_b$ and $G$ is connected, then $\mu(G) < \mu(G^1)$.

**Proof**

From Lemma 2.1, we have that

$$\mu(G^1) - \mu(G) = \max_{\|Y\| = 1} \frac{1}{2} [Y^t R(G^1) Y - X^t R(G) X] \geq \frac{1}{2} [R(G^1) - R(G)] X.$$  

Suppose that $x_u + x_v < x_a + x_b$. By (2.1), it is obvious that $\mu(G) < \mu(G^1)$.

Suppose that $x_u + x_v \leq x_a + x_b$ and $G$ is connected. From the Perron–Frobenius theorem, $X$ is a positive unit eigenvector. Assume $\mu(G) = \mu(G^1)$. From (2.1), we have that $\mu(G^1) = X^t R(G^1) X$. Again by Lemma 2.2, we get $R(G^1) X = \mu(G^1) X$, i.e. $[W(G^1) + A(G^1)] X = \mu(G^1) X$. Without loss of generality, assume $a \notin \{u, v\}$. Then

$$\mu(G^1) x_a = w_{G^1}(a) x_a + w_{G^1}(ab) x_b + \sum_{z \in N_G(a) \setminus \{b\}} w_{G^1}(za) x_z = \delta (x_a + x_b) + w_G(a) x_a + w_G(ab) x_b + \sum_{z \in N_G(a) \setminus \{b\}} w_G(za) x_z = \delta (x_a + x_b) + w_G(a) x_a + \sum_{z \in N_G(a)} w_G(za) x_z.$$  

Also from $R(G) X = \mu(G) X$, we have that

$$\mu(G) x_a = w_G(a) x_a + \sum_{z \in N_G(a)} w_G(za) x_z.$$  

So we have that $(\mu(G^1) - \mu(G)) x_a = \delta (x_a + x_b) = 0$. This implies that $x_a + x_b = 0$, a contradiction to $x_u + x_v > 0$. Therefore, $\mu(G) < \mu(G^1)$. $\blacksquare$

**COROLLARY 2.4**

Let $G$ and $G^1$ be the two weighted graphs defined in Theorem 2.3. Let $(x_1, x_2, \ldots, x_n)^t$ and $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^t$ be two nonnegative unit eigenvectors corresponding to $\mu(G)$ and $\mu(G^1)$, where $x_i$ and $\tilde{x}_i$ correspond to the vertex $v_i$ of $G$ and $G^1$, respectively. If $x_u + x_v \leq x_a + x_b$, then $\tilde{x}_u + \tilde{x}_v \leq \tilde{x}_a + \tilde{x}_b$.

In addition, if $x_u + x_v < x_a + x_b$ or $x_u + x_v \leq x_a + x_b$ and $G$ is connected, then $\tilde{x}_u + \tilde{x}_v < \tilde{x}_a + \tilde{x}_b$. 

---

**LEMMA 2.1** [8] (Rayleigh–Ritz Theorem) Let $M$ be a Hermitian matrix and let $\lambda_1(M)$ be the largest eigenvalue of $M$. Then $\lambda_1(M) = \max_{\|x\| = 1, x \in \mathbb{R}^n} X^t MX$, and $\lambda_1(M) = X^t MX$ if $X$ is a unit eigenvector corresponding to $\lambda_1(M)$.

**LEMMA 2.2** [7] Let $M$ be a nonnegative symmetric matrix and let $X$ be a unit vector of $\mathbb{R}^n$. If $\lambda_1(M) = X^t MX$, then $MX = \lambda_1(M) X$. 
Proof It is easy to see that $G$ can be obtained from $G^1$ in the following way:

$$w_G(uv) = w_G(ab) + \delta, \quad w_G(ab) = w_G(ab) - \delta, \quad w_G(e) = w_G(e), \quad e \in E(G^1) \setminus \{ab, uv\}.$$  

We first prove that $\tilde{x}_u + \tilde{x}_v \leq \tilde{x}_a + \tilde{x}_b$. Assume that $\tilde{x}_u + \tilde{x}_v > \tilde{x}_a + \tilde{x}_b$. Then by the additional claim of Theorem 2.3, we have $\mu(G^1) < \mu(G)$. On the other hand, since $x_u + x_v \leq x_a + x_b$, again from Theorem 2.3, we get $\mu(G) \leq \mu(G^1)$, a contradiction. Therefore, $\tilde{x}_u + \tilde{x}_v \leq \tilde{x}_a + \tilde{x}_b$.

We next prove the additional claim. Assume that $\tilde{x}_u + \tilde{x}_v \geq \tilde{x}_a + \tilde{x}_b$. Then by Theorem 2.3, we get $\mu(G^1) \leq \mu(G)$. On the other hand, since $x_u + x_v < x_a + x_b$ or $x_u + x_v \leq x_a + x_b$ and $G$ is connected, from the additional claim of Theorem 2.3, we have that $\mu(G) < \mu(G^1)$, a contradiction. Therefore, $\tilde{x}_u + \tilde{x}_v < \tilde{x}_a + \tilde{x}_b$.

Theorem 2.5 Let $u, v$ be two distinct vertices of a connected weighted graph $G$ and $u_1, u_2, \ldots, u_s$ ($u_i \neq v, s \neq 0$) be some vertices of $N_G(u) \setminus N_G(v)$. Let $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of $R(G)$, where $x_i$ corresponds to the vertex $v_i$ of $G$. Let $G^2$ be the weighted graph obtained from $G$ by deleting edges $uu_j$ and adding edges $vu_j$ such that $w_{G^2}(vu_j) = w_G(uu_j), \quad w_{G^2}(e) = w_G(e), \quad e \neq uu_j, \quad j = 1, 2, \ldots, s$.

If $x_u \leq x_v$, then $\mu(G) < \mu(G^2)$.

Proof Set $H_0 = G$. For $j = 1, 2, \ldots, s$, let $H_j$ be the weighted graph obtained from $H_{j-1}$ by deleting the edge $uu_j$ and adding the edge $vu_j$ such that $w_{H_j}(vu_j) = w_{H_{j-1}}(uu_j), \quad w_{H_j}(e) = w_{H_{j-1}}(e), \quad e \in E(H_{j-1}) \setminus \{uu_j\}.$

Since $w_{H_{j-1}}(uu_j) > w_{H_{j-1}}(vu_j) = 0$ for $j = 1, 2, \ldots, s$, set $\delta_j = w_{H_{j-1}}(uu_j)$, then $H_j$ can be obtained from $H_{j-1}$ in the following way:

$$w_{H_j}(vu_j) = w_{H_{j-1}}(vu_j) + \delta_j, \quad w_{H_j}(uu_j) = w_{H_{j-1}}(uu_j) - \delta_j, \quad w_{H_j}(e) = w_{H_{j-1}}(e), \quad e \in E(H_{j-1}) \setminus \{vu_j, uu_j\}.$$  

Let $X' = (x_1', x_2', \ldots, x_n')$ be a nonnegative unit eigenvector corresponding to $\mu(H_j)$, where $X' = X$ and $x_i'$ corresponds to the vertex $v_i$ of $H_j$. Then $x_{u_1}' + x_v' \geq x_{u_1} + x_v$, and by the additional claim of Corollary 2.4, we have that $x_{u_{j+1}}' + x_v' > x_{u_{j+1}} + x_v', \quad j = 1, 2, \ldots, s - 1$.

Since $H_s = G^2$, by the additional claim of Theorem 2.3, we get $\mu(G) = \mu(H_0) < \mu(H_1) < \cdots < \mu(H_s) = \mu(G^2)$.

Theorem 2.6 Let $a, b, u, v$ be four distinct vertices of a connected weighted graph $G$ and let $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of $R(G)$, where $x_i$ corresponds to the vertex $v_i$ of $G$. For $0 < \theta \leq w_G(ab), \quad 0 < \delta \leq w_G(uv)$, let $G^3$ be the weighted graph obtained from $G$ such that $w_{G^3}(ab) = w_G(ab) - \theta, \quad w_{G^3}(au) = w_G(au) + \theta, \quad w_{G^3}(uv) = w_G(uv) - \delta, \quad w_{G^3}(vb) = w_G(vb) + \delta, \quad w_{G^3}(e) = w_G(e), \quad e \in E(G) \setminus \{ab, uv, vb, au\}.$

If $(x_u - x_v)[(2x_u + x_b + x_a)\theta - (2x_v + x_b + x_a)\delta] \geq 0$, then $\mu(G) \leq \mu(G^3)$. 

\[\begin{array}{c}
\text{Proof} \\
\text{It is easy to see that } G \text{ can be obtained from } G^1 \text{ in the following way:}
\end{array} \]
In addition, $\mu(G) = \mu(G^3)$ if and only if $x_u = x_b$ and $(2x_u + x_b + x_a)\theta = (2x_v + x_b + x_a)\delta$.

**Proof** From Lemma 2.1, we have

$$
\mu(G^3) - \mu(G) = \max_{\|Y\|=1} Y^T R(G^3)Y - X^T R(G)X \geq X^T (R(G^3) - R(G))X
$$

$$
= (x_u - x_b)[(2x_u + x_b + x_a)\theta - (2x_v + x_b + x_a)\delta] \geq 0. \quad (2.2)
$$

Assume $\mu(G) = \mu(G^3)$. By (2.2), we have $\mu(G^3) = X^T R(G^3)X$. Again from Lemma 2.2, we get $R(G^3)X = \mu(G^3)X$, i.e. $[W(G^3) + A(G^3)]X = \mu(G^3)X$. Thus

$$
\mu(G^3)x_a = w_{G^3}(a)x_a + w_{G^3}(ba)x_b + w_{G^3}(ua)x_u + \sum_{z \in N_G(a) \setminus \{u, b\}} w_{G^3}(za)x_z
$$

$$
= \theta(x_u - x_b) + w_{G^3}(a)x_a + \sum_{z \in N_G(a)} w_{G^3}(za)x_z
$$

$$
= \theta(x_u - x_b) + \mu(G)x_a.
$$

So we obtain $x_u = x_b$. In the similar way, we can get

$$
\mu(G^3)x_b = \delta(x_b + x_v) - \theta(x_a + x_b) + \mu(G)x_b.
$$

Therefore, we have that $\theta(x_u + x_b) = \delta(x_b + x_v)$. Combining $x_u = x_b$, we can get

$$
(2x_u + x_b + x_a)\theta = (2x_v + x_b + x_a)\delta.
$$

Assume $x_u = x_b$, $(2x_u + x_b + x_a)\theta = (2x_v + x_b + x_a)\delta$, i.e. $x_u = x_b$, $\theta(x_a + x_b) = \delta(x_v + x_b)$, $\theta(x_a + x_u) = \delta(x_v + x_u)$. Then we easily get

$$
w_{G^3}(a)x_a + \sum_{z \in N_G^3(a)} w_{G^3}(za)x_z = \theta(x_u - x_b) + \mu(G)x_a = \mu(G)x_a,
$$

$$
w_{G^3}(v)x_v + \sum_{z \in N_G^3(v)} w_{G^3}(zv)x_z = \delta(x_b - x_u) + \mu(G)x_v = \mu(G)x_v,
$$

$$
w_{G^3}(b)x_b + \sum_{z \in N_G^3(b)} w_{G^3}(zb)x_z = \delta(x_b + x_v) - \theta(x_a + x_b) + \mu(G)x_b = \mu(G)x_b,
$$

$$
w_{G^3}(u)x_u + \sum_{z \in N_G^3(u)} w_{G^3}(zu)x_z = \theta(x_a + x_u) - \delta(x_v + x_u) + \mu(G)x_u = \mu(G)x_u.
$$

It is obvious that, for $p \in V(G) - \{a, b, u, v\}$, we have

$$
w_{G^3}(p)x_p + \sum_{z \in N_G^3(p)} w_{G^3}(zp)x_z = w_G(p)x_p + \sum_{z \in N_G(p)} w_G(zp)x_z = \mu(G)x_p.
$$

Thus $[W(G^3) + A(G^3)]X = \mu(G)X$, i.e. $R(G^3)X = \mu(G)X$. Since $X$ is the Perron vector of $R(G)$, by the Perron–Frobenius theorem, it follows that $\mu(G) = \mu(G^3)$. \hfill \blacksquare

For $v \in V(G)$, let $R_v(G)$ be the principal submatrix of $R(G)$ formed by deleting the row and column corresponding to vertex $v$. If $G = v$, then suppose $\phi(R_v(G)) = 1$. We have the following theorem which is a generalization of Theorem 2.2 in [14] and its proof is very similar to Lemma 8 in [6].
**Theorem 2.7** Let $G$ be the weighted graph formed from two weighted graphs $G_1$ and $G_2$ by joining the vertex $u$ of $G_1$ to the vertex $v$ of $G_2$ by an edge $e=uv$. Then
\[
\phi(G) = \phi(G_1)\phi(G_2) - w_G(e)[\phi(G_1)\phi(R_u(G_2)) + \phi(G_2)\phi(R_v(G_1))].
\] (2.3)

**Proof** Let $R(G_1^e)(R(G_2^e))$ be the principal submatrix obtained by deleting the row and column corresponding to the vertex $v(u)$ from $R(G_1 u: v)(R(G_2 v: u))$, where $G_1 u: v$ is the weighted graph formed from $G_1$ by joining a new pendant vertex $v$ to $u$. Without loss of generality, we may assume that
\[
R(G) = \begin{pmatrix} R(G_1^e) & E_{11} \\ E_{11}^t & R(G_2^e) \end{pmatrix},
\]
where $E_{11}$ is the $|V(G_1)| \times |V(G_2)|$ matrix whose unique nonzero entry is a $w_G(e)$ in position $(1, 1)$. By the Laplace theorem, we have
\[
\phi(G) = \phi(R(G_1^e))\phi(R(G_2^e)) - w_G(e)\phi(R_u(G_1))\phi(R_v(G_2)).
\] (2.4)

Since
\[
\phi(R(G_1^e)) = \phi(G_1) - w_G(e)\phi(R_u(G_1)), \quad \phi(R(G_2^e)) = \phi(G_2) - w_G(e)\phi(R_v(G_2)),
\]
by putting them into (2.4), it follows (2.3).

**Corollary 2.8**

1. Let $uva$ be a path of a weighted graph $G$ such that $u$ is a vertex of degree 1 and $v$ is a vertex of degree 2 in $G$. Then
\[
\phi(G) = (x - 2w_G(uv))\phi(G - u) - w_G(uv)w_G(va)\phi(G - u - v).
\] (2.5)

2. Let $z$ be a vertex of a weighted graph $G$. Let $\tilde{G}_i$ be the weighted graph obtained from $G$ by adding pendant edges $za_i$ with weights $w(za_i)$ ($i = 1, 2, \ldots, t$). Then
\[
\phi(\tilde{G}_i) = \left[\phi(G) - \sum_{i=1}^{t} \frac{xw(za_i)\phi(R(\tilde{G}_i))}{x - w(za_i)}\right] \prod_{i=1}^{t} (x - w(za_i)).
\]

**Proof**

1. By Theorem 2.7, we have
\[
\phi(G) = (x - w_G(uv))\phi(G - u) - w_G(uv)x\phi(R_u(G - u)),
\] (2.6)
\[
\phi(G - u) = (x - w_G(va))\phi(G - u - v) - w_G(va)x\phi(R_v(G - u - v)).
\] (2.7)

It is easy to see that
\[
\phi(R_u(G - u)) = \phi(G - u - v) - w_G(va)\phi(R_v(G - u - v)).
\] (2.8)

By removing $\phi(R_u(G - u))$ and $\phi(R_v(G - u - v))$ from (2.6) to (2.8), we get (2.5).
By induction on \( t \) we prove the result. By setting \( G_1 = G \), \( G_2 = \{a_1\} \), \( u = z \), \( v = a_1 \) in Theorem 2.7, we get
\[
\phi(\tilde{G}_1) = x\phi(G) - w(za_1)[\phi(G) + x\phi(R_z(G))]
= \left[ \phi(G) - \sum_{i=1}^{t-1} \frac{xw(za_i)\phi(R_z(G_i))}{x - w(za_i)} \right] \prod_{i=1}^{t-1} (x - w(za_i)),
\]
for \( x \geq \mu(T) \), where \( a_1 \) is the component containing \( u \) in \( T_0 \) and \( G_1 \) be the reminder of \( T_0 \).

Assume that the result holds for \( t - 1(t \geq 2) \). By the definition of \( R_z(G) \), we have
\[
\phi(R_z(\tilde{G}_t - 1)) = \phi(R_z(G)) \prod_{i=1}^{t-1} (x - w(za_i)).
\]
Taking \( G_1 = \tilde{G}_{t-1} \), \( G_2 = \{a_1\} \), \( u = z \), \( v = a_1 \) in Theorem 2.7, by (2.9) and the induction hypothesis, we get
\[
\phi(\tilde{G}_t) = x\phi(\tilde{G}_{t-1}) - w(za_1)[\phi(\tilde{G}_{t-1}) + x\phi(R_z(\tilde{G}_{t-1}))]
= (x - w(za_1))\phi(\tilde{G}_{t-1}) - xw(za_1)\phi(R_z(\tilde{G}_{t-1}))
= (x - w(za_1)) \left[ \phi(G) - \sum_{i=1}^{t-1} \frac{xw(za_i)\phi(R_z(G_i))}{x - w(za_i)} \right] \prod_{i=1}^{t-1} (x - w(za_i))
- xw(za_1)\phi(R_z(G)) \prod_{i=1}^{t-1} (x - w(za_i))
= \left[ \phi(G) - \sum_{i=1}^{t} \frac{xw(za_i)\phi(R_z(G_i))}{x - w(za_i)} \right] \prod_{i=1}^{t} (x - w(za_i)).
\]
This completes the proof.

**Theorem 2.9** Let \( H \) be a weighted proper spanning subgraph of a weighted tree \( T \). Then for \( x \geq \mu(T) \), we have \( \phi(H) > \phi(T) \).

**Proof** Let \( E(T) \setminus E(H) = \{u_1v_1, u_2v_2, \ldots, u_sv_s\} \), where \( s \geq 1 \). Write \( T_0 = T \) and
\[
T_i = T_{i-1} - u_iv_i, \quad i = 1, 2, \ldots, s.
\]
Then \( T_s = H \). Let \( G_1^i \) be the component containing \( u_i \) in \( T_i \) and \( G_2^i \) be the reminder of \( T_i \). Then for \( i = 1, 2, \ldots, s \), we have
\[
\lambda_1(R(T_{i-1})) \geq \lambda_1(R(T_i)) \geq \max[\lambda_1(R(G_1^i)), \lambda_1(R(G_2^i))].
\]
In particular, since \( R(T_0) \) is irreducible, we have \( \mu(T) = \lambda_1(R(T_0)) > \lambda_1(R(T_1)) \). Thus when \( x \geq \mu(T) \), all of \( \phi(G_1^i) \), \( \phi(R_1(G_1^i)) \), \( \phi(G_2^i) \) and \( \phi(R_0(G_1^i)) \) are positive. Note that \( \phi(T_i) = \phi(G_1^i)\phi(G_2^i) \). So by (2.3), when \( x \geq \mu(T) \), for \( i = 1, 2, \ldots, s \), we have
\[
\phi(T_{i-1}) = \phi(T_i) - w_T(u_iv_i)[\phi(G_1^i)\phi(R_{v_i}(G_1^i)) + \phi(G_2^i)\phi(R_{u_i}(G_1^i))] < \phi(T_i).
\]
Therefore, the required result follows.
3. Main results and proofs

In this section, we will apply the idea from [1,12] to determine the weighted tree with the largest (signless) Laplacian spectral radius in the set of all weighted trees with fixed diameter and positive weight set.

Let $\Gamma(d; m_1, m_2, \ldots, m_{n-1})$ be the set of all weighted trees with $n$ vertices, diameter $d$ and positive weight set $\{m_1, m_2, \ldots, m_{n-1}\}$. Let $T_M$ be a weighted tree in $\Gamma(d; m_1, m_2, \ldots, m_{n-1})$ with the largest signless Laplacian spectral radius. Without loss of generality, we also assume $d \geq 3$ (otherwise, $T_M$ is a weighted star, so the problem is trivial). Let $X=(x_1, x_2, \ldots, x_n)'$ be the Perron vector of $R(T_M)$, where $x_i$ corresponds to the vertex $v_i$ of $T_M$. Next we will investigate the spectral and structural properties of $T_M$.

**LEMMA 3.1** Let $ab$, $uv$ be two distinct edges of $T_M$.

1. If $x_a + x_b \geq x_u + x_v$, then $w_T(ab) \geq w_T(uv)$.
2. If $w_T(ab) > w_T(uv)$, then $x_a + x_b > x_u + x_v$.
3. If $x_a + x_b = x_u + x_v$, then $w_T(ab) = w_T(uv)$.

**Proof** It is obvious that (2) and (3) can be immediately deduced from (1) and (2), respectively. Hence we only give the proof of (1). Assume that $w_T(ab) < w_T(uv)$. Put $\delta = w_T(ab) - w_T(uv)$ and let $T'$ be the weighted tree obtained from $T_M$ such that

$$w_T(ab) = w_T(ab) + \delta, \quad w_T(uv) = w_T(uv) - \delta, \quad w_T(e) = w_T(e), \quad e \neq ab, uv,$$

i.e. $T'$ is the weighted tree obtained from $T_M$ by exchanging the weights of edges $ab$ and $uv$ while making the weights of other edges fixed. Then $T' \in \Gamma(d; m_1, m_2, \ldots, m_{n-1})$, and by the additional claim of Theorem 2.3, we have that $\mu(T') > \mu(T_M)$, a contradiction to the choice of $T_M$.

**LEMMA 3.2** Let $n > d+1$ and let $P = u_1u_2 \cdots u_du_{d+1}$ be a longest path of $T_M$.

1. $T_M$ has a unique vertex $z$ such that $d_{T_M}(z) \geq 3$.
2. $z \in V(P)$ and each vertex not in $P$ is a pendant vertex adjacent to $z$.
3. For each $v \in V(T_M) \setminus \{z\}$, we have that $x_z > x_v$. In addition, for each vertex $u_i (2 \leq i \leq d)$, we have that $x_u > \max\{x_{u_i}, x_{u_{i+1}}\}$.

**Proof** $T_M$ can be obtained from $P$ by attaching a proper weighted tree to the vertex $u_i$ for each $i = 2, 3, \ldots, d$.

1. It is obvious that $T_M$ has a vertex $z$ such that $d_{T_M}(z) \geq 3$. Apart from $z$, suppose that $T_M$ has another vertex with degree greater than 2. We will distinguish the two following cases.

**Case 1** $T_M$ is a caterpillar.

It is obvious that there are two distinct vertices $u_k$ and $u_l$ of $P$ such that $T_M$ has two pendant edges $au_k$ and $bu_l$ not in $P$. Without loss of generality, assume $x_{u_k} \geq x_{u_l}$. Let $T'$ be the weighted tree obtained from $T_M$ by deleting the edge $bu_l$ and adding the new edge $bu_k$ such that

$$w_T(bu_k) = w_T(bu_l), \quad w_T(e) = w_T(e), \quad e \in E(T_M) \setminus \{bu_l\}.$$
Then $T' \in \Gamma(d; m_1, m_2, \ldots, m_{n-1})$, and by Theorem 2.5, we have $\mu(T') > \mu(T_M)$, a contradiction to the choice of $T_M$.

\textbf{Case 2} $T_M$ is not a caterpillar.

It is clear that $T_M$ has a nonpendent edge $uv$ not in $P$. Without loss of generality, assume $x_u \geq x_v$. Let $N_{T_M}(v) = \{v_1, v_2, \ldots, v_k, u\}$ and let $T'$ be the weighted tree obtained from $T_M$ by deleting edges $vv_i$ and adding new edges $uv_i$ such that

$$w_{T'}(uv_i) = w_{T_M}(vv_i), \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq vv_i, \quad i = 1, 2, \ldots, k.$$  

Then $T' \in \Gamma(d; m_1, m_2, \ldots, m_{n-1})$, and by Theorem 2.5, we have $\mu(T') > \mu(T_M)$, a contradiction to the choice of $T_M$.

(2) It is obvious that $z$ is in $P$, i.e. there is a vertex $u_r(2 \leq r \leq d)$ such that $z = u_r$. The remainder can be proved in the similar way to Case 2 of Claim (1).

(3) Write $N_{T_M}(z) = \{z_1, z_2, \ldots, z_s, z_{s+1}, z_{s+2}\}$, where $z_{s+1} = u_{r+1}, z_{s+2} = u_{r-1}$.

First by the Perron–Frobenius theorem, we easily see that $\mu(T_M) > 2w_{T_M}(zz_i)$. So for $i = 1, 2, \ldots, s$, from $R(T_M)X = \mu(T_M)X$, we get

$$x_{zi} = \frac{w_{T_M}(zz_i)}{\mu(T_M) - w_{T_M}(z_i)} \cdot x_z = \frac{w_{T_M}(zz_i)}{\mu(T_M) - w_{T_M}(zz_i)} \cdot x_z < x_z.$$

Next suppose that there is a vertex $u_j \neq z$ such that $x_{u_j} \geq x_z$. If $u_j \notin \{u_1, u_{d+1}\}$, then let $T'$ be the weighted tree obtained from $T_M$ by deleting the edge $z_1z$ and adding the new edge $z_1u_j$ such that

$$w_{T'}(z_1u_j) = w_{T_M}(z_1z), \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) \setminus \{z_1z\}.$$

If $u_j \in \{u_1, u_{d+1}\}$, without loss of generality, assume $u_j = u_1$, then let $T'$ be the weighted tree obtained from $T_M$ by deleting edges $zz$ and adding edges $z_1u_j$ such that

$$w_{T'}(z_1u_j) = w_{T_M}(z_1z), \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq zz, \quad i = 1, 2, \ldots, s + 1.$$  

Then $T' \in \Gamma(d; m_1, m_2, \ldots, m_{n-1})$, and by Theorem 2.5, we have that $\mu(T') > \mu(T_M)$, a contradiction to the choice of $T_M$. The additional claim also follows in the similar method to the case $u_j \in \{u_1, u_{d+1}\}$. \hfill \blacksquare

By Lemma 3.2(1), (2), if we consider no weights of edges, then $T_M$ is the tree shown in Figure 1. In the following we always suppose that

$$P = 1 \sim 2 \sim 3 \sim \ldots \sim d \sim (d + 1)$$

is the path of $T_M$ and $e_i = i(i + 1) \in E(T_M), \quad i = 1, 2, \ldots, d$. For convenience, write $a_1 = d + 2, a_2 = d + 3, \ldots, a_i = n \ (i = n - d - 1)$ when $n > d + 1$.

![Figure 1. The tree $T_M$ without considering weights.](image-url)
LEMMA 3.3 Let \( a = k, b = k + 1, u = q, v = q + 1 \) be distinct vertices of \( P \). Then

1. \((x_u - x_b)[(2x_a + x_b + x_u)w_{T_M}(ab) - (2x_v + x_b + x_u)w_{T_M}(uv)] \leq 0\). In addition, \( x_u = x_b \) if and only if \((2x_a + x_b + x_u)w_{T_M}(ab) = (2x_v + x_b + x_u)w_{T_M}(uv)\).

2. \((x_v - x_a)[(2x_b + x_a + x_v)w_{T_M}(ab) - (2x_a + x_a + x_v)w_{T_M}(uv)] \leq 0\). In addition, \( x_v = x_a \) if and only if \((2x_b + x_a + x_v)w_{T_M}(ab) = (2x_a + x_a + x_v)w_{T_M}(uv)\).

**Proof** We only give the proof of (1). Assume the contrary, that is

\( (x_u - x_b)[(2x_a + x_b + x_u)w_{T_M}(ab) - (2x_v + x_b + x_u)w_{T_M}(uv)] > 0 \),

or only one between \( x_u = x_b \) and \((2x_a + x_b + x_u)w_{T_M}(ab) = (2x_v + x_b + x_u)w_{T_M}(uv)\) holds. Put \( \theta = w_{T_M}(ab), \delta = w_{T_M}(uv) \). Let \( T' \) be the weighted tree obtained from \( T_M \) such that

\[ w_{T'}(ab) = w_{T_M}(ab) - \theta, \quad w_{T'}(au) = w_{T_M}(au) + \theta, \quad w_{T'}(uv) = w_{T_M}(uv) - \delta, \]

\[ w_{T'}(vb) = w_{T_M}(vb) + \delta, \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) \setminus \{ab, uv\}, \]

i.e. \( T' \) is the weighted tree obtained from \( T_M \) by deleting the edges \( ab, uv \) and adding the new edges \( au, vb \) such that \( w_{T'}(au) = w_{T_M}(ab), w_{T'}(vb) = w_{T_M}(uv) \). Then \( T' \in \Gamma(d; m_1, m_2, \ldots, m_{n-1}) \), and from Theorem 2.6, we have that \( \mu(T') > \mu(T_M) \), a contradiction to the choice of \( T_M \). 

LEMMA 3.4 Let \( r, l \) \((r < l)\) be two distinct vertices of \( P \) such that \( x_r = x_l \). Then

1. \( r + l = d + 2 \).
2. \( x_i = x_{d-i+2}, \quad i = 1, 2, \ldots, \left[\frac{d+1}{2}\right] \).

**Proof**

1. Assume the contrary, namely \( r + l \neq d + 2 \). Suppose that \( r \geq 2 \) and \( l \leq d \). We first prove \( x_{r-1} = x_{l+1} \). Assume that \( x_{r-1} \neq x_{l+1} \), then, without loss of generality, assume that \( x_{r-1} > x_{l+1} \). Let \( a = r - 1, \quad b = r, \quad u = l, \quad v = l + 1 \). Then \( x_a + x_b > x_u + x_v \). By Lemma 3.1(1), we have \( w_{T_M}(ab) \geq w_{T_M}(uv) \).

If \( w_{T_M}(ab) = w_{T_M}(uv) \), then

\[ x_u = x_b, \quad (2x_a + x_b + x_u)w_{T_M}(ab) > (2x_v + x_b + x_u)w_{T_M}(uv). \]  

(3.1)

If \( w_{T_M}(ab) > w_{T_M}(uv) \), then

\[ x_a > x_v, \quad (2x_b + x_a + x_v)w_{T_M}(ab) = (2x_u + x_a + x_v)w_{T_M}(uv). \]

(3.2)

(3.1) and (3.2) contradict the additional claims of Lemma 3.3. Therefore, \( x_{r-1} = x_{l+1} \). If \( r \geq 3 \) and \( l \leq d - 1 \), then replacing the vertices \( r \) and \( l \) above by \( r - 1 \) and \( l + 1 \) respectively, in the similar way above we can get \( x_{r-2} = x_{l+2} \).

By proceeding in this way, we can get

\[ x_{r-k} = x_{l+k}, \quad k = 0, 1, \ldots, \min\{r - 1, d + 1 - l\}. \]

(3.3)

Clearly, (3.3) also holds for \( r = 1 \) or \( l = d + 1 \). If \( r + l \leq d + 1 \), then \( \min\{r - 1, d + 1 - l\} = r - 1 \). Take \( k = r - 1 \) in (3.3) and set \( v = r + l - 1 \). Then \( x_1 = x_v \).

It is easy to see that \( 2 \leq v \leq d \), so by Lemma 3.2(3), we get \( x_v > x_1 \), a contradiction. If \( r + l \geq d + 3 \), then \( \min\{r - 1, d + 1 - l\} = d + 1 - l \).
Take \( k = d + 1 - l \) in (3.3) and set \( v = r + l - d - 1 \). Then \( x_{d+1} = x_v \). It is easy to see that \( 2 \leq v \leq d \), so by Lemma 3.2, we get \( x_v > x_{d+1} \), a contradiction. Therefore, \( r + l = d + 2 \).

(2) From \( r + l = d + 2 \) and (3.3), it follows that \( x_i = x_{d-i+2}, \ i = 1, 2, \ldots, r \). If \( r = \lfloor \frac{d+1}{2} \rfloor \), then the results have followed. Hence now assume that \( r < \lfloor \frac{d+1}{2} \rfloor \) and prove that \( x_{r+1} = x_{d+1-r} \). Assume that \( x_{r+1} \neq x_{d+1-r} \), then, without loss of generality, assume that \( x_{r+1} > x_{d+1-r} \). Let \( a = r + 1, \ b = r, \ u = d + 2 - r, \ v = d + 1 - r \). In the similar way with the proof of (1), we will get (3.1) and (3.2), contradictions to the additional claims of Lemma 3.3. Therefore, \( x_{r+1} = x_{d+1-r} \). Repeat the above steps by replacing \( r \) with \( r + 1 \), we can get 

\[ x_i = x_{d-i+2}, \ i = r + 1, r + 2, \ldots, \lfloor \frac{d+1}{2} \rfloor. \]

From now on, we always suppose that the vertices of \( P \) are relabelled by \( v_1, v_2, \ldots, v_{d+1} \) such that \( v_1 \geq v_2 \geq \cdots \geq v_{d+1} \).

**Lemma 3.5** For the path \( P \), \( \{i, d-i+2\} = \{v_{d-2i+2}, v_{d-2i+3}\}, \ i = 1, 2, \ldots, \lfloor \frac{d+1}{2} \rfloor \).

**Proof** For \( 2 \leq i \leq d \), by the additional claim of Lemma 3.2, we have that \( x_i > x_1 \) and \( x_i > x_{d+1} \). It follows that \( \{x_1, x_{d+1}\} = \{v_{d+1}, v_{d+1}\} \). Therefore, \( \{1, d+1\} = \{v_1, v_{d+1}\} \), i.e. the results hold for \( i = 1 \).

**Case 1** Let \( d \) be an even number.

Suppose that the results hold for \( 1, 2, \ldots, i - 1 \), i.e.

\[ \{j, d-j+2\} = \{v_{d-2j+2}, v_{d-2j+3}\}, \ j = 1, 2, \ldots, i - 1. \]  

(3.4)

Next we will show that \( \{i, d-i+2\} = \{v_{d-2i+2}, v_{d-2i+3}\} \). By (3.4), we get

\[ \{1, 2, \ldots, i-1, d-i+3, \ldots, d, d+1\} = \{v_{d+1}, v_d, v_{d-1}, v_{d-2}, \ldots, v_{d-2i+5}, v_{d-2i+4}\}. \]  

(3.5)

Assume \( \{i, d-i+2\} \neq \{v_{d-2i+2}, v_{d-2i+3}\} \), then we have

\[ \{i, d-i+2\} \not\subseteq \{v_{d-2i+2}, v_{d-2i+3}\}, \{v_{d-2i+2}, v_{d-2i+3}\} \not\subseteq \{i, d-i+2\}. \]

So by combining (3.5), we have

\[ \{i, d-i+2\} \not\subseteq \{v_{d+1}, v_d, v_{d-1}, v_{d-2}, \ldots, v_{d-2i+5}, v_{d-2i+4}, v_{d-2i+3}, v_{d-2i+2}\}, \]

(3.6)

\[ \{v_{d-2i+2}, v_{d-2i+3}\} \not\subseteq \{1, 2, \ldots, i-1, i, d-i+2, d-i+3, \ldots, d, d+1\}. \]  

(3.7)

By (3.6), it is easy to see that there is a \( k (1 \leq k \leq d-2i+1) \) such that either \( i = v_k \) or \( d-i+2 = v_k \). By (3.7), it is easy to see that there is a \( l (i+1 \leq l \leq d-i+1) \) such that either \( l = v_{d-2i+2} \) or \( l = v_{d-2i+3} \).

First assume that \( i = v_k \). Write \( a = i - 1, b = i, u = l, v = l + 1 \). Then by Lemma 3.3, we have

\[ (x_u - x_b)(2x_u + x_b + x_a)w_{T,d}(ab) - (2x_v + x_b + x_a)w_{T,d}(uv) \leq 0, \]

(3.8)

\[ (x_v - x_a)(2x_b + x_a)w_{T,d}(ab) - (2x_u + x_a)w_{T,d}(uv) \leq 0. \]

(3.9)

Note that \( k \leq d-2i+1 < d-2i+2 < d-2i+3 \), so we have

\[ x_b = x_{v_k} \geq x_{v_{d-2i+1}} \geq x_{v_{d-2i+2}} \geq x_{v_{d-2i+3}}. \]
Therefore, by $u = l \in \{v_{d-2i+2}, v_{d-2i+3}\}$, we get
\[ x_b \geq x_u. \tag{3.10} \]

Since $l+1 \in \{i+2, i+3, \ldots, d-i+1, d-i+2\}$, by (3.5), we have
\[ v = l+1 \in \{v_1, v_2, \ldots, v_{d-2i+2}, v_{d-2i+3}\}. \]

By taking $j = i - 1$ in (3.4), we have
\[ a = i - 1 \in \{v_{d-2i+4}, v_{d-2i+5}\}. \]

Therefore, by the above two equations, we get
\[ x_v \geq x_a. \tag{3.11} \]

If $x_b > x_u$ and $x_v > x_a$, then by (3.8) and (3.9), we get, respectively,
\[ w_{T_u}(ab) \geq \frac{2x_v + x_b + x_u}{2x_a + x_b + x_u} \cdot w_{T_u}(uv) > w_{T_d}(uv), \tag{3.12} \]
and
\[ w_{T_u}(ab) \leq \frac{2x_u + x_a + x_v}{2x_b + x_a + x_v} \cdot w_{T_u}(uv) < w_{T_d}(uv), \tag{3.13} \]
a contradiction. Thus we must have that $x_b \leq x_u$ or $x_v \leq x_a$. Again by (3.10) and (3.11), we have that $x_b = x_u$ or $x_v = x_a$. By Lemma 3.4(1), we get $b + u = d + 2$ or $a + v = d + 2$. These contradict $b + u = a + v = i + l \leq d + 1$.

Next assume that $d - i + 2 = v_k$. Write $a = d - i + 3$, $b = d - i + 2$, $u = l$, $v = l - 1$. Then (3.8)–(3.10) hold. Since $l - 1 \in \{i, i+1, \ldots, d-i-1, d-i\}$, by (3.5), we have
\[ v = l - 1 \in \{v_1, v_2, \ldots, v_{d-2i+2}, v_{d-2i+3}\}. \]

By taking $j = i - 1$ in (3.4), we have
\[ a = d - i + 3 \in \{v_{d-2i+4}, v_{d-2i+5}\}. \]

Therefore, by the above two equations, (3.11) still holds. Note that
\[ b + u = a + v = d + (l - i) + 2 \geq d + 3, \]
so in the similar way to the case $i = v_k$, we will also get contradictions.

**Case 2** Let $d$ be an odd number.

Set $d = 2r - 1$ and suppose the results hold for $1, 2, \ldots, i-1(2 \leq i \leq r - 1)$, i.e.
\[ \{j, 2r - j + 1\} = \{v_{2r-2j+1}, v_{2r-2j+2}\}, \quad j = 1, 2, \ldots, i - 1. \]

In the similar way to Case 1 we can show that $\{i, 2r - i + 1\} = \{v_{2r-2i+1}, v_{2r-2i+2}\}$. So
\[ \{j, 2r - j + 1\} = \{v_{2r-2j+1}, v_{2r-2j+2}\}, \quad j = 1, 2, \ldots, r - 1. \]

Again from $\{1, 2, \ldots, 2r\} = \{v_1, v_2, \ldots, v_{2r}\}$, we get $\{r, r + 1\} = \{v_1, v_2\}$. ■

Let $T$ be a weighted tree shown Figure 1, and without loss of generality, assume that
\[ w_T(z_{a_1}) \geq w_T(z_{a_2}) \geq \cdots \geq w_T(z_{a_i}). \]
Then $T$ (any weighted tree isomorphic to it) is called an alternating weighted tree in edge weights if $d = 2r$, $z = r + 1$ (if $t \neq 0$) and the weights of all edges of $T$ satisfy

$$w_T(e_{r+1}) \geq w_T(e_r) \geq w_T(za_1) \geq w_T(za_2) \geq \cdots \geq w_T(za_t) \geq w_T(e_{r+2}) \geq w_T(e_{r+1}) \geq \cdots \geq w_T(e_2) \geq w_T(e_1),$$

(3.14)

or $d = 2r - 1$, $z = r + 1$ (if $t \neq 0$) and the weights of all edges of $P$ satisfy

$$w_T(e_r) \geq w_T(e_{r+1}) \geq w_T(za_1) \geq w_T(za_2) \geq \cdots \geq w_T(za_t) \geq w_T(e_{r+2}) \geq w_T(e_{r+1}) \geq \cdots \geq w_T(e_{2r-1}) \geq w_T(e_1).$$

(3.15)

In particular, when $n = d + 1$ (i.e. $T$ is a weighted path), $T$ is called an alternating weighted path in edge weights.

By Lemma 3.5, we have $\{1, d + 1\} = \{v_{d+1}, v_{d}\}$. Without loss of generality, now assume that (for the other case the proof is quite analogous and the resulting weighted tree is isomorphic)

$$v_{d+1} = 1, \quad v_d = d + 1.$$  

(3.16)

**Lemma 3.6** \(P\) is an alternating weighted path in edge weights and \(v_1 = \left\lfloor \frac{d+3}{2} \right\rfloor\).

**Proof** We will distinguish the two following cases.

**Case 1** Assume that \(x_1, x_2, \ldots, x_{d+1}\) are distinct, i.e. \(x_1 > x_2 > \cdots > x_{d+1}\).

We first prove the following claim:

$$v_{d-2i+3} = i, \quad v_{d-2i+2} = d - i + 2, \quad i = 1, 2, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor.$$  

(3.17)

By (3.16), the results hold for \(i = 1\). For \(2 \leq i \leq \left\lfloor \frac{d+1}{2} \right\rfloor\), suppose that

$$v_{d-2j+3} = j, \quad v_{d-2j+2} = d - j + 2, \quad j = 1, 2, \ldots, i - 1.$$  

Next we prove that \(v_{d-2i+3} = i\) or \(v_{d-2i+2} = d - i + 2\). Assume the contrary, i.e. either \(v_{d-2i+3} \neq i\) or \(v_{d-2i+2} \neq d - i + 2\). By Lemma 3.5, \(\{i, d - i + 2\} = \{v_{d-2i+3}, v_{d-2i+2}\}\). So we must have that \(v_{d-2i+3} = d - i + 2\). Write \(a = i - 1, b = i, u = d - i + 2, v = d - i + 3\). Then (3.8) and (3.9) hold. It is easy to see that

$$x_b = x_i = x_{v_{d-2i+2}} > x_{v_{d-2i+3}} = x_{d-i+2} = x_a.$$  

By the assumptions of induction (take \(j = i - 1\)), we have

$$x_y = x_{d-i+3} = x_{v_{d-2i+4}} > x_{v_{d-2i+3}} = x_{d-i+2} = x_{d-i+3}.$$  

So by (3.8) and (3.9), we get (3.12) and (3.13), a contradiction. Therefore, \(v_{d-2i+3} = i\), \(v_{d-2i+2} = d - i + 2\). By induction principle, the claim (3.17) holds.

First assume that \(d = 2r\). From (3.17), we get \(v_1 = r + 1 = \left\lfloor \frac{d+3}{2} \right\rfloor\). For convenience, set \(v_0 = v_1\). Then by (3.17), for \(i = r, r - 1, \ldots, 2, 1\), we have

$$e_i = i(i + 1) = v_{2r-2i+3}v_{2r-2i+1}, \quad e_{2r-i+1} = (2r - i + 1)(2r - i + 2) = v_{2r-2i}v_{2r-2i+2}. $$

It is easy to see that \(x_{2r} + x_{2r+1} > x_1 + x_2\), and for \(i = r, r - 1, \ldots, 3, 2\),

$$x_{2r-i+1} + x_{2r-i+2} > x_i + x_{i+1} > x_{2r-i+2} + x_{2r-i+3}. $$


So by Lemma 3.1(1), we get \( w_{T_d}(e_{2r}) \geq w_{T_d}(e_1) \), and for \( i = r, r - 1, \ldots, 3, 2, \)
\[
w_{T_d}(e_{2r-i+1}) \geq w_{T_d}(e_i) \geq w_{T_d}(e_{2r-i+2}).
\]

Hence it is easy to see that the weights of all edges of \( P \) satisfies (3.14) (by taking \( t = 0 \)), i.e. \( P \) is an alternating weighted path in edge weights.

Next assume that \( d = 2r - 1 \). By (3.17), we get \( v_1 = r + 1 = \lceil \frac{d+3}{2} \rceil \). For convenience, set \( v_0 = v_1, v_{-1} = v_2 \). Then by (3.17), for \( i = r, r - 1, \ldots, 2, 1 \), we have that
\[
e_i = (i+1) = v_{2r-2i+2}v_{2r-2i-2}, \quad e_{2r-i} = (2r-i)(2r-i+1) = v_{2r-2i-1}v_{2r-2i+1}.
\]

It is easy to see that \( x_{2r-i} + x_{2r-i+1} > x_i + x_{i+1} \), and for \( i = r, r - 1, \ldots, 3, 2, \)
\[
x_{2r-i} + x_{2r-i+1} > x_i + x_{i+1} > x_{2r-i+1} + x_{2r-i+2}.
\]

So by Lemma 3.1(1), we get \( w_{T_d}(e_{2r-i}) \geq w_{T_d}(e_i) \), and for \( i = r, r - 1, \ldots, 3, 2, \)
\[
w_{T_d}(e_{2r-i}) \geq w_{T_d}(e_{2r-i+1}).
\]

Hence it is easy to see that the weights of all edges of \( P \) satisfies (3.15) (by taking \( t = 0 \)), i.e. \( P \) is an alternating weighted path in edge weights.

**Case 2** Assume that at least two of \( x_1, x_2, \ldots, x_{d+1} \) are equal.

By Lemmas 3.4(2) and 3.5, we get, respectively,
\[
x_i = x_{d-i+2}, \quad i = 1, 2, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor,
\]
\[
\{i, d-i+2\} = \{v_{d-2i+2}, v_{d-2i+3}\}, \quad i = 1, 2, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor.
\]

First assume that \( d = 2r \). By (3.19), we get \( v_1 = r + 1 = \lceil \frac{d+3}{2} \rceil \). By (3.18), we get
\[
x_i + x_{i+1} = x_{2r-i+1} + x_{2r-i+2}, \quad i = 1, 2, \ldots, r.
\]

So by Lemma 3.1(3), we get
\[
w_{T_d}(e_i) = w_{T_d}(e_{2r-i+1}), \quad i = 1, 2, \ldots, r.
\]

Again by (3.18) and (3.19), we get \( x_i = x_{v_{2r-2i+3}}, \quad i = 1, 2, \ldots, r + 1 \). Therefore, for \( i = 1, 2, \ldots, r - 1 \), we have
\[
x_i + x_{i+1} = x_{v_{2r-2i+3}} + x_{v_{2r-2i+2}} \leq x_{v_{2r-2i+1}} + x_{v_{2r-2i+1}} = x_{i+1} + x_{i+2}.
\]

So by Lemma 3.1(1), we get
\[
w_{T_d}(e_i) \leq w_{T_d}(e_{i+1}), \quad i = 1, 2, \ldots, r - 1.
\]

By (3.20) and (3.21), it is easy to see that the weights of all edges of \( P \) satisfies (3.14) (by taking \( t = 0 \)), i.e. \( P \) is an alternating weighted path in edge weights.

Next assume that \( d = 2r - 1 \). From (3.18) and (3.19), we have \( x_r = x_{r+1} \), \( \{r, r+1\} = \{v_1, v_2\} \). It follows \( x_{v_1} = x_{v_2} \). So, without loss of generality, we may take \( v_1 = r + 1 = \lceil \frac{d+3}{2} \rceil \). Again by (3.18), we have
\[
x_i + x_{i+1} = x_{2r-i} + x_{2r-i+1}, \quad i = 1, 2, \ldots, r - 1.
\]
So by Lemma 3.1(3), we get
\[ w_{T_M}(e_i) = w_{T_M}(e_{2r-i}), \quad i = 1, 2, \ldots, r - 1. \] (3.22)

For convenience, set \( v_0 = v_1 \). Again by (3.18) and (3.19), we get \( x_i = x_{2r-2r+2} \), \( i = 1, 2, \ldots, r + 1 \). Therefore, for \( i = 1, 2, \ldots, r - 1 \), we have
\[ x_i + x_{i+1} = x_{2r-2r+2} + x_{2r-2} \leq x_{2r-2} + x_{2r-2} = x_{i+1} + x_{i+2}. \]

So by Lemma 3.1(1), we get
\[ w_{T_M}(e_i) \leq w_{T_M}(e_{i+1}), \quad i = 1, 2, \ldots, r - 1. \] (3.23)

By (3.22) and (3.23), it is easy to see that the weights of all edges of \( P \) satisfies (3.15) (by taking \( t = 0 \)), i.e. \( P \) is an alternating weighted path in edge weights.

**Lemma 3.7** Suppose that \( n > d + 1 \) and \( w_{T_M}(za_1) \geq w_{T_M}(za_2) \geq \cdots \geq w_{T_M}(za_i) \). Then \( w_{T_M}(e_{z-1}) \geq w_{T_M}(za_1) \) and \( w_{T_M}(e_z) \geq w_{T_M}(za_1) \).

**Proof** By Lemmas 3.2(3) and 3.6, we have \( z = v_1 = \left[ \frac{d+3}{2} \right] \geq 3 \).

First prove \( w_{T_M}(e_{z-1}) \geq w_{T_M}(za_1) \). Assume the contrary. By Lemma 3.1(2), it follows that \( x_{z-1} + x_z < x_z + x_{a_1} \), i.e. \( x_{z-1} < x_{a_1} \). Let \( T' \) be the weighted tree obtained from \( T_M \) by deleting the edge \( e_{z-2} \) and adding the edge \( (z-2) \), such that
\[ w_{T'}((z-2)a_1) = w_{T_M}(e_{z-2}), \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) \setminus \{e_{z-2}\}. \]

Then \( T' \) is obtained by \( T_M \). By Theorem 2.5, we have \( \mu(T') > \mu(T_M) \), contradicting the choice of \( T_M \).

Next prove \( w_{T_M}(e_z) \geq w_{T_M}(za_1) \). If \( d \geq 4 \), in the similar way above, we can show \( w_{T_M}(e_z) \geq w_{T_M}(za_1) \). If \( d = 3 \), then \( e_z, za_1, \ldots, za_i \) are symmetric pendant edges. Thus without loss of generality, we may assume \( w_{T_M}(e_z) \geq w_{T_M}(za_1) \).

**Lemma 3.8** Suppose that \( n > d + 1 \) and \( w_{T_M}(za_1) \geq w_{T_M}(za_2) \geq \cdots \geq w_{T_M}(za_i) \). Then \( w_{T_M}(za_i) \geq w_{T_M}(e_{z-2}) \) for \( d \geq 3 \) and \( w_{T_M}(za_i) \geq w_{T_M}(e_{z+1}) \) for \( d \geq 4 \).

**Proof** Assume \( d \geq 3 \). Now we prove that \( w_{T_M}(za_i) \geq w_{T_M}(e_{z-2}) \). By Lemmas 3.2(3) and 3.6, \( z = v_1 = \left[ \frac{d+3}{2} \right] \geq 3 \). Let \( P(k, l) \) be the weighted subpath between the vertex \( k \) and the vertex \( l \) in \( P \), including \( k \) and \( l \). In particular, \( P(i, i) \) is an isolated vertex \( P_i \). Let \( H \) be the weighted tree containing the vertex \( z \) in \( T_M - e_{z-2} \). For convenience, denote \( w_{T_M}(e) \) by \( w(e) \). Set \( H_1 = H - a_1, q = d + 1, \alpha = f(R_0(P(z, q)), \]
\[ g(k) = \prod_{i=k}^{t-1} (x - w(za_i)), \quad h(k) = \frac{x \cdot \alpha - w(za_i)}{x - w(za_i)}, \]
where for \( k \geq t \), set \( g(k) = 1 \) and \( h(k) = 0 \); while for \( s = 0 \), set \( f(P(1, s)) = 1 \) and \( w(e_s) = f(R_0(P(1, s))) = 0 \). Write \( u = z - 2, v = z - 1 = a_0 \). By Theorem 2.7, we get
\[ \phi(T_M) = \phi(P(1, u)) \phi(H) - w(e_u) [\phi(P(1, u)) \phi(R_0(H))] + \phi(H) \phi(R_0(P(1, u)))], \]
\[ \phi(H) = x \phi(H_1) - w(za_i) [\phi(H_1) + x \phi(0)]. \]

By the definition of \( R_0(G) \), we get
\[ \phi(R_0(H)) = \phi(H - a_0) - w(za_0) [\phi(R_0(H - a_0)) (use \ Theorem 2.7 to \ \phi(H - a_0))] \]
\[ = (x - w(za_i)) \phi(H_1 - a_0) - [w(za_i)x + w(za_0)(x - w(za_i))] \alpha \phi(1), \]
\[ \phi(R_0(P(1, u))) = \phi(P(1, z - 3)) - w(e_{z-3}) \phi(R_{z-3}(P(1, z - 3))). \]
Let $\beta = w(e_u)w(za_i)$. Then by the above four equations, we get
\[
\phi(T_M) = F_1(w(e_u), w(za_i)) + F_2(w(e_u), w(za_i)),
\] where
\[
F_1(w(e_u), w(za_i)) = x\phi(P(1, u))\phi(H_1)
+ \beta\phi(P(1, u))\phi(H_1 - a_0) + x\alpha g(1) - w(za_0)\alpha g(1)
+ \beta(\phi(P(1, z - 3)) - w(e_{z-3})\phi(R_{z-3}(P(1, z - 3))))\phi(H_1) + x\alpha g(0),
\]
\[
F_2(w(e_u), w(za_i)) = -w(za_i)\phi(P(1, u))\phi(H_1) - xw(e_u)\phi(P(1, u))\phi(H_1 - a_0)
+ x\alpha\phi(P(1, u))[-w(za_i)g(0) + w(e_u)w(za_0)g(1)]
- xw(e_u)\phi(H_1)\phi(P(1, z - 3)) - w(e_{z-3})\phi(R_{z-3}(P(1, z - 3))).
\] (3.25)

Let $\eta = w(e_u) + w(za_i)$. By the definition of $g(k)$, we have
\[
-w(za_i)g(0) + w(e_u)w(za_0)g(1) = [-xw(za_i) + \eta w(za_0)]g(1).
\] (3.26)

By Theorem 2.7, we have
\[
\phi(P(1, u)) = x\phi(P(1, z - 3)) - w(e_{z-3})\phi(P(1, z - 3)) + x\phi(R_{z-3}(P(1, z - 3))).
\]
i.e.
\[
x[\phi(P(1, z - 3)) - w(e_{z-3})\phi(R_{z-3}(P(1, z - 3)))] = \phi(P(1, u)) + w(e_{z-3})\phi(P(1, z - 3)).
\] (3.27)

By placing (3.26) and (3.27) into (3.25), it follows that
\[
F_2(w(e_u), w(za_i)) = \eta[-\phi(H_1) + x\alpha w(za_0)g(1)]\phi(P(1, u))
- xw(e_u)\phi(P(1, u))\phi(H_1 - a_0) - x^2\alpha w(za_i)\phi(P(1, u))g(1)
- w(e_u)w(e_{z-3})\phi(P(1, z - 3))\phi(H_1).
\] (3.28)

Since $F_1(w(e_u), w(za_i))$ contains no $w(e_u)$ and $w(za_i)$ apart from $\beta$, it follows that
\[
F_1(w(e_u), w(za_i)) = F_1(w(za_i), w(e_u)).
\] (3.29)

Assume $w(e_{z-2}) > w(za_i)$, i.e. $w(e_u) > w(za_i)$. Let $T'$ be the weighted graph formed from $T_M$ by exchanging the weights of $e_u$ and $za_i$ while keeping the weights of other edges not changed. By (3.24), (3.28) and (3.29), we get
\[
\phi(T_M) - \phi(T') = F_2(w(e_u), w(za_i)) - F_2(w(za_i), w(e_u)) = (w(e_u) - w(za_i))\Delta,
\]
where
\[
\Delta = x\phi(P(1, z - 2))[-\phi(H_1 - a_0) + x\alpha g(1)] - w(e_{z-3})\phi(P(1, z - 3))\phi(H_1).
\] (3.30)

By setting $G = P(z, q)$ in Corollary 2.8(2), we get
\[
\phi(H_1) = [\phi(P(z, q)) - h(0)]g(0).
\] (3.31)
\[
\phi(H_1 - a_0) = [\phi(P(z, q)) - h(1)]g(1).
\] (3.32)
By placing (3.31) and (3.32) into (3.30), we get \( \triangle = g(1)\Theta \), where

\[
\Theta = \frac{x\phi(P(1, z - 2))}{w(za_0)} [-w(za_0)\phi(P(z, q)) + x\alpha w(za_0)] \\
- w(e_{z-3})\phi(P(1, z - 3))[(x - w(za_0))\phi(P(z, q)) - x\alpha w(za_0)] \\
+ w(e_{z-3})(x - w(za_0))\phi(P(1, z - 3))h(1) + xh(1)\phi(P(1, z - 2)).
\]

Now assume \( x \geq \mu(T_m) \). Then \( x > 2w(za_0) \) and \( \phi(P(k, l))h(1) \geq 0 \), so we get

\[
\Theta \geq \frac{x\phi(P(1, z - 2))}{w(za_0)} [-w(za_0)\phi(P(z, q)) + x\alpha w(za_0)] \\
- w(e_{z-3})\phi(P(1, z - 3))[(x - w(za_0))\phi(P(z, q)) - x\alpha w(za_0)]. \tag{3.33}
\]

By using Theorem 2.7 to \( \phi(P(z - 1, q)) \), we get

\[
\phi(P(z - 1, q)) = (x - w(za_0)\phi(P(z, q)) - x\alpha w(za_0). \tag{3.34}
\]

By using Corollary 2.8(1) to \( \phi(P(z - 1, q)) \) in (3.34), we get

\[
w(za_0)w(e_z)\phi(P(z + 1, q)) = -w(za_0)\phi(P(z, q)) + x\alpha w(za_0). \tag{3.35}
\]

So by (3.33) to (3.35), it follows that

\[
\Theta \geq xw(e_z)\phi(P(1, z - 2))\phi(P(z + 1, q)) - w(e_{z-3})\phi(P(1, z - 3))\phi(P(z - 1, q)).
\]

Since \( P_1 \cup P(z + 1, q) \) is a proper spanning subgraph of \( P(z, q) \), from Theorem 2.9,

\[
x\phi(P(z + 1, q)) = \phi(P_1 \cup P(z + 1, q)) > \phi(P(z, q)).
\]

Therefore, by combining \( w(e_z) \geq w(e_{z-3}) \), we get

\[
\Theta > xw(e_z)\phi(P(1, z - 2))\phi(P(z, q)) - w(e_{z-3})\phi(P(1, z - 3))\phi(P(z - 1, q)) \\
\geq w(e_{z-3})\Phi,
\]

where

\[
\Phi = \phi(P(1, z - 2))\phi(P(z, q)) - \phi(P(1, z - 3))\phi(P(z - 1, q)).
\]

Define \( \prod_{i=1}^{n}w^2(e_i) = 1 \). Since \( w(e_1) \leq w(e_2) \leq \cdots \leq w(e_{z-2}) \leq w(e_{z-1}) \), by repeating using Corollary 2.8(1), we get

\[
\Phi = [(x - 2w(e_{z-3}))\phi(P(1, z - 3)) - w(e_{z-3})w(e_{z-4})\phi(P(1, z - 4))]\phi(P(z, q)) \\
- \phi(P(1, z - 3))[(x - 2w(e_{z-1}))\phi(P(z, q)) - w(e_{z-1})w(e_z)\phi(P(z + 1, q))] \\
= 2[w(e_{z-1}) - w(e_{z-3})]\phi(P(1, z - 3))\phi(P(z, q)) \\
+ w(e_{z-1})w(e_z)\phi(P(1, z - 3))\phi(P(z + 1, q)) \\
- w(e_{z-3})w(e_{z-4})\phi(P(1, z - 4))\phi(P(z, q)) \\
\geq w(e_{z-1})w(e_z)\phi(P(1, z - 3))\phi(P(z + 1, q)) \\
- w(e_{z-3})w(e_{z-4})\phi(P(1, z - 4))\phi(P(z, q)) \\
\geq [\phi(P(1, z - 3))\phi(P(z + 1, q)) - \phi(P(1, z - 4))\phi(P(z, q))]w^2(e_{z-3})
\]

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\[\geq \cdots \]
\[\geq [\phi(P(1, 1))\phi(P(2z - 3, q)) - \phi(P(1, 0))\phi(P(2z - 4, q))] \prod_{i=1}^{z-3} w^2(e_i)\]
\[= [\phi(P_1 \cup P(2z - 3, q)) - \phi(P(2z - 4, q))] \prod_{i=1}^{z-3} w^2(e_i).\]

Note that \(P_1 \cup P(2z - 3, q)\) is a weighted proper subgraph of \(P(2z - 4, q)\), by Theorem 2.9, for \(x \geq \mu(P(2z - 4, q))\), we have \(\phi(P_1 \cup P(2z - 3, q)) - \phi(P(2z - 4, q)) > 0\). But \(\mu(T_M) > \mu(P(2z - 4, q))\), so for \(x \geq \mu(T_M), \Phi > 0\), i.e. \(\Delta = g(1)\Theta > 0\). Thus when \(x \geq \mu(T_M), \phi(T_M) > \phi(T')\). This implies \(\mu(T_M) < \mu(T')\), a contradiction to the choice of \(T_M\). Therefore, \(w_{T_M} = \phi(T_M)\geq w_T(e_{z-2})\). In the similar way, we can prove that \(w_{T_M}(e_i) \geq w_{T_M}(e_{z-1})\) for \(d \geq 4\).

Combining Lemmas 3.6–3.8, it is easy to see that the weights of all edges of \(T_M\) satisfy (3.14) or (3.15), i.e. \(T_M\) is an alternating weighted tree in edge weights. Note that, for given \(n, d\) and positive weight set \(\{m_1, m_2, \ldots, m_{n-1}\}\), the alternating weighted tree in edge weights is uniquely determined. Therefore, by the choice of \(T_M\), we immediately get the following result.

**Theorem 3.9** The alternating weighted tree in edge weights is the unique graph in \(\Gamma(n; m_1, m_2, \ldots, m_{n-1})\) having the largest signless Laplacian spectral radius.

**Remark** Let \(G = (V_1, V_2, E)\) be a connected bipartite weighted graph with \(n\) vertices and suppose that \(V_1 = \{v_1, v_2, \ldots, v_k\}, V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}\). Let \(U = (u_{ij})\) be the \(n \times n\) diagonal matrix with \(u_{ii} = 1\) if \(1 \leq i \leq k\), and \(u_{ii} = -1\) if \(k + 1 \leq i \leq n\). It is easy to show that \(U^{-1}L(G)U = R(G)\). This indicates that \(L(G)\) and \(R(G)\) have the same spectrum. Hence by Theorem 3.9, we get the main result in the article.

**Theorem 3.10** The alternating weighted tree in edge weights is the unique graph in \(\Gamma(n; m_1, m_2, \ldots, m_{n-1})\) having the largest Laplacian spectral radius.

**Example** In Figure 2, two alternating weighted trees are displayed, where the numbers denote the weights corresponding to the edges. The first has the largest Laplacian spectral radius in \(\Gamma(8; 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)\), while the second one has the largest Laplacian spectral radius in \(\Gamma(7; 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)\).

![Figure 2. Examples of two alternating weighted trees.](image-url)
THEOREM 3.11  Let $T_M = T_{n,d}$. Then $\rho(T_{n,d})$ is strictly decreasing in $d(2 \leq d \leq n - 1)$.

Proof  Assume $d \geq 3$. By Lemma 3.6, we have $v_1 = \lfloor \frac{d+3}{2} \rfloor \geq 3$. Let $T'$ be the weighted tree formed from $T_{n,d}$ by deleting the edge $e_1 = 12$ and adding the edge $v_11$ such that $w_{T'}(v_11) = w_{T_{n,d}}(e_1)$, $w_{T'}(e) = w_{T_{n,d}}(e)$, $e \in E(T_{n,d}) \setminus \{e_1\}$.

Then $T' \in \Gamma(d-1; m_1, m_2, \ldots, m_{n-1})$. Since $x_{v_1} \geq x_2$, by Remark, Theorems 2.5 and 3.10, we get

$$\rho(T_{n,d}) = \mu(T_{n,d}) < \mu(T') \leq \mu(T_{n,d-1}) = \rho(T_{n,d-1})$$

By Theorems 3.10 and 3.11, we immediately get the following results.

COROLLARY 3.12  Let $T$ be a weighted tree with $n$ vertices, diameter at least $d$ and positive weight set. Then $\rho(T) \leq \rho(T_{n,d})$, with equality if and only if $T = T_{n,d}$.

COROLLARY 3.13  Let $T$ be a weighted tree with $n \geq 3$ vertices and positive weight set. Then $\rho(T) \leq \rho(T_{n,2})$, with equality if and only if $T = T_{n,2}$ (a weighted star).

COROLLARY 3.14  Let $T \neq T_{n,2}$ be a weighted tree with $n \geq 4$ vertices and positive weight set. Then $\rho(T) \leq \rho(T_{n,3})$, with equality if and only if $T = T_{n,3}$.

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