The Laplacian spectral radius of bicyclic graphs with a given girth

Mingqing Zhai, Guanglong Yu, Jinlong Shu

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Abstract

Let \( \mathcal{B}(n, g) \) be the class of bicyclic graphs on \( n \) vertices with girth \( g \). Let \( \mathcal{B}_1(n, g) \) be the subclass of \( \mathcal{B}(n, g) \) consisting of all bicyclic graphs with two edge-disjoint cycles and \( \mathcal{B}_2(n, g) = \mathcal{B}(n, g) \setminus \mathcal{B}_1(n, g) \). This paper determines the unique graph with the maximal Laplacian spectral radius among all graphs in \( \mathcal{B}_1(n, g) \) and \( \mathcal{B}_2(n, g) \), respectively. Furthermore, the upper bound of the Laplacian spectral radius and the extremal graph for \( \mathcal{B}(n, g) \) are also obtained.

1. Introduction

All graphs considered here are connected and simple. Let \( G \) be a graph. The vertex set and edge set are denoted by \( V(G) \) and \( E(G) \), respectively. The set of vertices adjacent to a vertex \( v \) is denoted by \( N_G(v) \). A vertex of degree \( k \) is called a \( k \)-vertex. The girth \( g(G) \) of \( G \) is the length of the shortest cycle in \( G \). Let \( A(G) \) be the adjacency matrix of \( G \) and \( D(G) \) be the diagonal matrix of vertex degrees. The matrix \( D(G) - A(G) \) is called the Laplacian matrix of \( G \) and is denoted by \( L(G) \). The Laplacian spectral radius \( \mu(G) \), is the largest eigenvalue of \( L(G) \). A principle eigenvector of \( G \) is a unit eigenvector of \( L(G) \) corresponding to \( \mu(G) \).

In [1], R.A. Brualdi and E.S. Solheid posed the problem of maximizing the spectral radius for a given class of graphs. Much attention has been paid to this question in the past decades. Recently, the problems about determining the extremal graph with the maximal Laplacian spectral radius among all graphs with the given order and the number of pendant vertices. This paper focuses on \( \mathcal{B}(n, g) \), namely the class of bicyclic graphs with order \( n \) and girth \( g \).

Let \( \mathcal{B}_1(n, g) \) be the subclass of \( \mathcal{B}(n, g) \) consisting of all bicyclic graphs with two edge-disjoint cycles and \( \mathcal{B}_2(n, g) = \mathcal{B}(n, g) \setminus \mathcal{B}_1(n, g) \). Let \( P_n\) (resp. \( C_g \)) be the path (cycle) on \( n \) vertices. Denote by \( B_{n,q,g}^k \) the graph obtained from two disjoint cycles \( C_p \) and \( C_q \) by identifying a vertex \( u \) of \( C_p \) with a vertex \( v \) of \( C_q \) and attaching \( k \) pendant edges to \( u(v) \). Denote by \( P_{n,q,g}^{k,p} \) the graph consisting of three pairwise internal disjoint paths \( P_{p+1}, P_{q+1}, P_{r+1} \) with common endpoints, and \( k \) pendant edges at one of the common endpoints (see Fig. 1). The main result of this paper is as follows:

**Theorem 1.1.** (i) For every pair of positive integers \( n, g \) with \( 3 \leq g \leq \frac{n+1}{2} \), \( B_{n,q,g}^{n-2g+1} \) is the unique graph with the maximal Laplacian spectral radius among all graphs in \( \mathcal{B}_1(n, g) \).

(ii) For every pair of positive integers \( n, g \) with \( 3 \leq g \leq \frac{2(n+1)}{3} \), \( P_{n,\lfloor \frac{n}{2} \rfloor,\lfloor \frac{n}{2} \rfloor}^{n-1,\lfloor \frac{n}{2} \rfloor,\lfloor \frac{n}{2} \rfloor} \) is the unique graph with the maximal Laplacian spectral radius among all graphs in \( \mathcal{B}_2(n, g) \).

Furthermore, the upper bound of Laplacian spectral radius and the extremal graph for \( \mathcal{B}(n, g) \) are also obtained.

* Corresponding author at: Department of Mathematics, East China Normal University, Shanghai, 200241, China.
E-mail addresses: mqzhai@126.com (M. Zhai), yglong01@163.com (G. Yu), jsju@math.ecnu.edu.cn (J. Shu).

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2. Preliminaries

Let $G, H$ be two disjoint graphs with $u \in V(G)$ and $w \in V(H)$. The graph obtained from $G$ and $H$ by identifying $u$ with $w$ is denoted by $GuwH$ (see Fig. 2).

Lemma 2.1 ([5]). Let $G, H$ be two disjoint nontrivial connected graphs with $u, v \in V(G)$ and $w \in V(H)$. Let $X$ be a principal eigenvector of $GuwH$ with component $x_i$ corresponding to vertex $i$. If $|x_i| \leq |x_u|$, then
(i) $\mu(GuwH) \leq \mu(GuwH)$. If the equality holds, $|x_u| = |x_v|$, and either $X$ or $X'$ is a principal eigenvector of $GuwH$, where
\[
(X')_i = \begin{cases} 
-x_i & i \in V(H) \setminus \{u\}, \\
0 & \text{otherwise.}
\end{cases}
\]
(ii) In particular, if $H$ is bipartite, the equality holds if and only if $X$ is a principal eigenvector of $GuwH$ such that $x_u = x_v = \sum_{i \in N_H(u)} x_i = 0$ if and only if $X'$ is a principal eigenvector of $GuwH$ such that $x_u = x_v = \sum_{i \in N_H(u)} x_i = 0$.

Lemma 2.2. Let $G$ be a connected graph, $\Delta$ be its maximum degree, $d_i$ be the degree of vertex $v_i$ and $m_i = \sum_{j \in N(v_i)} d_j / d_i$. Then
(i) $|18| \mu(G) \geq \Delta + 1$, the equality holds if and only if $\Delta = n - 1$;
(ii) $|19| \mu(G) \leq \max \{d_i + d_j | v_i, v_j \in E(G)\}$, the equality holds if and only if $G$ is either a regular bipartite graph or a semiregular bipartite graph;
(iii) $|90| \mu(G) \leq \max \{d_i + m_i | v \in V(G)\}$, the equality holds if and only if $G$ is either a regular bipartite graph or a semiregular bipartite graph;
(iv) $|121| \mu(G) \leq \max \{d_i + d_j + m_i | v_i, v_j \in E(G)\}$, the equality holds if and only if $G$ is either a regular bipartite graph or a semiregular bipartite graph.

3. Maximizing the Laplacian spectral radius in $B(n, g)$

Let $G$ be a bicyclic graph. The base of $G$, denoted by $B(G)$, is the minimal bicyclic subgraph of $G$. Clearly, $B(G)$ is the unique bicyclic subgraph of $G$ containing no pendant vertices, and $G$ can be obtained from $B(G)$ by planting trees to some vertices of $B(G)$.

Bicyclic graphs have two types of bases (see Fig. 3). Denote by $B(p, l, q)$ the graph obtained by joining a new path $u_1u_2 \ldots u_l$ between two vertex-disjoint cycles $C_p$ and $C_q$, where $u_1 \in V(C_p)$ and $u_l \in V(C_q)$. In particular, $B(p, 1, q) \cong C_p \cup vC_q$ for some $u \in V(C_p)$ and $v \in V(C_q)$. Denote by $P(p, q, r)$ the graph consisting of three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{r+1}$ with common endpoints, that is, $P(p, q, r) \cong P_{p,q,r}$.

Clearly, $B_1(n, g)$ and $B_2(n, g)$ can also be defined as follows:
$B_1(n, g) = \{G \in B(n, g) | B(G) = B(p, l, q) \text{ for some } l \geq 1 \text{ and } p, q \geq 3\}$,
$B_2(n, g) = \{G \in B(n, g) | B(G) = P(p, q, r) \text{ for some } p, q, r \geq 1\}$.

Lemma 3.1. Let $G^*$ have the maximal Laplacian spectral radius among all graphs in $B_1(n, g)$ (resp. $B_2(n, g)$). Then $G^*$ is obtained from $B(G^*)$ by attaching some pendant edges (if exist) to a unique vertex.

Proof. Let $X$ be a principal eigenvector of $G^*$. First, we claim that all pendant vertices of $G^*$ have a unique neighbor. Otherwise, let $u_1, u_2$ be two pendant vertices with different neighbors $u'_1$ and $u'_2$, respectively. Since $G^*$ is an extremal graph, by Lemma 2.1, $x_{u'_1} = x_{u'_2} = 0$ and hence $x_{u_1} = x_{u_2} = 0$. Now let $u_3$ be a vertex of $G^*$ with $x_{u_3} \neq 0$. Then $|x_{u'_1}| < |x_{u_3}|$ and by Lemma 2.1, $\mu(G^*) < \mu(G^* - u_1u'_1 + u_1u_3)$, a contradiction.
The claim above implies that $G^*$ is the graph obtained from $B(G^*)$ and a star $S$ by joining a path $P_{v_1v}$ between a vertex $v(=u)$ of $B(G^*)$ and the center $v$ of the star (see Fig. 4). Now it suffices to show that $v = v'$. Assume to the contrary that $v \neq v'$. If $|x_u| < |x_{v'}|$, then by Lemma 2.1, $\mu(G^*) < \mu(B(G^*)uv'(S + P_{v_1v}))$, a contradiction. So $|x_u| \geq |x_{v'}|$ and by Lemma 2.1, $\mu(G^*) \leq \mu(G)$, where $G = G^* - \sum_{i \in V} \nu_i + \sum_{i \in V} Iv$ and $V$ is the set of pendant vertices. Since $G^*$ is an extremal graph and $S$ is bipartite, $\mu(G^*) = \mu(G)$ and hence $x_u = x_{v'} = 0$. Now let $w$ be a vertex of $G^*$ with $x_w \neq 0$. Then $|x_v| < |x_w|$ and by Lemma 2.1, $\mu(G^*) < \mu(G^* - \sum_{i \in V} \nu_i + \sum_{i \in V} Iw)$, a contradiction. This completes the proof. □

**Lemma 3.2.** Let $G^*$ have the maximal Laplacian spectral radius among all graphs in $\mathcal{B}_1(n, g)$. Then $G^* \cong B_{g,q}^{n-g+q-1}$ for some $q \geq g$.

**Proof.** Let $C_g$ and $C_g$ be the two cycles connected by a path $u_1u_2 \ldots u_l$ in $G^*$, where $u_1 \in V(C_g)$ and $u_l \in V(C_g)$. First consider the case $G^* \cong B(G^*)$. Now $G^*$ has no pendant vertices. It suffices to show $l = 1$. Suppose to the contrary that $l \geq 2$. Then by Lemma 2.2, $\mu(G^*) < \max \{|d_i| + d_j|v_i| \in E(G^*)\} \leq 6$ since in this case $G^*$ cannot be regular and semiregular. However, $\mu(B_{g,q}^{1}) \geq \Delta + 1 = 6$. Since in this case we can find a graph $G \in \mathcal{B}_1(n, g)$ that contains $B_{g,q}^{1}$ as a subgraph, $\mu(G) \geq \mu(B_{g,q}^{1})$, a contradiction.

Now by Lemma 3.1, it remains the case that $G^* \cong B(G^*)uvS$, namely the graph obtained from $B(G^*)$ and a star $S$ by identifying a vertex $u$ of $B(G^*)$ with the center $v$ of $S$. In this case, it suffices to prove $u = u_1 = u_l$. Considering the symmetry of $u_1$ and $u_l$, we next only show that $u = u_1$. Assume to the contrary that $u \neq u_1$. Let $X$ be a principle eigenvector of $G^*$. If $|x_u| > |x_{u_1}|$, then by Lemma 2.1, $\mu(G^*) < \mu(V(C_guvS - u_1))u_1u(C_guvS)$

when $u \in V(C_g)$ and

$$\mu(G^*) < \mu(C_gu_1u(G^* - V(C_g - u_1)))$$

otherwise. This will be in contrary to the assumption that $G^*$ is an extremal graph. So $|x_u| \leq |x_{u_1}|$, and by Lemma 2.1, $\mu(G^*) \leq \mu(B(G^*)u_1vS)$. Since $G^*$ is an extremal graph and $S$ is bipartite, $\mu(G^*) = \mu(B(G^*)u_1vS)$ and hence $x_u = x_{u_1} = 0$. Now let $w$ be a vertex of $B(G^*)$ with $x_w \neq 0$. Then $|x_u| < |x_w|$ and by Lemma 2.1, $\mu(G^*) < \mu(B(G^*)wvS)$, a contradiction. Thus $u = u_1$ and the proof is completed. □

**Theorem 3.3.** Let $G^*$ have the maximal Laplacian spectral radius among all graphs in $\mathcal{B}_1(n, g)$, where $n \geq 2g - 1$. Then $G^* \cong B_{g,q}^{n-2g+1}$ and $\mu(G^*) < n - 2g + 6 + \frac{4}{n-2g+5}$.

**Proof.** By Lemma 3.2, $G^* \cong B_{g,q}^{n-2g+1}$ for some $q \geq g$. Now suppose that $q \geq g + 1$, then $n \geq 2g$. For convenience, set $k = n - (g + q - 1)$, namely the number of pendant edges in $G^*$. Then $k \leq n - 2g$. Note that $G^*$ cannot be regular and semiregular, by Lemma 2.2, we have

$$\mu(G^*) < \max \{d_i + m_i | v_i \in V(G^*)\} = k + 4 + \frac{k + 8}{k + 4} = k + 5 + \frac{4}{k + 4}.$$

Note that $k + 5 + \frac{4}{k + 4}$ is increasing with nonnegative number $k$. Thus

$$\mu(G^*) < n - 2g + 5 + \frac{4}{n - 2g + 4} \leq n - 2g + 6$$

since $n \geq 2g$. However, by Lemma 2.2, $\mu(B_{g,q}^{n-2g+1}) \geq \Delta + 1 = n - 2g + 6 > \mu(G^*)$, a contradiction. Therefore $G^* \cong B_{g,q}^{n-2g+1}$.

And since $B_{g,q}^{n-2g+1}$ is neither regular nor semiregular,

$$\mu(B_{g,q}^{n-2g+1}) < \max \{d_i + m_i | v_i \in V(B_{g,q}^{n-2g+1})\} = n - 2g + 6 + \frac{4}{n - 2g + 5}.$$

□
Next we determine the extremal graph for the class $\mathcal{B}_2(n, g)$. Let $G^*$ have the maximal Laplacian spectral radius among all graphs in $\mathcal{B}_2(n, g)$. Lemma 3.1 implies that $G^*$ is obtained from $P(p, q, r)$ by attaching $n - (p + q + r) + 1$ pendant edges to a unique vertex. Further, we can get the following result.

**Lemma 3.4.** Let $G^*$ have the maximal Laplacian spectral radius among all graphs in $\mathcal{B}_2(n, g)$, where $n \geq \lceil \frac{3k}{2} \rceil - 1$. Then $G^*$ is obtained from $P\left(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil \right)$ by attaching $n - \lceil \frac{3k}{2} \rceil + 1$ pendant edges to a unique vertex.

**Proof.** The proof need not distinguish two cases that $g$ is even or not. However for convenience, we prefer to give only the proof for even $g$. Set $a = \frac{g}{2}$ and suppose that $B(G^*) = P(p, q, r)$, where $p \leq q \leq r$ and $p + q = 2a$.

It suffices to show that $p = q = r = a$. Note that $p + q + r \geq 3a$ and $n \geq (p + q + r) - 1$.

If $n = \frac{3k}{2} - 1 = 3a - 1$, then $G^*$ cannot contain pendant edges and $p + q + r = 3a$. Since $p + q = 2a$, $a = r$ and hence $p \leq q \leq a$. If $p \leq a - 1$, then $q \geq a + 1$, which contradicts $q \leq r$. Thus $p = q = r = a$.

If $n = \frac{3k}{2} = 3a$, then $p + q + r \leq 3a + 1$. Since $p + q = 2a$, $r \leq a + 1$. This implies that $q \leq a + 1$ and $p \geq a - 1$. So $(p, q, r) \in \{(a, a, a), (a, a, a + 1), (a - 1, a + 1, a - 1)\}$. Moreover, if $(p, q, r) \in \{(a, a, a + 1), (a - 1, a + 1, a + 1)\}$, then $G^*$ cannot contain pendant edges, in other words, $G^*$ is isomorphic to one of $P(a, a, a + 1)$ and $P(a + 1, a - 1, a - 1)$. When $a = 2$, straightforward calculations show that $\max\{\mu(P(2, 2, 3)), \mu(P(1, 3, 3))\} < \mu(P_{2, 2, 2})$, a contradiction. When $a \geq 3$, $G^*$ does not contain a pair of adjacent 3-vertices. So by Lemma 2.2,

$$\mu(G^*) \leq \max\{d_i + d_j|v_i, v_j \in E(G^*)\} = 5.$$ 

However, $\mu(P_{a,a,a}) < \Delta + 1 = 5$ since in this case $\Delta(P_{a,a,a}) \neq n - 1$. Thus $\mu(G^*) < \mu(P_{a,a,a})$, a contradiction. Therefore, $p = q = r = a$.

Now it remains the case $n \geq \frac{3k}{2} + 1 = 3a + 1$. If $p = q = r = a$ since $p + q = 2a$ and $p \leq q \leq r$. Next suppose that $r \geq a + 1$ and set $k = n - (p + q + r) + 1$, namely the number of pendant edges in $G^*$. Then $k \leq n - 3a$. We can find that $\max\{d_i + m_i|v_i \in V(G^*)\}$ attains the maximum just when $p = 1$ and $k$ pendant edges of $G^*$ are incident to a 3-vertex of $P(p, q, r)$. In this case,

$$\max\{d_i + m_i|v_i \in V(G^*)\} = k + 3 + \frac{k + 7}{k + 3} = k + 4 + \frac{4}{k + 3}.$$ 

By Lemma 2.2, we have

$$\mu(G^*) < \max\{d_i + m_i|v_i \in V(G^*)\} = k + 4 + \frac{4}{k + 3},$$ 

since in this case $G^*$ cannot be regular and semiregular. Note that $k + 4 + \frac{4}{k + 3}$ is increasing with nonnegative number $k$. Thus

$$\mu(G^*) < n - 3a + 4 + \frac{4}{n - 3a + 3} \leq n - 3a + 5$$

since $n \geq 3a + 1$. However, by Lemma 2.2, $\mu(P_{a,a,a}^n) \geq \Delta + 1 = n - 3a + 5 > \mu(G^*)$, a contradiction. □

**Theorem 3.5.** Let $G^*$ have the maximal Laplacian spectral radius among all graphs in $\mathcal{B}_2(n, g)$, where $n \geq \lceil \frac{3k}{2} \rceil - 1$. Then $G^* \simeq P_{\frac{n - \lceil \frac{3k}{2} \rceil + 1}{\lceil \frac{k}{2} \rceil}, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$ and $\mu(G^*) < n - \lceil \frac{3k}{2} \rceil + 5 + \frac{4}{n - \lceil \frac{3k}{2} \rceil + 4}$.

**Proof.** By Lemma 3.4, $G^*$ is obtained from $P\left(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil \right)$ by attaching $n - \lceil \frac{3k}{2} \rceil + 1$ pendant edges to a unique vertex $u$. Next we only need to show that $u$ is a 3-vertex of $P\left(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil \right)$. Assume to the contrary that $u$ is a 2-vertex of $P\left(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil \right)$.

For convenience, set $k = n - \lceil \frac{3k}{2} \rceil + 1$. If $k = 0$, clearly $G^* \simeq P_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}.$

Now consider the case $k = 1$. If $g \in \{3, 4, 5\}$, then $G^*$ is isomorphic to one of $G_i (i \in \{1, 2, 3, 4\}$, see Fig. 5). Straightforward calculations show that $\mu(G_1) < \mu(P_{1,2,2})$, $\mu(G_2) < \mu(P_{2,2,2})$ and $\max\{\mu(G_3), \mu(G_4)\} < \mu(P_{3,3,3})$, a contradiction. If $g \geq 6$, then $u$ cannot be simultaneously adjacent to the two 3-vertices of $P\left(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil \right)$ and these 2-vertices are not adjacent. So $d_u m_u \leq 1 + 2 + 3 = 6$. Besides, $d_i m_i \leq 2 + 2 + 3 = 7$ if $v_i$ is a 3-vertex different from $u$, $d_i m_i \leq 3 + 3 = 6$ if $v_i$ is a 2-vertex and $d_i m_i = 3$ if $v_i$ is a pendant vertex. This implies that

$$\max\left\{\frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} \middle| v_i, v_j \in E(G^*)\right\} \leq \max\left\{\frac{31}{6}, \frac{26}{5}, \frac{20}{4}\right\} = 5 \frac{1}{5},$$

since $d_i + d_j \in \{4, 5, 6\}$ for each edge $v_iv_j$. By Lemma 2.2 (iv), we have $\mu(G^*) \leq 5 \frac{1}{5}$. However when $g \geq 6$, $G_5$ (see Fig. 5) is a subgraph of $P_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$. Thus $\mu(P_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}) \geq \mu(G_5) = 3 + \sqrt{5} > 5 \frac{1}{5}$. That is, $\mu(G^*) < \mu(P_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil})$, a contradiction.
Next consider the case $k \geq 2$. We can find that \( \max\{d_i + m_i | v_i \in V(G^*)\} \) attains the maximum just when \( u \) is simultaneously adjacent to two 3-vertices of \( P(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil) \). In this case,

\[
\max\{d_i + m_i | v_i \in V(G^*)\} = k + 2 + \frac{k + 6}{k + 2} \leq k + 4
\]

since $k \geq 2$. By Lemma 2.2 (iii), we have $\mu(G^*) < k + 4$, since $G^*$ cannot be regular and semiregular. However, by Lemma 2.2 (i), $\mu(P(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil)) \geq k + 1 = k + 4 > \mu(G^*)$, a contradiction. Therefore $u$ is a 3-vertex of \( P(\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil) \).

In other words, $G^* \cong P_{n-\lceil \frac{3g}{2} \rceil + 1}^{\lceil \frac{g}{2} \rceil}$. Note that $P_{n-\lceil \frac{3g}{2} \rceil + 1}^{\lceil \frac{g}{2} \rceil}$ is neither regular nor semiregular except that \( P_{2,2}^0 \cong K_{2,3} \) is a semiregular bipartite graph. Thus if \( (n, g) \neq (5, 4) \), we know that

\[
\mu(P_{2,2,2}^0) = 5 \leq 5 \frac{1}{3} = n - \left\lceil \frac{3g}{2} \right\rceil + 5 + \frac{4}{n - \left\lceil \frac{3g}{2} \right\rceil + 4}. \quad \Box
\]

**Theorem 3.6.** Let $G^*$ have the maximal Laplacian spectral radius among all graphs in $\mathcal{B}(n, g)$, where \( n \geq \lceil \frac{3g}{2} \rceil - 1 \).

(i) If $g = 3$, then $\mu(G^*) = n$ and $G^*$ is isomorphic to one of $B_{3,3}^{n-5}$ and $P_{1,2,2}^{n-4}$.

(ii) If $g \geq 4$, then $\mu(G^*) < n - \left\lceil \frac{3g}{2} \right\rceil + 5 + \frac{4}{n - \left\lceil \frac{3g}{2} \right\rceil + 4}$ and $G^* \cong P_{n-\lceil \frac{3g}{2} \rceil + 1}^{\lceil \frac{g}{2} \rceil}$.

**Proof.** (i) By Theorems 3.3 and 3.5, the extremal graph $G^*$ must be $B_{3,3}^{n-5}$ or $P_{1,2,2}^{n-4}$. Since the maximum degrees of $B_{3,3}^{n-5}$ and $P_{1,2,2}^{n-4}$ are $n - 1$, by Lemma 2.2 (i), $\mu(B_{3,3}^{n-5}) = \mu(P_{1,2,2}^{n-4}) = n$. So both $B_{3,3}^{n-5}$ and $P_{1,2,2}^{n-4}$ are extremal graphs for $\mathcal{B}(n, 3)$.

(ii) For $g \geq 4$, we have to show $\mu(B_{g,8}^{n-2g+1}) < \mu(P_{g,8}^{n-\lceil \frac{3g}{2} \rceil + 1})$. According to Theorem 3.3, $\mu(B_{g,8}^{n-2g+1}) < n - 2g + 6 + \frac{4}{n - 2g + 5} \leq n - 2g + 7$ since in this case $n \geq 2g - 1$. However $\mu(P_{g,8}^{n-\lceil \frac{3g}{2} \rceil + 1}) \geq \Delta + 1 = n - \left\lceil \frac{3g}{2} \right\rceil + 5$. Since $g \geq 4$,

\[
\begin{align*}
 &n - \left\lceil \frac{3g}{2} \right\rceil + 5 - (n - 2g + 7) = \left\lceil \frac{g}{2} \right\rceil - 2 \geq 0.
\end{align*}
\]

Thus $\mu(B_{g,8}^{n-2g+1}) < \mu(P_{g,8}^{n-\lceil \frac{3g}{2} \rceil + 1})$. $\Box$

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