The spectral characterization of graphs of index less than 2 with no path as a component

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Abstract

A graph is said to be determined by the adjacency and Laplacian spectrum (or to be a DS graph, for short) if there is no other non-isomorphic graph with the same adjacency and Laplacian spectrum, respectively. It is known that connected graphs of index less than 2 are determined by their adjacency spectrum. In this paper, we focus on the problem of characterization of DS graphs of index less than 2. First, we give various infinite families of cospectral graphs with respect to the adjacency matrix. Subsequently, the results will be used to characterize all DS graphs (with respect to the adjacency matrix) of index less than 2 with no path as a component. Moreover, we show that most of these graphs are DS with respect to the Laplacian matrix.

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1. Introduction

Let $G$ be an undirected finite simple graph with $n$ vertices and the adjacency matrix $A(G)$. Let $D(G)$ be the diagonal matrix with degrees of the corresponding vertices of $G$ on the main diagonal and zero elsewhere. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of $G$. Since $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the adjacency and the Laplacian eigenvalues of $G$, respectively. The multiset of the eigenvalues of $A(G)$ and $L(G)$ are called the...
adjacency and Laplacian spectrum of $G$, respectively. The maximum eigenvalue of $A(G)$ is called the index of $G$. Two graphs are said to be cospectral with respect to the adjacency (Laplacian, respectively) matrix if they have the same adjacency (Laplacian, respectively) spectrum. A graph is said to be determined (DS for short) by its adjacency and Laplacian spectrum if there is no other non-isomorphic graph with the same spectrum with respect to the adjacency and Laplacian matrix, respectively.

In [8], all graphs of index at most 2 are identified. Most of connected graphs of index at most 2 are known to be DS with respect to adjacency matrix [3]. In this paper, we focus on the problem of characterization of DS graphs of index less than 2 and we give various infinite families of cospectral graphs with respect to the adjacency matrix. The results are used to characterize all DS graphs (with respect to the adjacency matrix) of index less than 2 which does not have any path as a component. Moreover, we give new infinite families of DS graphs with respect to the Laplacian matrix.

Notation. The path and cycle with $n$ vertices are denoted by $P_n$ and $C_n$, respectively. For $a, b, c \geq 1$, we denote the graph shown in Fig. 1 (left) by $T(a, b, c)$. In particular, $Z_n$ ($n \geq 2$) stands for $T(1, n - 1, 1)$. For $n \geq 2$, we denote the graph shown in Fig. 1 (right) by $W_n$.

2. Graphs of index less than 2

In [8], all connected graphs of index at most 2 are identified. Among them all connected graphs of index 2 are well known. So using this we can determine all graphs with index less than 2.

Theorem 1 [8]. The list of all connected graphs of index at most 2 includes precisely the following graphs:

(i) $P_n, C_n, Z_n$ ($n \geq 2$), $W_n$ ($n \geq 2$),
(ii) $T(a, b, c)$ for $(a, b, c) = \{(1, 2, 2), (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 3), (2, 2, 2)\}$,
(iii) $K_{1, 4}$.

We will denote the graphs $T(1, 2, 2), T(1, 2, 3), T(1, 2, 4), T(1, 2, 5), T(1, 3, 3)$ and $T(2, 2, 2)$ by $T_i$ for $i = 1, 2, \ldots, 6$, respectively. Among graphs of index at most 2, the graphs $C_n, W_n$ ($n \geq 2$), $K_{1, 4}$ and $T_i$ for $i = 4, 5, 6$ have 2 as an eigenvalue.

Corollary 1. The list of all connected graphs of index less than 2 consist of precisely the following graphs:

(i) $P_n, Z_n$ ($n \geq 2$),
(ii) $T_i$ for $i = 1, 2, 3$. 

Fig. 1.
In [3], it was proved that all connected graphs of index at most 2 are DS with respect to the adjacency matrix except for \( W_n \) (\( n \geq 2 \)), \( T(2,2,2) \) and \( K_{1,4} \). So all connected graphs of index less than 2 are DS with respect to the adjacency matrix.

The adjacency spectrum of the union of two graphs is obviously the union of their spectra (having in view the multiplicities of the eigenvalues). The expressions \( G_1 + G_2 \) and \( \hat{G}_1 + \hat{G}_2 \) will denote the union of the graphs \( G_1 \) and \( G_2 \) and the union of their adjacency spectra, respectively. The expressions \( kG \) and \( \hat{k}G \) denote the union of \( k \) copies of \( G \) and \( \hat{G} \), respectively. If \( \hat{G}_1 \subseteq \hat{G}_2 \), the expression \( \hat{G}_2 - \hat{G}_1 \) denote the difference of systems of numbers of \( \hat{G}_1 \) from \( \hat{G}_2 \).

Let \( G \) be a graph of index less than 2. Then using Corollary 1, for some natural numbers \( i_1, i_2, \ldots, i_l \) and \( j_1, j_2, \ldots, j_k \), \( G \) can be represented in a unique way as a linear combination of the form

\[
P_{i_1} + P_{i_2} + \cdots + P_{i_l} + Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1 T_1 + t_2 T_2 + t_3 T_3.
\]

So any graph of index less than 2 with no path as a component can be represented in a unique way as a linear combination of the form

\[
G = Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1 T_1 + t_2 T_2 + t_3 T_3.
\]

3. New infinite families of cospectral graphs

In [2], infinite families of cospectral graphs of index at most 2 with respect to the adjacency spectrum are identified. In this section, we give new infinite families of cospectral graphs which contain most previously known families of cospectral graphs of index less than 2. In the next section the results will be used to characterize all DS graph of index less than 2 with no path as a component.

The adjacency spectra of graphs of index at most 2 are well known [2], and we have

\[
\hat{P}_n = \left\{ 2 \cos \frac{j \pi}{n+1} \mid j = 1, 2, \ldots, n \right\},
\]

\[
\hat{Z}_n = \left\{ 2 \cos \frac{(2j+1) \pi}{2(n+1)} \mid j = 0, 1, 2, \ldots, n \right\} + \{0\},
\]

\[
\hat{W}_n = \left\{ 2 \cos \frac{j \pi}{n+1} \mid j = 1, 2, \ldots, n \right\} + \{2, 0, 0, -2\},
\]

\[
\hat{C}_n = \left\{ 2 \cos \frac{2j \pi}{n} \mid j = 1, 2, \ldots, n \right\},
\]

\[
\hat{T}_1 = \left\{ 2 \cos \frac{j \pi}{12} \mid j = 1, 4, 5, 7, 8, 11 \right\},
\]

\[
\hat{T}_2 = \left\{ 2 \cos \frac{j \pi}{18} \mid j = 1, 5, 7, 9, 11, 13, 17 \right\},
\]

\[
\hat{T}_3 = \left\{ 2 \cos \frac{j \pi}{30} \mid j = 1, 7, 11, 13, 17, 19, 23, 29 \right\},
\]

\[
\hat{T}_4 = \left\{ 2 \cos \frac{j \pi}{3} \mid j = 1, 2, 3, 4, 5, 6 \right\} + \{0\},
\]

\[
\hat{T}_5 = \left\{ 2 \cos \frac{j \pi}{4} \mid j = 1, 2, 3 \right\} + \{2, 1, 0, -1, -2\}.
\]
\[
\hat{T}_6 = \left\{ 2 \cos \frac{j\pi}{5} \mid j = 1, 2, 3, 4 \right\} + \{2, 1, 0, -1, -2\}.
\]

The following Lemma presents some cospectral graphs of index at most 2 with respect to the adjacency matrix.

**Lemma 1** [2]. The following can be obtained from the above quoted facts:

(i) \( \hat{Z}_n + \hat{P}_n = \hat{P}_{2n+1} + \hat{P}_1 \),
(ii) \( \hat{W}_n = \hat{C}_4 + \hat{P}_n \),
(iii) \( \hat{C}_{2n} + 2\hat{P}_1 = \hat{C}_4 + 2\hat{P}_{n-1} \),
(iv) \( \hat{T}_1 + \hat{P}_3 + \hat{P}_5 = \hat{P}_1 + \hat{P}_2 + \hat{P}_{11} \),
(v) \( \hat{T}_2 + \hat{P}_3 + \hat{P}_5 = \hat{P}_1 + \hat{P}_2 + \hat{P}_{11} \),
(vi) \( \hat{T}_3 + \hat{P}_{14} + \hat{P}_9 + \hat{P}_5 = \hat{P}_29 + \hat{P}_4 + \hat{P}_2 + \hat{P}_1 \),
(vii) \( \hat{T}_4 + \hat{P}_1 = \hat{C}_4 + 2\hat{P}_2 \),
(viii) \( \hat{T}_5 + \hat{P}_1 = \hat{C}_4 + \hat{P}_3 + \hat{P}_2 \),
(ix) \( \hat{T}_6 + \hat{P}_1 = \hat{C}_4 + \hat{P}_4 + \hat{P}_2 \).

So using Lemma 1, the adjacency spectrum of any graphs of index less than 2 can be obtained from the union of spectra of some paths.

**Theorem 2** [2]. The adjacency spectrum of any bipartite graph of index at most 2 can uniquely be represented as a linear combination of the form

\[ \delta_0\hat{C}_4 + \delta_1\hat{P}_1 + \delta_2\hat{P}_2 + \cdots + \delta_m\hat{P}_m. \]

The number \( m \) is bounded by a function of the number of vertices. Moreover, \( \delta_0 \) is always non-negative and the non-vanishing coefficient \( \delta_i \), with the greatest \( i \), is positive.

**Lemma 2.** Let \( a \) and \( b \) be non-negative integers. We have

(i) \( \hat{H}_1 = (a + b)\hat{Z}_2 + b\hat{Z}_4 + a\hat{T}_2 + b\hat{T}_3 = (2b + a)\hat{P}_1 + a\hat{Z}_8 + b\hat{Z}_{14} \),
(ii) If \( a \geq b \), then \( \hat{H}_2 = (a - b)\hat{Z}_2 + a\hat{Z}_4 + b\hat{Z}_8 + a\hat{T}_3 = (2a - b)\hat{P}_1 + a\hat{Z}_{14} + b\hat{T}_2 \),
(iii) If \( 2b \geq a \geq b \), then \( \hat{H}_3 = b\hat{Z}_4 + a\hat{Z}_8 + b\hat{T}_3 = (2b - a)\hat{P}_1 + (a - b)\hat{Z}_2 + b\hat{Z}_{14} + a\hat{T}_2 \),
(iv) If \( a \geq 2b \), then \( \hat{H}_4 = (a - b)\hat{Z}_2 + b\hat{Z}_{14} + a\hat{T}_2 = (a - 2b)\hat{P}_1 + b\hat{Z}_4 + a\hat{Z}_8 + b\hat{T}_3 \).

**Proof.** Using Lemma 1, we can represent each side of these equations as a linear combination of the form

\[ \delta_1\hat{P}_1 + \delta_2\hat{P}_2 + \cdots + \delta_m\hat{P}_m. \]

It is clear that the two sides of each equation have the same representation. \( \square \)

4. **The characterization of DS graphs of index less than 2 with no path as a component**

The spectral characterization of graphs of index less than 2 with respect to the adjacency matrix is an important problem which seems to be challenging. In [9], it was shown that the disjoint union of \( k \) disjoint paths \( P_{n_1} + P_{n_2} + \cdots + P_{n_k} \) is determined by its spectrum with respect to the adjacency matrix as well as the Laplacian matrix, where \( n_1, n_2, \ldots, n_k \) are integers greater
than 1. Moreover, in [7], Shen and others showed that $\sum_{k} Z_{n_k}$ is determined by its adjacency spectrum, where $n_1, n_2, \ldots, n_k$ are integers at least 2. In this section we identify all DS graphs of type $G = Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1 T_1 + t_2 T_2 + t_3 T_3$.

**Lemma 3** [9]. Let $G$ be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum:

(i) The number of vertices:

For the adjacency matrix, the following follows from the spectrum:

(ii) The number of edges:

For the Laplacian matrix, the following follows from the spectrum:

(iii) The number of closed walks of any length:

(iv) The number of spanning trees:

(v) The number of components:

(vi) The sum of squares of degrees of vertices.

**Theorem 3.** Let $G = Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1 T_1 + t_2 T_2 + t_3 T_3$ be a graph of index less than 2. Then $G$ can be determined by its adjacency spectrum if and only if $G$ does not have some components where the spectrum of their union is $\hat{H}_i (i = 1, \ldots, 4)$.

**Proof.** Suppose that $G$ has $z_i$ components of type $Z_i$. Then for some natural number $l$, $G$ can be represented as

$$G = \sum_{i=2}^{l} z_i Z_i + t_1 T_1 + t_2 T_2 + t_3 T_3.$$ 

Let $H$ be cospectral to $G$. Since $G$ has index less than 2, $H$ can be represented in a unique way as a linear combination of the form

$$H = P_1 + P_2 + \cdots + P_t + Z_{c_1} + Z_{c_2} + \cdots + Z_{c_k} + \tilde{t}_1 T_1 + \tilde{t}_2 T_2 + \tilde{t}_3 T_3.$$ 

Again, suppose that $H$ has $\tilde{p}_i$ components of type $P_i$ and $\tilde{z}_i$ components of type $Z_i$. Then for some natural numbers $\tilde{l}$ and $\tilde{t}$, $H$ can be represented as

$$H = \sum_{i=1}^{\tilde{l}} \tilde{p}_i P_i + \sum_{i=2}^{\tilde{t}} \tilde{z}_i Z_i + \tilde{t}_1 T_1 + \tilde{t}_2 T_2 + \tilde{t}_3 T_3.$$ 

By Lemma 3, $H$ and $G$ have the same numbers of vertices and edges. Since their components are trees, they have the same number of components. So we have

$$\sum_{i=2}^{l} z_i + t_1 + t_2 + t_3 = \sum_{i=1}^{\tilde{l}} \tilde{p}_i + \sum_{i=2}^{\tilde{t}} \tilde{z}_i + \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3.$$ 

(1)

Since $G$ and $H$ are cospectral and bipartite graphs of index at most 2, by Theorem 2, their spectrum can be represented in a unique way as a linear combination of the form

$$\delta_0 \hat{P}_0 + \delta_1 \hat{P}_1 + \delta_2 \hat{P}_2 + \cdots + \delta_m \hat{P}_m.$$ 

(2)

Using Lemma 1, we can calculate $\delta_i$ for any $i$.

$$\delta_5 = z_2 - z_5 - t_1 - t_2 - t_3 = \tilde{p}_5 + \tilde{z}_2 - \tilde{z}_5 - \tilde{t}_1 - \tilde{t}_2 - \tilde{t}_3,$$ 

(3)
\[
\delta_1 = \sum_{i=2}^l z_i + t_1 + t_2 + t_3 = \tilde{p}_1 + \sum_{i=2}^l \tilde{z}_i + \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3. \tag{4}
\]

By the relations (1) and (4) we have \( \tilde{p}_1 = 0 \) for \( i \neq 1 \). So
\[
\delta_2 = -z_2 + t_1 + t_2 + t_3 = -\tilde{z}_2 + \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3. \tag{5}
\]

By (3) and (5), we have \(-z_5 = \tilde{p}_5 - \tilde{z}_5\) and so \( z_5 = \tilde{z}_5 \).

\[
\delta_3 = -z_3 - t_1 = -\tilde{z}_3 - \tilde{t}_1, \tag{6}
\]
\[
\delta_4 = -z_4 + t_1 = -\tilde{z}_4 + \tilde{t}_1, \tag{7}
\]
\[
\delta_{11} = z_5 - z_{11} + t_1 = \tilde{z}_5 - \tilde{z}_{11} + \tilde{t}_1, \tag{8}
\]
\[
\delta_8 = -z_8 - t_2 = -\tilde{z}_8 - \tilde{t}_2, \tag{9}
\]
\[
\delta_{14} = -z_{14} - t_3 = -\tilde{z}_{14} - \tilde{t}_3, \tag{10}
\]
\[
\delta_{29} = -z_{29} + z_{14} + t_3 = -\tilde{z}_{29} + \tilde{z}_{14} + \tilde{t}_3. \tag{11}
\]

By (10) and (11), we have \( z_{29} = \tilde{z}_{29} \).

\[
\delta_9 = z_4 - z_9 - t_3 = \tilde{z}_4 - \tilde{z}_9 - \tilde{t}_3, \tag{12}
\]
\[
\delta_7 = z_3 - z_7 = \tilde{z}_3 - \tilde{z}_7, \tag{13}
\]
\[
\delta_{19} = z_9 - z_{19} = \tilde{z}_9 - \tilde{z}_{19}. \tag{14}
\]

Let \( S = \{1, 2, 3, 4, 5, 8, 9, 11, 14, 17, 29\} \) and \( x \in N - S \). Then for \( n \geq 0 \) we have
\[
\delta((2^n x + 2^n - 1) = \tilde{z}(2^n x + 2^n - 1) - \tilde{z}(2^n x + 2^n - 1) = \tilde{z}(2^n x + 2^n - 1) - \tilde{z}(2^n x + 2^n - 1). \]

If \( z_x \neq \tilde{z}_x \), then for any \( n \geq 1 \), \( z((2^n x + 2^n - 1) \neq \tilde{z}(2^n x + 2^n - 1) \) and so at least one of \( z(2^n x + 2^n - 1) \) and \( \tilde{z}(2^n x + 2^n - 1) \) is non-zero. Since \( G \) is a finite graph, this is impossible. Therefore, \( z_x = \tilde{z}_x \). Since \( z_7 = \tilde{z}_7 \), by (13), \( z_3 = \tilde{z}_3 \) and so by (6), \( t_1 = \tilde{t}_1 \). Moreover, by (8), \( t_{11} = \tilde{t}_{11} \). Since \( z_{19} = \tilde{z}_{19} \), by (14), \( z_9 = \tilde{z}_9 \). Moreover, \( z_{35} = \tilde{z}_{35} \) and \( \delta_{35} = z_{17} - z_{35} = \tilde{z}_{17} - \tilde{z}_{35} \), so \( z_{17} = \tilde{z}_{17} \).

By deleting the similar corresponding components of \( G \) and \( H \), the remaining graphs are cospectral and so
\[
\sum_{i \in S} z_i \hat{Z}_i + t_2 \hat{T}_2 + t_3 \hat{T}_3 = \hat{p}_1 \hat{P}_1 + \sum_{i \in S} \tilde{z}_i \hat{Z}_i + \tilde{t}_2 \hat{T}_2 + \tilde{t}_3 \hat{T}_3,
\]
where \( S = \{2, 4, 8, 14\} \). Since by deleting the similar corresponding components of \( G \) and \( H \) the remaining graphs are cospectral, we can suppose that at most one of the \( t_i \) and \( \tilde{t}_i \) (\( z_i \) and \( \tilde{z}_i \)) is non-zero. We have the following cases:

(1) Let \( t_3 \geq 0 \) and \( \tilde{t}_3 = 0 \). Then by (7) and (10), \( z_{14} = \tilde{z}_4 = 0 \) and \( z_4 = t_3 = \tilde{z}_{14} \). So we have
\[
z_2 \hat{Z}_2 + z_4 \hat{Z}_4 + z_8 \hat{Z}_8 + t_2 \hat{T}_2 + t_3 \hat{T}_3 = \hat{p}_1 \hat{P}_1 + z_2 \hat{Z}_2 + z_8 \hat{Z}_8 + \tilde{t}_2 \hat{T}_2.
\]

If \( t_2 \geq 0 \) and \( \tilde{t}_2 = 0 \), then by (9), \( z_8 = 0 \) and \( \tilde{z}_8 = t_2 \). So by (3), \( z_2 = \tilde{z}_2 + t_2 + t_3 \). Hence \( \tilde{z}_2 = 0 \) and \( z_2 = t_2 + t_3 \). By (1) we have \( \hat{p}_1 = 2t_3 + t_2 \) and so:
\[
(t_3 + t_2) \hat{Z}_2 + t_3 \hat{Z}_4 + t_2 \hat{T}_2 + t_3 \hat{T}_3 = (2t_3 + t_2) \hat{P}_1 + t_2 \hat{Z}_8 + t_3 \hat{Z}_{14}.
\]

Which means that \( G \) has some components where the spectrum of their union is \( \hat{H}_1 \) for \( a = t_2 \) and \( b = t_3 \).
If $t_2 > 0$ and $t_2 = 0$, then by (9), $z_8 = t_2$ and $\bar{z}_8 = 0$. So by (3), $z_2 + \bar{t}_2 = \bar{z}_2 + t_3$. So we have the following cases:

(i) If $t_3 > \bar{t}_2$, then $\bar{z}_2 = 0$ and we have

$$(t_3 - \bar{t}_2)\hat{Z}_2 + t_3\hat{Z}_4 + \bar{t}_2\hat{Z}_8 + t_3\hat{T}_3 = (2t_3 - \bar{t}_2)\hat{P}_1 + t_3\hat{Z}_{14} + \bar{t}_2\hat{T}_2.$$  

Therefore, $G$ has some components where the spectrum of their union is $\hat{H}_2$ for $a = t_3$ and $b = \bar{t}_2$.

(ii) If $t_3 < \bar{t}_2$, then $z_2 = 0$ and we have

$$t_3\hat{Z}_4 + \bar{t}_2\hat{Z}_8 + t_3\hat{T}_3 = (2t_3 - \bar{t}_2)\hat{P}_1 + (\bar{t}_2 - t_3)\hat{Z}_2 + t_3\hat{Z}_{14} + \bar{t}_2\hat{T}_2.$$  

Hence $G$ has some components where the spectrum of their union is $\hat{H}_3$ for $a = \bar{t}_2$ and $b = t_3$.

(2) Let $\bar{t}_3 > 0$ and $t_3 = 0$. Then by (7) and (10), $z_{14} = \bar{z}_4 = \bar{t}_3$ and $z_4 = t_3 = \bar{z}_{14} = 0$. If $t_2 \geq 0$ and $\bar{t}_2 = 0$, then by (9), $z_8 = 0$ and $\bar{z}_8 = t_2$ and we have the following cases:

(i) If $\bar{t}_3 > t_2$, then by (3), $\bar{z}_2 = \bar{t}_3 - t_2$ and $z_2 = 0$. Hence by (1), $\bar{p}_1 = t_2 - 2\bar{t}_3 < 0$, this is impossible.

(ii) If $\bar{t}_3 \leq t_2$, then by (1) and (3), $z_2 = t_2 - \bar{t}_3$, $\bar{z}_2 = 0$ and $\bar{p}_1 = t_2 - 2\bar{t}_3$. Hence

$$(t_2 - \bar{t}_3)\hat{Z}_2 + \bar{t}_3\hat{Z}_{14} + t_2\hat{T}_2 = (t_2 - 2\bar{t}_3)\hat{P}_1 + \bar{t}_3\hat{Z}_4 + t_2\hat{Z}_8 + \bar{t}_3\hat{T}_3.$$  

So $G$ has some components where the spectrum of their union is $\hat{H}_4$ for $a = t_2$ and $b = \bar{t}_3$. If $t_2 \geq 0$ and $t_2 = 0$, then by (9), $z_8 = \bar{t}_2$ and $\bar{z}_8 = 0$. So by (5), $z_2 = 0$ and $\bar{z}_2 = \bar{t}_2 + t_3$. Hence by (1), $\bar{p}_1 = -(2\bar{t}_3 + \bar{t}_2) < 0$ this is impossible. □

**Corollary 2** [7]. Let $G = Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k}$. Then $G$ can be determined by its adjacency spectrum.

**Proof.** The assertion holds by substituting $t_1 = t_2 = t_3 = 0$, in Theorem 3. □

**Corollary 3.** Let $G = t_1T_1 + t_2T_2 + t_3T_3$. Then $G$ can be determined by its adjacency spectrum.

**Proof.** The assertion holds by substituting $z_i = 0$, in Theorem 3. □

5. The graph $G = Z_{i_1} + Z_{i_2} + \cdots + Z_{i_k}$ is determined by its Laplacian spectrum

Up to now only a few infinite families of $DS$ graphs are known, so finding any new infinite family of these graphs is of large interest. In [9], it was shown that the disjoint union of $k$ disjoint paths $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$ is determined by its spectrum with respect to the adjacency matrix as well as the Laplacian matrix, where $n_1, n_2, \ldots, n_k$ are integers at least 2. Moreover, in [7], Shen and others showed that the graph $Z_n$ is determined by its Laplacian spectrum. In this section we show that the graph $G = Z_{i_1} + Z_{i_2} + \cdots + Z_{i_k}$ is determined by its Laplacian spectrum.

**Lemma 4** [9]. (Interlacing) Suppose that $A$ is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ of a principal submatrix of $A$ of size $m$ satisfy $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$. 

Lemma 5 [9]. Let $T$ be a tree with $n$ vertices and let $L(T)$ be its line graph. Then for $i = 1, \ldots, n$, we have $\mu_i(T) = \lambda_i(L(T)) + 2$.

Corollary 4. If two trees $T$ and $\overline{T}$ are cospectral with respect to the Laplacian matrix, then $L(T)$ and $L(\overline{T})$ are cospectral with respect to the adjacency matrix.

Lemma 6 [1]. Let $G$ be a connected graph, and let $H$ be a proper subgraph of $G$. Then $\lambda_1(H) < \lambda_1(G)$.

Theorem 4 [6]. Let $G$ be a starlike tree. Then $G$ is determined by its Laplacian spectrum.

Corollary 5 [10]. Let $G$ be a $T$-shape tree. Then $G$ is determined by its Laplacian spectrum.

Theorem 5 [1]. Let $G$ and $H$ be connected graphs and $\{G, H\} \neq \{K_{1,3}, K_3\}$. Then $G$ and $H$ are isomorphic if and only if their line graphs $L(G)$ and $L(H)$ are isomorphic.

Lemma 7 [4]. Let $G$ be a connected graph that is not isomorphic to $W_n$ (see Fig. 1) and let $G_{uv}$ be the graph obtained from $G$ by subdividing the edge $uv$ of $G$. If $uv$ lies on an internal path of $G$, then $\lambda_1(G_{uv}) \leq \lambda_1(G)$.

Let $N_G(H)$ be the number of subgraphs of a graph $G$ which are isomorphic to $H$ and let $N_G(i)$ be the number of closed walks of length $i$ in $G$. Let $N'_H(i)$ be the number of closed walks of length $i$ of $H$ which contain all edges and let $S_i(G)$ be the set of connected graph $H$ with $N'_H(i) \neq 0$ where $G$ has at least one subgraph isomorphic to $H$. Then we have

$$N_G(i) = \sum_{H \in S_i(G)} N_G(H)N'_H(i).$$  \hfill (15)

The following Lemma provides some formulae for calculating the number of closed walks of small lengths.

Lemma 8. The following can be obtained from Eq. (15):

(i) $N_G(2) = 2m$, $N_G(3) = 6N_G(K_3)$,
(ii) $N_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4)$, $N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(G_1)$.

(see Fig. 2).

![Fig. 2.](image-url)
Lemma 9 [5]. Let $G$ be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then $\Delta(G) + 1 \leq \mu_{\text{max}} \leq \max \left\{ \frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v} : uv \in E(G) \right\}$, where $\Delta(G)$, $\mu_{\text{max}}$ and $m_v$ denote the maximum vertex degree of $G$, the largest Laplacian eigenvalue of $G$ and the average of degrees of vertices adjacent to the vertex $v$ in $G$, respectively.

Theorem 6. Let $G = Z_{i_1} + Z_{i_2} + \cdots + Z_{i_r}$ be a graph of index less than 2. Then $G$ can be determined by its spectrum with respect to the Laplacian matrix.

Proof. We give the proof by induction on the number of components of $G$. If $G$ is a connected graph, then $G$ is a $T$-shape tree and by Corollary 5, $G$ can be determined by its spectrum with respect to the Laplacian matrix. Now let $G$ has $r > 1$ components and let $H$ be cospectral to $G$ with respect to the Laplacian matrix. By Lemma 3, $G$ and $H$ have the same number of components, vertices and edges. Therefore, $H$ has $r$ components and all of them are trees. Now let $H = H_1 + H_2 + \cdots + H_r$. Without loss of generality we can assume that $i_r \geq i_{r-1} \geq \cdots \geq i_1$. So $\lambda_1(L(G)) = \lambda_1(L(Z_{i_1}))$. By Corollary 4, $L(G)$ and $L(H)$ are cospectral with respect to the adjacency matrix. Applying Lemma 9, we find that $4 \leq \mu_{\text{max}}(G) \leq 4.8$. So $H$ has no vertex of degree at least 4. If one of the components of $H$ such as $H_1$ has more than one vertices of degree 3, then its line graph $L(H_1)$ has at least two triangles such as $D_1$ and $D_2$. So one can successively subdivide certain edges of the internal path between $D_1$ and $D_2$ of $L(H_1)$ in an appropriate way, to obtain graph $\tilde{H}$, such that $L(Z_{i_r})$ can be embedded in $\tilde{H}$ as a proper subgraph. So by Lemma 6, $\lambda_1(L(Z_{i_r})) < \lambda_1(\tilde{H})$ and by Lemma 7, $\lambda_1(\tilde{H}) \leq \lambda_1(L(H_i))$. Hence $\lambda_1(L(G)) = \lambda_1(L(Z_{i_1})) < \lambda_1(L(H_i)) \leq \lambda_1(L(H))$, this is a contradiction to the fact that $L(H)$ and $L(G)$ are cospectral with respect to the adjacency matrix. So every component $L(H_i)$ of $L(H)$ has at most one triangle. Therefore, every component $L(H_1)$ of $L(H)$ has at least two triangles, which is impossible. So $H$ is of type $Z_{d}$.

Let $X_i$ be the number of vertices of degree $i$ of $H$. By Lemma 3, $G$ and $H$ have the same number of vertices and edges. Hence

$$X_1 + X_2 + X_3 = n,$$

and

$$X_1 + 2X_2 + 3X_3 = 2(n - r),$$

where $n$ is the number of vertices of $G$. Again by using Lemma 3, the sum of squares of degrees of vertices of $G$ and $H$ are equal. So

$$X_1 + 4X_2 + 9X_3 = 4(n - r).$$

Hence $X_1 = 3r$, $X_2 = n - 4r$ and $X_3 = r$. We know that every component $L(H_1)$ of $L(H)$ has at most one triangle. On the other hand $X_3 = r$ and $G$ and $H$ have the same number of components. Therefore, every component $H_i$ of $H$ has one vertex of degree 3. By Lemma 8,

$$N_{L(G)}(5) = 30N_{L(G)}(K_3) + 10N_{L(G)}(C_5) + 10N_{L(G)}(G_1) = 40r.$$ 

So $N_{L(H)}(5) = 40r$ and $N_{L(H)}(G_1) = r$. Therefore, $L(H)$ does not have $G_2$ as a subgraph and any component of $H$ is of type $Z_{d}$. Without loss of generality suppose that $H_r = Z_{d}$ is the component of $H$ where $\lambda_1(L(H)) = \lambda_1(L(H_r))$. Then $\lambda_1(L(G)) = \lambda_1(L(H)) = \lambda_1(L(Z_{i_r})) = \lambda_1(L(Z_d))$ and by Lemma 6, $L(Z_{i_r})$ and $L(Z_d)$ are isomorphic. Hence by Theorem 5, $Z_{i_r}$ and
$Z_d$ are isomorphic and $i_r = d$. So $H_r = Z_{i_r}$ and two graphs $G - Z_{i_r} = Z_1 + Z_2 + \cdots + Z_{r-1}$ and $H - H_r = H_1 + H_2 + \cdots + H_{r-1}$ are cospectral with respect to the Laplacian matrix. Using induction on the number of components of $G$, $G - Z_{i_r}$ is determined by its spectrum with respect to the Laplacian matrix, so the two graphs $G - Z_{i_r}$ and $H - Z_{i_r}$ are isomorphic. Therefore, $G$ and $H$ are isomorphic. □

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References