Graph $Z_n$ and some graphs related to $Z_n$
are determined by their spectrum

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Abstract

It is proved that graph $Z_n$ is determined by its adjacency spectrum as well as its Laplacian spectrum; $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ is determined by its adjacency spectrum, where $n_1, n_2, \ldots, n_k$ are integers at least 2; $W_n$ is not determined by its adjacency spectrum but is determined by its Laplacian spectrum; $kZ_n, T_n$ are determined by their Laplacian spectrum, respectively, where $k$ is a positive integer.

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1. Introduction

We consider undirected graphs having no loops or parallel edges. All notions on graphs that are not defined here can be found in [1].

Let $G$ be a graph with $n$ vertices, $V(G)$ and $E(G)$ be the sets of vertices and edges of $G$, respectively. We assume $V(G) \neq \emptyset$ (and so $n > 0$). Let matrix $A(G)$ be the adjacency matrix of $G$, $d_G(v)$ be the degree of vertex $v$ in $G$, and $D(G)$ be
the diagonal matrix with degrees of the corresponding vertices of $G$ on the main diagonal. Matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of $G$. Denote the characteristic polynomial of the adjacency matrix $A(G)$ (Laplacian matrix $L(G)$) by $P_{A(G)}(\lambda) \, (P_{L(G)}(\mu))$. The eigenvalues of $A(G)$ ($L(G)$) and the spectrum (which consists of eigenvalues) of $A(G)$ ($L(G)$) are also called the adjacency (Laplacian) eigenvalues of $G$ and the adjacency (Laplacian) spectrum of $G$. Since both matrices $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are all real numbers. So we can assume that $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)(= 0)$ are the adjacency eigenvalues and the Laplacian eigenvalues of $G$, respectively.

About the background of the question “which graphs are determined by their spectrum?”, we refer to [3]. It seems hard to prove a graph to be determined by its spectrum. Only few graphs have been proved to be determined by their spectrum.

The following known results can be found in [3,4]:

(i) Graphs with the number of vertices less than 5, the path with $n$ vertices $P_n$, the complete graph $K_n$, the regular complete bipartite graph $K_{m,m}$, the cycle $C_n$ and their complements, the disjoint union of $k$ disjoint paths $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$ are determined by their spectrum with respect to the adjacency matrix as well as the Laplacian matrix.

(ii) The disjoint union of $k$ complete graph, $K_{n_1} + K_{n_2} + \cdots + K_{n_k}$, is determined by their adjacency spectrum.

Remark. If we view an isolated vertex as $P_1$, the result ‘the disjoint union of $k$ disjoint paths is determined by its adjacency spectrum’ would be wrong. For example, $P_7 + P_1$ is cospectral with $Z_3 + P_3$ with respect to the adjacency matrix ($Z_3$ is a tree defined in the following). The result holds only for all integers $n_1, \ldots, n_k$ greater than 1. For convenience, we refer an isolated vertex as $K_1$ not $P_1$ in this paper.

The following question is proposed in [3]: which trees are determined by their spectrum? We still do not know the answer. In this paper, three special graphs are involved. The following three graphs were denoted by $Z_n$ ([2], p. 77), $T_n$ and $W_n$, respectively (Fig. 1).

![Fig. 1. Three special graphs.](image)
Clearly, graph $\mathbb{Z}_n$, $T_n$ are trees with $n + 2$ vertices and $n + 1$ edges, respectively. $W_n$ is a tree with $n + 4$ vertices and $n + 3$ edges.

This paper is constructed as following: In Section 2, we will prove that $\mathbb{Z}_n$ is determined by its adjacency spectrum and get a more general result. In Section 3, graphs $\mathbb{Z}_n$, $k\mathbb{Z}_n$, $T_n$ and $W_n$ will be proved to be determined by their Laplacian spectrum, respectively, where $k$ is a positive integer.

2. $\mathbb{Z}_n$ is determined by its adjacency spectrum

The following lemmas will be frequently used throughout this paper.

**Lemma 2.1** [3]. For $n \times n$ matrices $A$ and $B$, the following are equivalent:

(i) $A$ and $B$ are cospectral;
(ii) $A$ and $B$ have the same characteristic polynomial;
(iii) $\text{tr}(A^i) = \text{tr}(B^i)$ for $i = 1, 2, \ldots, n$.

If $A$ is the adjacency matrix of a graph, then $\text{tr}(A^i)$ gives the total number of closed walks of length $i$. So cospectral graphs have the same number of closed walks of a given length $i$. In particular, they have the same number of edges (take $i = 2$) and triangles (take $i = 3$).

**Lemma 2.2** [3]. In a graph without 4 cycles, the number of closed walks of length 4 equals twice the number of edges plus four times the number of induced paths of length 2.

**Lemma 2.3** [5]. Let $Y$ be a subgraph of $X$, then $\lambda_{\text{max}}(Y) \leq \lambda_{\text{max}}(X)$. Furthermore, when $Y$ is a proper subgraph, equality can hold only when $X$ is not connected.

Other useful tool is the following statement.

A tree in which exactly one vertex has degree greater than 2 is said to be starlike (see [11]). For starlike trees, we have:

**Lemma 2.4** [8]. No two non-isomorphic starlike trees are cospectral with respect to their adjacency matrices.

Since the adjacency spectrum of $W_n$ is the union of the spectra of the circuit $C_4$ and the path $P_n$ ([2], p. 77), then the largest eigenvalue of $W_n$ is 2 and $W_n$ cannot be determined by its adjacency spectrum.

Now we prove our first result:

**Theorem 2.5.** Graph $\mathbb{Z}_n$ is determined by its adjacency spectrum.
Proof. The adjacency eigenvalues of $Z_n$ are $0, 2 \cos \left( \frac{(2i+1)\pi}{2(n+1)} \right), i = 0, 1, \ldots, n$ ([2], p. 77). It gives $\lambda_1(Z_n) < 2$. For $n = 1$, graph $Z_n$ is $P_3$ (the path with three vertices), it is determined by its spectrum. The result holds. For $n > 1$, suppose a graph $T$ is cospectral with $Z_n$ with respect to the adjacency spectrum. By Lemma 2.1, $T$ is a graph with $n + 2$ vertices and $n + 1$ edges. Since the circuit has an eigenvalue 2, it cannot be an induced subgraph of $T$ because of Lemma 2.3. Therefore $T$ is a tree. Similarly, the star $K_{1,4}$ has an eigenvalue 2, so $K_{1,4}$ is not a subgraph of $T$. Also graph $W_n$ has an eigenvalue 2, so $T$ is a tree without any vertex of degree at least 4 and at most one vertex of degree 3. Since the path is determined by its spectrum, $T$ is not a path. Therefore $T$ is a starlike tree with the largest vertex degree 3. By Lemma 2.4, $T$ is isomorphic to $Z_n$. □

We denote the disjoint union of $k$ graphs $Z_{n_1}, Z_{n_2}, \ldots, Z_{n_k}$ by $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$; denote the disjoint union of $k$ disjoint graphs $Z_n$ by $kZ_n$; denote the following three graphs by $G_1, G_2, G_3$, which share a common property with their largest adjacency eigenvalues less than 2.

By using Maple, we find the characteristic polynomial $P_{A(G_2)}(\lambda)$ is a factor of the characteristic polynomial $P_{A(Z_8)}(\lambda)$. Hence the adjacency eigenvalues of $G_2$ is a part of that of $Z_8$. Thus we get the adjacency spectrum of $G_2$ is

$$2 \cos \frac{\pi}{18}, 2 \cos \frac{5\pi}{18}, 2 \cos \frac{7\pi}{18}, 0, 2 \cos \frac{11\pi}{18}, 2 \cos \frac{13\pi}{18}, 2 \cos \frac{17\pi}{18}.$$ 

Similarly, $G_3$ has eigenvalues

$$2 \cos \frac{\pi}{30}, 2 \cos \frac{7\pi}{30}, 2 \cos \frac{11\pi}{30}, 2 \cos \frac{13\pi}{30}, 2 \cos \frac{17\pi}{30}, 2 \cos \frac{19\pi}{30}, 2 \cos \frac{23\pi}{30}, 2 \cos \frac{29\pi}{30}.$$ 

Similar to the proof of Theorem 2.5, we get:

**Corollary 2.6.** Graphs $G_1, G_2, G_3$ are determined by their adjacency spectrum, respectively.

For $n = 3$, $T_n$ is $Z_3$; for $n = 4$, $T_n$ is $G_1$; for $n = 5$, $T_n$ is $G_2$; for $n = 6$, $T_n$ is $G_3$. Theorem 2.5 and Corollary 2.6 imply that $T_n$ is determined by its adjacency spectrum for $n < 7$. For $n = 7$, $T_n$ is $G_6$ (see Fig. 3), from the spectra which displayed in [2] from p. 272 to p. 306, we know that it is also determined by its adjacency spectrum. But for $n > 7$, we do not know the answer with our skills.

The following can be deduced directly from their spectrum:

**Corollary 2.7**

(i) $Z_8 + K_1$ is cospectral with $G_2 + Z_2$ with respect to the adjacency matrix;
(ii) $Z_{14} + 2K_1$ is cospectral with $G_3 + Z_4 + Z_2$ with respect to the adjacency matrix;
(iii) More generally, \( kZ_{14} + lZ_8 + (2k + l)K_1 \) is cospectral with \( kG_3 + lG_2 + kZ_4 + (k + l)Z_2 \) with respect to the adjacency matrix, where \( k, l \) are positive integers.

The following gives a more general result.

**Theorem 2.8.** Graph \( Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k} \) is determined by its adjacency spectrum, where all \( n_1, n_2, \ldots, n_k \) are greater than 1.

**Proof.** Suppose a graph \( G \) is cospectral with \( Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k} \) with respect to the adjacency matrix. Similar to the proof of Theorem 2.5, we find \( G \) and \( Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k} \) have the same number of vertices, edges, and closed walks of length 4. At the same time, \( G \) is a forest with \( k \) components, and each component has no vertex of degree at least 4 and at most one vertex of degree 3.

First, we declare that \( G \) has no path component and the possible components of \( G \) are \( G_2, G_3, K_1 \) (see Fig. 2). Assume that there exist \( t \) (\( t \geq 1 \)) path components in \( G \), then the number of induced paths of length 2 in \( G \) is less than that in \( Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k} \) by \( t \). By Lemma 2.2, the number of closed walks of length 4 of \( G \) is clearly less than that of \( Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k} \). So no path is a component of \( G \). Therefore each component of \( G \) contains exactly one vertex of degree 3. Furthermore, \( G_1 \) (see Fig. 2) is not a component of \( G \) because the spectrum of \( G_1 \) contains an eigenvalue 1 ([2], p. 276), which is not an eigenvalue of \( Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k} \). Since the following three graphs (see Fig. 3) all have the largest adjacency eigenvalue 2 ([2], p. 276), then \( G_4, G_5, G_6 \) cannot be induced subgraphs of \( G \) by Lemma 2.3. So the possible components of \( G \) are \( G_2, G_3 \) (see Fig. 2), \( Z_m \) (\( m > 1 \)) or \( K_1 \). From their adjacency spectrum, we know that it is impossible that all components of \( G \) are \( G_2, G_3, \) or \( K_1 \).

Second, we prove that \( G_2, G_3, K_1 \) are also not components of \( G \). Suppose \( G \) is \( aG_2 + bG_3 + Z_{m_1} + \cdots + Z_{m_l} + cK_1 \), where \( a + b + c + l = k \), \( a, b, l \) are positive integers, \( c \) is a nonnegative integer. We declare that each component of \( Z_{m_1} + \cdots + Z_{m_l} \) is just one component of \( Z_{n_1} + \cdots + Z_{n_k} \). Assume that there are \( t \) (\( 0 \leq t < l \)) pair-wise isomorphic components between \( Z_{m_1} + \cdots + Z_{m_l} \) and \( Z_{n_1} + \cdots + Z_{n_k} \). Deleting these \( t \) pair-wise isomorphic components simultaneously from \( G \) and \( Z_{n_1} + \cdots + Z_{n_k} \), and denote the remaining by \( S \) and \( S' \), respectively. Obviously, \( S \) and \( S' \) are cospectral. So there are \( x (= k - a - b - c - t) \) components (say \( Z_{m_{i_1}}, \)

![Fig. 2. Three graphs with their largest adjacency eigenvalue less than 2.](image-url)
is not a component of \(Z_{m_1} + \cdots + Z_{m_k}\) which are not isomorphic to any components of \(S'\), and assume \(Z_n\) is one of these \(x\) components with the largest size. Then the spectrum of \(Z_n\) is part (not all) of the spectrum of some component(s) of \(S'\). Since \(2 \cos \frac{\pi}{2(n+1)}\) is an eigenvalue of \(Z_n\), there are at least one eigenvalue \(\lambda = 2 \cos \frac{\pi}{2(n+1)}(m\) is some positive integer in the spectrum of \(S'\) and \(S\), respectively. Since \(Z_n\) has the largest size among \(Z_{m_1}, \ldots, Z_{m_k}\), it follows \(\lambda\) is an eigenvalue of \(G_2\), or \(G_3\).

If \(\lambda\) is an eigenvalue of \(G_3\), then the equality \(\lambda = 2 \cos \frac{\pi}{30}\) holds. It implies \(n = 2, m = 2, n = 4, m = 1\). If \(n = 2, m = 1\), then each component of \(Z_{m_1} + \cdots + Z_{m_k}\) is \(Z_2\).

So both \(S'\) and \(S\) have the largest eigenvalue \(2 \cos \frac{\pi}{30}\) with multiplicity \(b\) and another eigenvalue \(2 \cos \frac{\pi}{18}\) (which is the largest eigenvalue of \(G_2\)) with multiplicity \(a\). It follows that there are exactly \(a\) components \(Z_8\) and \(b\) components \(Z_{14}\) in \(S'\). So \(2 \cos \frac{\pi}{18}\) is an eigenvalue of \(S'\), but it is not an eigenvalue of \(S\). A contradiction! If \(n = 4\), the possible components of \(Z_{m_1} + \cdots + Z_{m_k}\) are \(Z_2, Z_3,\) or \(Z_4\). Similarly, to the proof above, it is impossible for all components of \(Z_{m_1} + \cdots + Z_{m_k}\) to be \(Z_2\) (or \(Z_4\)). Assume that \(Z_3\) is a component of \(Z_{m_1} + \cdots + Z_{m_k}\), since \(2 \cos \frac{\pi}{8}\) is an eigenvalue of \(Z_3\), then \(2 \cos \frac{\pi}{8(2m'+1)}\) \((m'\) is some positive integer\) is an eigenvalue of \(S'\) for the same reason above. But it is never an eigenvalue of \(Z_8\) nor \(Z_{14}\). So \(Z_3\) is not a component of \(Z_{m_1} + \cdots + Z_{m_k}\). Hence both \(Z_2\) and \(Z_4\) are components of \(Z_{m_1} + \cdots + Z_{m_k}\). It follows that there are exactly \(a\) components \(Z_8\) and \(b\) components \(Z_{14}\) in \(S'\). Since the \(x\) components of \(Z_{m_1} + \cdots + Z_{m_k}\) are not isomorphic to any component of \(S'\), it forces the remaining \(x + c\) components in \(S'\) are \(K_1\). It contradicts our hypothesis! Similarly, \(\lambda\) is also not an eigenvalue of \(G_2\). Thus each component of \(Z_{m_1} + \cdots + Z_{m_k}\) is just one component of \(Z_{n_1} + \cdots + Z_{n_l}\). It follows that \(aG_2 + bG_3 + cK_1\) cospectral with the remaining of \(Z_{n_1} + \cdots + Z_{n_l}\) by deleting \(Z_{n_1} + \cdots + Z_{n_l}\). But from their spectrum, we know it is impossible. Similarly, the graph \(G\) is neither the form \(aG_2 + Z_{m_1} + \cdots + Z_{m_k} + cK_1\) nor the form \(bG_3 + Z_{m_1} + \cdots + Z_{m_k} + cK_1\), where \(a, b, l\) are positive integers, \(c\) is a nonnegative integer.

Finally, we easily find that the graph \(G\) is also not the form \(Z_{m_1} + Z_{m_2} + \cdots + Z_{m_k} + cK_1\) from the spectrum of \(Z_n\), where \(l, c\) are positive integers. So \(G\) is isomorphic to some component(s) of \(S'\), the largest integer (say \(n\)) in \(\{m_1, \ldots, m_k\}\) follows from the largest eigenvalue. Then the other \(m_i\) follows recursively by deleting \(Z_n\) from the graph and the eigenvalues of \(Z_n\) from the spectrum. \(\square\)
From their spectrum, we easily find the following result:

**Corollary 2.9**

(i) \(P_{2n+1}\) (the path with \(2n+1\) vertices) + \(K_1\) is cospectral with \(P_n + Z_n\); in particular, \(P_{4n+3} + 2K_1\) is cospectral with \(Z_{2n+1} + Z_n + P_n\);
(ii) \(P_{2r-1} + (r - 2)K_1\) is cospectral with \(Z_{2r-1-1} + \cdots + Z_7 + Z_5 + P_3, r \geq 3\).

3. **Graphs** \(Z_n, W_n, T_n\) **are determined by their Laplacian spectrum**

We write the characteristic polynomial \(P_{L(G)}(\mu) = |\mu I - L(G)| = q_0\mu^n + q_1\mu^{n-1} + \cdots + q_{n-1}\mu + q_n\) and summarize some results in [3,10] in the following lemma.

**Lemma 3.1**

(i) Let \(G\) be a graph with \(n\) vertices and \(m\) edges and let \(d = (d_1, \ldots, d_n)\) be its non-increasing degree sequence. Then some of the coefficients in \(P_{L(G)}(\mu)\) are:

\[
q_0 = 1; \quad q_1 = -2m; \quad q_2 = 2m^2 - m - \frac{1}{2} \sum_{i=1}^{n} d_i^2; \\
q_{n-1} = (-1)^{n-1}nS(G); \quad q_n = 0;
\]

where \(m\) is the number of edges of \(G\). \(S(G)\) is the number of spanning trees in \(G\).

(ii) For the Laplacian matrix of a graph, the following follows from its spectrum:

(a) the number of components.
(b) the number of spanning trees.

The following lemma can be found in [7,9]

**Lemma 3.2.** Let \(G\) be a graph with \(V(G) \neq \emptyset\) and \(E(G) \neq \emptyset\). Then

\[
\Delta(G) + 1 \leq \mu_{\text{max}} \leq \max \left\{ \frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v}, uv \in E(G) \right\}
\]

where \(\Delta(G)\) denotes the maximum vertex degree of \(G\), \(\mu_{\text{max}}\) denotes the largest Laplacian eigenvalue of \(G\), \(m_v\) denotes the average of the degrees of the vertices adjacent to vertex \(v\) in \(G\).
Lemma 3.3 [6]. Let T be a tree with n vertices and X its line graph. Then, for \( i = 1, 2, \ldots, n - 1 \), \( \mu_i(T) = \lambda_i(X) + 2 \).

Lemma 3.4 [5, Theorem 13.6.2]. Let \( G' \) be a graph obtained by deleting an edge from the graph G. Then, for \( i = 1, 2, \ldots, n - 1 \), \( \mu_i(G) \geq \mu_i(G') \geq \mu_{i+1}(G) \).

Lemma 3.5. All starlike trees with the largest vertex degree 3 have the second largest Laplacian eigenvalue less than 4.

**Proof.** Note each starlike tree with the largest vertex degree 3 is the disjoint union of two paths (or a path plus \( K_1 \)) by deleting the edge adjacent to the vertex of degree 3 from its edge set. The Laplacian spectrum of \( P_n \) is \( 2 + 2 \cos \frac{2\pi}{n+1}, i = 1, \ldots, n \). [3]. Thus \( \mu_1(P_n) < 4 \), by Lemma 3.4, the result follows.

Theorem 3.6. Graph \( Z_n \) is determined by its Laplacian spectrum.

**Proof.** Suppose that a graph \( G \) and \( Z_n \) are cospectral with respect to the Laplacian matrix, then \( G \) has \( n + 2 \) vertices. By Lemma 2.1, \( G \) and \( Z_n \) share the same characteristic polynomial of \( L(G) \). So \( G \) and \( Z_n \) have the same number of edges and spanning trees by Lemma 3.1(i). Since \( Z_n \) contains one spanning tree, then \( G \) is a tree. Applying Lemma 3.2, we find that \( 4 \leq \mu_1(Z_n) \leq 4.414 \). So \( G \) is a tree with no vertex of degree at least 4 by Lemma 3.2. At the same time, Lemma 3.1 implies

\[
\sum_{i=1}^{n+2} d_i^2 = \frac{n+2}{2} \sum_{i=1}^{n+2} d_i^2
\]

where \( d_i \) are degrees of vertex \( v_i \) in \( G \) and \( Z_n \), respectively. It follows that \( G \) is a starlike tree with the largest vertex degree 3. Furthermore, we declare that graph \( G_4 \) (see Fig. 3) is not an induced subgraph of \( G \). Let \( L' \) be the Laplacian matrix of \( G_4 \). By using Maple, we get the largest Laplacian eigenvalue of \( G_4 \) is about 4.414. If graph \( G_4 \) is a induced subgraph of \( G \), then \( L' + D \) is a principle submatrix of \( L(G) \) for some diagonal matrix \( D \) with non-negative entries. But then \( L' + D \) has the largest eigenvalue at least 4.414, a contradiction. Suppose \( G \) is non-isomorphic to \( Z_n \), then \( G \) must be isomorphic to one of the following \( \lfloor \frac{n}{2} \rfloor - 1 \) graphs (Fig. 4(a)).

By Lemma 3.3, the line graph of \( G \) is cospectral with the line graph of \( Z_n \) with respect to the adjacency matrix. Therefore the line graph of \( G \) and the line graph of \( Z_n \) should have the same number of closed walks of length 4 by Lemma 2.1. But we can easily find that the numbers of closed walks of length 4 in the \( \lfloor \frac{n}{2} \rfloor - 1 \) line graphs (see Fig. 4(b)) are all greater than that of the line graph of \( Z_n \) (all those line graphs contain no 4 cycles, the number of induced paths of length two in the former are greater than that of the latter by 1). Hence \( G \) is isomorphic to \( Z_n \).
Similarly, we derive:

**Theorem 3.7.** Graph $W_n$ is determined by its Laplacian spectrum.

**Proof.** For $n = 1$, $W_n$ is $K_{1,4}$. Suppose a graph $X$ and $K_{1,4}$ are cospectral with respect to the Laplacian matrix, then $X$ is a tree with five vertices by Lemma 3.1. But all trees with five vertices are $P_5$, $Z_3$ and $K_{1,4}$. Since $P_5$, $Z_3$ are determined by their Laplacian spectrum, respectively, so $X$ is $K_{1,4}$. Therefore $K_{1,4}$ is determined by its Laplacian spectrum. Let $n \geq 2$. Suppose that $G$ and $W_n$ are cospectral with respect to the Laplacian matrix. Similar to the proof of Theorem 3.6, $G$ is a tree without any vertex of degree at least 4 and exactly two vertices of degree 3. So the line graph of $G$ and the line graph of $W_n$ are cospectral with respect to the adjacency matrix by Lemma 3.3. Therefore they have the same number of closed walks of length 4 by Lemma 2.1. For $n = 2$, obviously, $G$ is isomorphic to $W_n$. For $n = 3, n = 4$, we can easily get $G$ isomorphic to $W_n$ by counting the number of closed walks of length 4 in their line graphs of $G$ and $W_n$, respectively. For $n \geq 5$. Assume that $G$ is non-isomorphic to $W_n$. Similarly to the proof of Theorem 3.6, the inequalities $4 \leq \mu_1(W_n) = \mu_1(G) \leq 4.4$ hold and $G_4$ is not an induced subgraph of $G$. Then the line graph of $G$ is one of the following graphs (Fig. 5).

Clearly, all the number of closed walk of length 4 in these graphs are greater than that of the line graph of $W_n$ (the number of induced paths of length two in the former are all greater than that of the latter). Thus $G$ is isomorphic to $W_n$. $\square$

Similarly, we obtain:

**Corollary 3.8.** Graph $T_n$ is determined by its Laplacian spectrum.
**Proof.** Similar to the proof of Theorem 3.6 and only the number of closed walks of length 6 in line graph is involved in additional. □

**Theorem 3.9.** $kZ_n$ is determined by its Laplacian spectrum.

**Proof.** Suppose a graph $G$ is cospectral with $kZ_n$ with respect to the Laplacian matrix. Lemma 3.1 implies that graph $G$ has $k(n + 2)$ vertices, $k(n + 1)$ edges and $k$ components. So $G$ is a forest. From the proof of Theorem 3.6, we have that $4 \leq \mu_1(kZ_n) = \mu_2(kZ_n) = \cdots = \mu_k(kZ_n) \leq 4.4$. Furthermore, by Lemma 3.5 $\mu_2(Z_n) < 4$, so $\mu_{k+1}(kZ_n) = \cdots = \mu_{2k}(kZ_n) < 4$. Similar to the proof of Theorem 3.6, $G$ has no vertex with degree more than 3. We declare that there are exactly $k$ vertices of degree 3 in $G$. Suppose that there exist $x$ vertices of degree one, $y$ vertices of degree two, $z$ vertices of degree three, by Lemma 3.1 and $\sum_{v \in V(G)} d(v) = 2\varepsilon$, where $\varepsilon$ is the number of edges in $G$, we then have the following equations:

$$x + y + z = k(n + 2),$$
$$x + 2y + 3z = 2k(n + 1),$$
$$x + 4y + 9z = 3k + 4k(n - 2) + 9k.$$

Solving these equations simultaneously, we find $z = k$. Assume that there exists one path component in $G$, then there must exist one component with two vertices of degree 3 in $G$. Since the largest Laplacian eigenvalue of any path is less than 4, it forces the largest Laplacian eigenvalue and the second largest Laplacian eigenvalue are equivalent and all greater than 4 in the spectrum of one of the components except the path component in $G$. However, it is impossible by Lemmas 2.3, 3.3 and 3.4. Therefore each component of $G$ contains exactly one vertex of degree 3. Furthermore, each component has the same number of vertices. Assume that there exists a component $C$ which has $n + 2 + k$ ($k \geq 1$) vertices, then $\mu_1(C) \geq \mu_1(Z_{n+2+k}) > Z_n$ by Lemmas 2.3, 3.3 and 3.4. Hence there exists an eigenvalue greater than $\mu_1(kZ_n)$ in the Laplacian spectrum of $G$, a contradiction. From the proof of Theorem 3.6, each component of $G$ is $Z_n$. The result follows. □

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement, so the complements of graphs $Z_n$, $W_n$ and $T_n$ are determined by their Laplacian spectrum, respectively.

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References