On the Laplacian spectral radii of bicyclic graphs

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Received 30 June 2006; received in revised form 9 November 2007; accepted 12 November 2007
Available online 21 December 2007

\textbf{Abstract}

A graph $G$ of order $n$ is called a bicyclic graph if $G$ is connected and the number of edges of $G$ is $n + 1$. Let $B(n)$ be the set of all bicyclic graphs on $n$ vertices. In this paper, we obtain the first four largest Laplacian spectral radii among all the graphs in the class $B(n)$ ($n \geq 7$) together with the corresponding graphs.

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\textit{Keywords:} Bicyclic graph; Laplacian spectral radius; Characteristic polynomial

\section{1. Introduction}

We shall use the standard terminology of graph theory, as it is introduced in most textbooks on the theory of graphs. Our graphs $G = (V, E)$ are undirected finite graphs without loops and multiple edges. Having chosen a fixed ordering $v_1, v_2, \ldots, v_n$ of the set $V$, let $D = D(G)$ be the diagonal matrix of vertex degrees, and $A(G)$ be the adjacent matrix of the graph $G$. The Laplacian matrix $L(G)$ is defined to be $L(G) = D(G) - A(G)$. It is easy to see that $L(G)$ is a singular, semi-positive, symmetric matrix and its rows sum to 0. Denote its eigenvalues by

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0,$$

which are always enumerated in non-increasing order and repeated according to their multiplicity. We call the largest eigenvalue $\mu_1(G)$ of $L(G)$ the Laplacian spectral radius of $G$, denoted by $\mu(G)$. The Laplacian characteristic polynomial of $G$ is just $\det(x I - L(G))$, and denoted by $\Phi(G, x)$ or $\Phi(G)$. Up to now, many results on the Laplacian spectral radii of graphs have been obtained (see [1–10]).

Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one.

In this paper, we study the Laplacian spectral radii of bicyclic graphs. We will determine the first four largest Laplacian spectral radii in the class $B(n)$ ($n \geq 7$) together with the corresponding graphs.
2. Three classes of bicyclic graphs and some basic lemmas

**Definition 2.1.** A graph $G$ of order $n$ is called a bicyclic graph if $G$ is connected and the number of edges of $G$ is $n + 1$.

It is easy to see from the definition that $G$ is a bicyclic graph if and only if $G$ can be obtained from a tree $T$ (with the same order) by adding two new edges to $T$.

A pendant vertex of a graph is a vertex of degree 1.

Let $G$ be a bicyclic graph. The base of $G$, denoted by $\hat{G}$, is the (unique) minimal bicyclic subgraph of $G$. It is easy to see that $\hat{G}$ is the unique bicyclic subgraph of $G$ containing no pendant vertices, while $G$ can be obtained from $\hat{G}$ by attaching trees to some vertices of $G$.

It is well known that there are the following three types of bicyclic graphs containing no pendant vertices:

Let $B(p, q)$ be the bicyclic graph obtained from two vertex-disjoint cycles $C_p$ and $C_q$ by identifying vertices $u$ of $C_p$ and $v$ of $C_q$ (see Fig. 2.1).

Let $B(p, l, q)$ be the bicyclic graph obtained from two vertex-disjoint cycles $C_p$ and $C_q$ by joining vertices $u$ of $C_p$ and $v$ of $C_q$ by a new path $uu_1u_2\cdots u_{l−1}v$ with length $l$ ($l ≥ 1$) (see Fig. 2.1).

Let $B(P_k, P_l, P_m)$ ($1 ≤ m ≤ \min\{k, l\}$) be the bicyclic graph obtained from three pairwise internal disjoint paths from a vertex $x$ to a vertex $y$. These three paths are $xu_1u_2\cdots u_{k−1}y$ with length $k$, $xu_1u_2\cdots u_{l−1}y$ with length $l$ and $xw_1w_2\cdots w_{m−1}y$ with length $m$ (see Fig. 2.2).

Now we can define the following three classes of bicyclic graphs of order $n$:

$$B_1(n) = \{G ∈ \mathcal{B}(n) | \hat{G} = B(p, q) \text{ for some } p ≥ 3 \text{ and } q ≥ 3\},$$

$$B_2(n) = \{G ∈ \mathcal{B}(n) | \hat{G} = B(p, l, q) \text{ for some } p ≥ 3, q ≥ 3 \text{ and } l ≥ 1\},$$

$$B_3(n) = \{G ∈ \mathcal{B}(n) | \hat{G} = B(P_k, P_l, P_m) \text{ for some } 1 ≤ m ≤ \min\{k, l\}\}.$$

It is easy to see that

$$\mathcal{B}(n) = B_1(n) \cup B_2(n) \cup B_3(n).$$

Now we quote some basic lemmas which will be used in the proofs of our main results.

**Lemma 2.1** ([8]). Let $G$ be a connected graph with at least one edge and $\Delta(G)$ be the maximum degree of $G$; then $\mu(G) ≥ \Delta(G) + 1$, with equality if and only if $\Delta(G) = n − 1$.

**Lemma 2.2** ([3]). Let $G$ be a connected graph on $n ≥ 2$ vertices and $v$ be a vertex of $G$. Let $G_{k,l}$ be the graph obtained from $G$ by attaching two new paths $P : vv_1v_2\cdots v_k$ and $Q : uu_1u_2\cdots u_l$ of length $k$ and $l$ at $v$, respectively. If $k ≥ l ≥ 1$, then

$$\mu(G_{k,l}) ≥ \mu(G_{k+1,l-1}).$$
Lemma 2.3 ([4]). Suppose u, v are two vertices of a connected graph H. Let G be the graph obtained from H by attaching t new paths \{vv_i|v_i|2, \ldots, v_i|t\} (i = 1, \ldots, t) at v and suppose \(\Delta(G) \geq 3\). Let X be a unit eigenvector of G corresponding to \(\mu(G)\). Let

\[ R_G(u, \{v_1, \ldots, v_i\}, v) = G - vv_1 - \cdots - vv_i + uv_1 + \cdots + uv_i. \]

If \(|X_u| \geq |X_v|\), then \(\mu(R_G(u, \{v_1, \ldots, v_i\}, v)) \geq \mu(G)\). Furthermore, if \(|X_u| > |X_v|\), then \(\mu(R_G(u, \{v_1, \ldots, v_i\}, v)) > \mu(G)\).

Definition 2.2. Let \(G = G_1u : vG_2\) be the graph obtained from two disjoint graphs \(G_1\) and \(G_2\) by joining a vertex u of the graph \(G_1\) to a vertex v of the graph \(G_2\) by an edge. We call G a connected sum of \(G_1\) at u and \(G_2\) at v.

Definition 2.3. For any graph G and v \(\in V(G)\). Let \(L_v(G)\) denote the principal submatrix of \(L(G)\) obtained by deleting the row and column corresponding to the vertex v.

Lemma 2.4 ([5]). If \(G = G_1u : vG_2\) is a connected sum of \(G_1\) at u and \(G_2\) at v, then

\[ \Phi(G) = \Phi(G_1) \Phi(G_2) - \Phi(G_1) \Phi(L_v(G_2)) - \Phi(G_2) \Phi(L_u(G_1)). \]

Corollary 2.1. Let G be a connected graph with n vertices which consists of a subgraph H (with at least two vertices) and \(n - |H|\) distinct pendant edges (not in H) attached to a vertex v in H. Then

\[ \Phi(G) = (x - 1)^{n - |H|} \Phi(H) - (n - |H|) x (x - 1) \Phi(L_v(H)). \] (1)

Proof. By induction on \(n - |H|\). When \(n - |H| = 1\), i.e., there is one pendant edge (not in H) attached to v, denoted as vv_1. We regard G as a connected sum of an isolated vertex v_1 and H at v. By Lemma 2.4, we have

\[ \Phi(G) = x \Phi(H) - \Phi(H) - x \Phi(L_v(H)) = (x - 1) \Phi(H) - x \Phi(L_v(H)). \]

So the equality (1) holds.

For \(n - |H| = m \geq 2\), let the pendant edges (not in H) attached to v be vv_1, \ldots, vv_m. We regard G as a connected sum of an isolated vertex v_m and H’ at v, where H’ is the graph obtained from H by attaching m - 1 pendant edges vv_1, \ldots, vv_{m-1}. Then by Lemma 2.4,

\[ \Phi(G) = (x - 1) \Phi(H’) - x \Phi(L_v(H’)). \] (2)

By the inductive hypothesis, we have

\[ \Phi(H’) = (x - 1)^{m - 1} \Phi(H) - (m - 1) x (x - 1)^{m - 2} \Phi(L_v(H)). \] (3)

On the other hand,

\[ \Phi(L_v(H’)) = (x - 1)^{m - 1} \Phi(L_v(H)). \] (4)

Combining the equalities (2)–(4), we obtain the equality (1).

Lemma 2.5 ([7]). Let G be a connected graph with degree sequence \(d_1 \geq d_2 \geq \cdots \geq d_n\), and let \(\mu(G)\) be the Laplacian spectral radius of G. Then

\[ \mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}. \]

If the equation \(P(x) = 0\) has only real roots, we use \(\lambda(P(x))\) to denote its largest root.

Lemma 2.6. Let \(H\) be a connected graph with at least one edge, \(v \in V(H)\). Let \(H_k(v)\) be the graph obtained from \(H\) by attaching \(k + 1\) distinct pendant edges (not in \(H\)) to v, \(H_k’(v)\) be the graph obtained from \(H\) by attaching \(k - 1\) distinct pendant edges (not in \(H\)) and one new path of length 2 (not in \(H\)) to v, \(H_k’’(v) = Hv : uK_{1,k}\), where \(K_{1,k}\) is the star of order \(k + 1\) with center u (see Fig. 2.3). Then \(\mu(H_k(v)) \geq \mu(H_k’(v)) \geq \mu(H_k’’(v))\).
Lemma 2.2

Lemma 2.4

(6)

(9)

Corollary 2.1

and

(5)

Let $G$ holds.

$T$ is not a star with center $G$ has at least one edge,

It is easy to obtain $\mu(G)\geq \mu(H_k(v))$ by using Lemma 2.2. So in the following we only need to prove $\mu(H_k(v))\geq \mu(H''_k(v))$. By Lemma 2.4 and Corollary 2.1, we have

$$\Phi(H''_k(v)) = (x - 1)^k - [(x - 1)(x^2 - 3x + 1) - \frac{x(x - 1)^k - 1}{k} f(x)],$$

$$\Phi(H''_k(v)) = (x - 1)^k - [(x - 1)(x^2 - (k + 2)x + 1) - \frac{x(x - 1)^k - 1}{k} f(x)].$$

(5)

(6)

Let $a(x) = x^2 - (k + 2)x + 1, b(x) = x^2 - 3x + 1, \mu'' = \mu(H''_k(v))$.

From Lemma 2.1, and the fact that $H$ has at least one edge, we have $\mu'' = \Delta(H''_k(v)) + 1 \geq k + 2$. Then from the equality (6), we have

$$\Phi(L_v(H)) = \frac{a(x)}{x(x - k - 1)} \Phi(H). \quad (x = \mu'')$$

(7)

Combining (5) and (7), we have

$$f(x) = \left[ (x - 1)b(x) - \frac{(kb(x) + 1)a(x)}{x - k - 1} \right] \Phi(H). \quad (x = \mu'')$$

(8)

Now we show that when $x = \mu'' > \lambda(b(x))$,

$$x - 1)b(x) - \frac{[kb(x) + 1]a(x)}{x - k - 1} \leq 0,$$

(9)

or equivalently,

$$[ka(x) - a(x) - k]b(x) + a(x) \geq 0. \quad (10)$$

Write

$$g(x) = [ka(x) - a(x) - k]b(x) + a(x) = (k - 1)x^3 - (k + 5)x^2 + (3k + 7)x - k - 3].$$

By Corollary 2.1, we have

$$\Phi(T(1,k)) = x(x - 1)^k - [(x - 1)^3 - (k + 5)x^2 + (3k + 7)x - k - 3] = \frac{1}{k - 1} \left( x - \frac{1}{k - 1} g(x) \right),$$

where $T(1,k)$ is the tree obtained by attaching a pendant edge to a non-center vertex of a star of order $k + 2$. Then we know that $\lambda(g(x)) = \mu(T(1,k))$. Now $\mu'' = \mu(T(1,k))$ (since $H$ has at least one edge, $T(1,k)$ is a subgraph of $H''_k(v)$), so $g(\mu'' \geq 0$, and thus the inequality (9) holds.

Since $H$ is a proper induced subgraph of $H''_k(v), \mu(H) \leq \mu'', \Phi(H,\mu'') \geq 0$. So by (8) we have $f(\mu'') \leq 0$. Therefore we have $\Phi(H''_k(v), \mu'') \leq 0$ by (5). It follows that $\mu(H''_k(v)) \geq \mu' = \mu(H''_k(v))$. \qed

Lemma 2.7. Let $G_T(v)$ be a connected graph which consists of a connected subgraph $G$ and a tree $T$ which satisfies the following conditions:

(1) $G$ has at least one edge,

(2) $|T| = k + 2$,

(3) $T$ and $G$ have a unique common vertex $v$,

(4) $T$ is not a star with center $v$.

Form $G'_k(v)$ (as in Lemma 2.6) from $G$ by attaching $k - 1$ distinct pendant edges and one new path of length 2 to $v$. Then $\mu(G'_k(v)) \geq \mu(G_T(v))$. 

Lemma 2.6

For all graphs of order \( n \) with center \( v \), we have \( \mu(G) \geq \mu(H) \) whenever \( G \) results (R1)–(R6) later:

(R1) \( \mu(G_3) < \mu(G_2) \).

(R2) \( \mu(G_4) < \mu(G_3) \).

(R3) \( \mu(G'_4) = \mu(G_4) \).

(R4) For any \( G \in B_1(n) \setminus \{G_1, G_4\} \), we have \( \mu(G) < \mu(G_4) \).

(R5) For any \( G \in B_2(n) \), we have \( \mu(G) < \mu(G_4) \).

(R6) For any \( G \in B_3(n) \setminus \{G'_1, G_2, G_3, G'_4\} \), we have \( \mu(G) < \mu(G_4) \).

We will prove the results (R2) and (R3) in this section, prove the result (R4) in Section 4, prove the result (R5) in Section 5, prove the result (R6) in Section 6, prove the result (R1) and prove our main result of this paper in Section 7.
The proof of (R2). By Corollary 2.1, we have
\[ \phi(G_3) = x(x - 1)^{n-6} [x^3 - (n + 8)x^2 + (9n + 18)x + (6 + 27n)x^2 + (31n - 10)x - 11n], \]
\[ \phi(G_4) = x(x - 1)^{n-6} (x - 2)(x - 3)[x^3 - (n + 3)x^2 + (4n - 2)x - 2n] \]
\[ \text{Def} \]
\[ x(x - 1)^{n-6} (x - 2)(x - 3)f(x). \] (11)

Then \( \mu(G_4) = \lambda(f(x)). \) Since
\[ \phi(G_4) = \mu(x(x - 1)^{n-6} [x^3 - (n + 2)x^2 + (3n - 2)x - 2n]) = x(x - 1)^{n-6} g(x), \]
then \( g(x) = f(x) + x^2 - nx + n. \)

4. The proof of (R4)

Definition 4.1. Let \( G \) be a bicyclic graph, \( v \in V(\widehat{G}) \), if \( v \) is adjacent to some vertices not in \( V(\widehat{G}) \), then we call \( v \) a divarication-vertex of \( G \).

For convenience, we define
\[ B_{ij} = \{ G \ | \ G \in B_i(n) \text{ with exactly } j \text{ divarication-vertices}, 1 \leq i \leq 3, j \geq 0 \}. \]

Lemma 4.1. For any \( G \in B_{i,j}(n) \) \((1 \leq i \leq 3, j \geq 2)\), there exists \( H \in B_{i,j-1}(n) \) such that \( \widehat{H} = \widehat{G} \) and \( \mu(H) \geq \mu(G) \).

Proof. Let \( v_1, v_2 \) be two divarication-vertices of \( G \) \((j \geq 2)\), and \( T_1, T_2 \) be the two trees in \( G \) attached to \( v_1 \) and \( v_2 \) of orders, say \( n_1 \) and \( n_2 \), respectively. Let
\[ G^* = G - (T_1 - v_1) - (T_2 - v_2) + v_1x_1 + \cdots + v_1x_{n_1-1} + v_2y_1 + \cdots + v_2y_{n_2-1}, \]
where \( x_1, \ldots, x_{n_1-1}, y_1, \ldots, y_{n_2-1} \notin V(\widehat{G}) \). By Lemmas 2.6 and 2.7, \( \mu(G^*) \geq \mu(G) \).

Then \( \{G_1^*, G_2^*\} \subseteq B_{i,j-1}(n) \). By Lemma 2.3, \( \max\{\mu(G_1^*), \mu(G_2^*)\} \geq \mu(G^*) \geq \mu(G) \).

Let \( H \) be the graph with the larger Laplacian spectral radius among \( G_1^* \) and \( G_2^* \), then \( H \) satisfies the conditions in our assertion.

Lemma 4.2. Let \( G \in B_1(n) \) with \( \widehat{G} = B(p, q) \). If \( p + q \geq 8 \), then \( \mu(G) < \mu(G_4) \).

Proof. Since \( p + q \geq 8 \), we have \( n \geq 7 \). If \( G \in B_{10}(n) \), then \( d_1 = 4, d_2 = d_3 = 2 \). By Lemma 2.5, \( \mu(G) \leq 2 + \sqrt{(4 + 2 - 2)^2} = 6 \leq n - 1 < \mu(G_4) \). Then it suffices to consider \( G \in B_1(n) \setminus B_{10}(n) \). By Lemmas 2.6 and 2.7, we only need to consider the case that the trees attached to \( \widehat{G} \) are stars and the divarication-vertices are the centers of these stars. We distinguish the following two cases.

Case 1. \( G \in B_{11}(n) \).

Then \( G \neq G_1, G_4 \), since \( p + q \geq 8 \).

Subcase 1.1 The common vertex of \( C_p \) and \( C_q \) is the divarication-vertex. Then \( d_1 \leq n - 3, d_2 = d_3 = 2 \). By Lemma 2.5,
\[ \mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq n - 1 < \mu(G_4). \]
Subcase 1.2 The common vertex of $C_p$ and $C_q$ is not the divarication-vertex. Then $d_1 \leq n - 5$, $d_2 = 4$, $d_3 = 2$. By Lemma 2.5,

$$\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} < n - 1 < \mu(G_4).$$

Case 2. $G \in B_{1j}(n) \ (j \geq 2)$.

From Lemma 4.1, there exists $H \in B_{11}(n)$ with $\widehat{H} = \widehat{G}$ and $\mu(H) \geq \mu(G)$. Since $p + q \geq 8$ and $\widehat{H} = \widehat{G}$, we have $H \neq G_1, G_4$. By Case 1, we have $\mu(H) < \mu(G_4)$. Thus, $\mu(G) \leq \mu(H) < \mu(G_4)$.

Lemma 4.3. Let $G \in B_1(n) \setminus \{G_4\} \ (n \geq 7)$ with $\widehat{G} = B(p, q)$. If $p + q = 7$, then $\mu(G) < \mu(G_4)$.

Proof. Since $n \geq 7$, we have $G \in B_{1j}(n)$ with $j \geq 1$. We distinguish the following three cases.

Case 1. $G \in B_{11}(n) \setminus \{G_4\}$.

Subcase 1.1 The common vertex of $C_3$ and $C_4$ is the divarication-vertex.

Since $G \neq G_4$, by Lemma 2.7, $\mu(G) \leq \mu(B_1)$ ($B_1$ is shown in Fig. 4.1). Now $d_1(B_1) \leq n - 3$, $d_2(B_1) = d_3(B_1) = 2$. By Lemma 2.5,

$$\mu(B_1) \leq 2 + \sqrt{(d_1(B_1) + d_2(B_1) - 2)(d_1(B_1) + d_3(B_1) - 2)} \leq n - 1 < \mu(G_4).$$

Subcase 1.2 The common vertex of $C_3$ and $C_4$ is not the divarication-vertex. By Lemmas 2.6 and 2.7, it suffices to consider the case that the tree attached to $\widehat{G}$ is a star, and the divarication-vertex is the center of the star. Then $d_1 = n - 4$, $d_2 = 4$, $d_3 = 2$. By Lemma 2.5,

$$\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq n - 1 < \mu(G_4).$$

Case 2. $G \in B_{12}(n)$.

By Lemmas 2.6 and 2.7, we only need to consider the case that the trees attached to $\widehat{G}$ are stars and the divarication-vertices are the centers of the two stars respectively.

Subcase 2.1 The common vertex of $C_3$ and $C_4$ is not a divarication-vertex. By Lemma 4.1, there exists a graph $H \in B_{11}(n) \setminus \{G_4\}$ such that $\mu(G) \leq \mu(H)$ and $\widehat{H} = \widehat{G}$, by Subcase 1.2, $\mu(H) < \mu(G_4)$, which implies $\mu(G) < \mu(G_4)$.

Subcase 2.2 The common vertex of $C_3$ and $C_4$ is a divarication-vertex.

We use $v$ to denote the common vertex of $C_3$ and $C_4$, $u$ to denote the other divarication-vertex.

1. If $|X_u| \geq |X_v|$. By Lemma 2.3, there exists $H \in B_{11}(n) \setminus \{G_4\}$ such that $\widehat{H} = \widehat{G}$ and $\mu(G) \leq \mu(H)$, by Subcase 1.2, $\mu(H) < \mu(G_4)$, and then $\mu(G) < \mu(G_4)$.

2. If $|X_u| < |X_v|$. By Lemma 2.3, $\mu(G) < \mu(G_4)$.

Case 3. $G \in B_{1j}(n) \ (j \geq 3)$.

From Lemma 4.1, there exists $H \in B_{12}(n)$ with $\widehat{H} = \widehat{G}$ and $\mu(G) \leq \mu(H)$. By Case 2, we have $\mu(H) < \mu(G_4)$. Thus, $\mu(G) \leq \mu(H) < \mu(G_4)$.

Lemma 4.4. Let $G \in B_1(n) \setminus \{G_1\} \ (n \geq 7)$ with $\widehat{G} = B(p, q)$. If $p + q = 6$, then $\mu(G) < \mu(G_4)$.

Proof. Since $n \geq 7$, we have $G \in B_{1j}(n)$ with $j \geq 1$. Also $p + q = 6$ implies $p = q = 3$. Thus $\widehat{G} = B(3, 3)$. We distinguish the following three cases.

Case 1. $G \in B_{11}(n) \setminus \{G_1\}$.

Subcase 1.1 The common vertex of the two $C_3$ is the divarication-vertex. By Lemma 2.7, $\mu(G) \leq \mu(B_2)$ ($B_2$ is shown in Fig. 4.1). Now we show that $\mu(B_2) < \mu(G_4)$, which implies $\mu(G) < \mu(G_4)$. By Corollary 2.1, we have

$$\Phi(B_2) = x(x - 1)^{8-8}[x^7 - (n + 10)x^6 + (11n + 36)x^5 - (52 + 47n)x^4 + (98n + 13)x^3 + (30 - 103n)x^2 + (51n - 18)x - 9n].$$
Then
\[
\Phi(B_2) - \Phi(G_4) = x(x - 1)^{n-6}(x - 3)(2x - n),
\]
for \( x = \mu(B_2) > n - 1 \), \( \Phi(G_4) < 0 \), which implies \( \mu(G_4) > \mu(B_2) \).

**Subcase 2.1** The common vertex of the two \( C_3 \) is not the divarication-vertex. By **Lemmas 2.6 and 2.7**, \( \mu(G) \leq \mu(B_3) \) (\( B_3 \) is shown in **Fig. 4.1**). Now we show that \( \mu(B_3) < \mu(G_4) \). By **Corollary 2.1**, we have
\[
\Phi(B_3) = x(x - 1)^{n-7}[x^6 - (n + 9)x^5 + (11n + 20)x^4 + (14 - 42n)x^3
+ (68n - 77)x^2 + (51 - 45n)x + 9n].
\]
Then
\[
\Phi(B_3) - \Phi(G_4) = x(x - 1)^{n-5}[(n - 7)x^2 + (13 - n)x - n]) > 0,
\]
for \( x = \mu(B_3) > n - 1 \) and \( n \geq 7 \). Then \( \Phi(G_4) < 0 \) for \( x = \mu(B_3) \), which implies \( \mu(G_4) > \mu(B_3) \).

**Case 2.** \( G \in \mathcal{B}_{12}(n) \).

By **Lemmas 2.6 and 2.7**, we only need to consider the case that the trees attached to \( \hat{G} \) are stars and the divarication-vertices are the centers of the two stars.

**Subcase 2.1** The common vertex of the two \( C_3 \) is not a divarication-vertex. By **Lemma 2.3**, \( \mu(G) \leq \mu(B_3) < \mu(G_4) \).

**Subcase 2.2** The common vertex of the two \( C_3 \) is a divarication-vertex.

First, we show that \( \mu(B_4) < \mu(G_4) \) (\( B_4 \) is shown in **Fig. 4.1**). By **Corollary 2.1**, \( \Phi(B_4) = x(x - 1)^{n-6}(x - 3)[x^4 - (n + 5)x^3 + (6n + 3)x^2 + (5 - 9n)x + 3n] \).

Then \( \Phi(B_4) - \Phi(G_4) = x(x - 1)^{n-5}(x - 3)(n - x) > 0 \), for all \( 3 < x < n \), which implies \( \mu(G_4) > \mu(B_4) \).

Now we consider the remaining graphs in this subcase. We use \( v \) to denote the common vertex of two \( C_3 \), and \( u \) to denote the other divarication-vertex.

\begin{enumerate}
  \item If \( |X_u| \geq |X_v| \). By **Lemma 2.3**, \( \mu(G) \leq \mu(B_3) < \mu(G_4) \).
  \item If \( |X_u| < |X_v| \). By **Lemma 2.3**, \( \mu(G) \leq \mu(B_4) < \mu(G_4) \).
\end{enumerate}

**Case 3.** \( G \in \mathcal{B}_{11}(n) \) \( (j \geq 3) \).

From **Lemma 4.1**, there exists \( H \in \mathcal{B}_{12}(n) \) with \( \hat{H} = \hat{G} \) and \( \mu(H) \geq \mu(G) \). By **Case 2**, we have \( \mu(H) < \mu(G_4) \). Thus, \( \mu(G) \leq \mu(H) < \mu(G_4) \). \( \Box \)

Combining the above lemmas, we obtain the result (R4). Namely,

**Theorem 4.1.** For any \( G \in \mathcal{B}_1(n) \setminus \{G_1, G_4\} \) \( (n \geq 7) \), \( \mu(G) < \mu(G_4) \).

**5. The proof of (R5)**

In this section, we will prove the result (R5). Namely,

**Theorem 5.1.** Let \( G \in \mathcal{B}_2(n) \), where \( n \geq 7 \), then \( \mu(G) < \mu(G_4) \).

**Proof.** Since \( G \in \mathcal{B}_2(n) \), we may assume that \( \hat{G} = B(p, l, q) \) for some \( p \geq 3, q \geq 3 \) and \( l \geq 1 \). By **Lemmas 2.6 and 2.7**, it suffices to consider the case that the trees attached to \( \hat{G} \) are stars and the divarication-vertices are the centers of the stars.

If \( G \in \mathcal{B}_{20}(n) \), then \( d_1 = d_2 = 3, d_3 = 2 \). By **Lemma 2.5**, we have \( \mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} < 6 \leq n - 1 < \mu(G_4) \), so it suffices to consider \( G \in \mathcal{B}_2(n) \setminus \mathcal{B}_{20}(n) \).

**Case 1.** \( p + q + l \geq 8 \).

**Subcase 1.1** \( G \in \mathcal{B}_2(n) \).
Since $p + q + l \geq 8$, then $d_1 \leq n - 4$, $d_2 \leq 3$, $d_3 \leq 3$. By Lemma 2.5,
\[
\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq n - 1 < \mu(G_4).
\]

**Subcase 1.2** $G \in B_{2j}(n)$ $(j \geq 2)$.

From Lemma 4.1, there exists $H \in B_{21}(n)$ with $\widehat{H} = \widehat{G}$ and $\mu(H) \geq \mu(G)$. Then by Subcase 1.1, we have $\mu(G) \leq \mu(H) < \mu(G_4)$.

**Case 2.** $p + q + l = 7$.

Since $n \geq 7$, we have $G \in B_{2j}(n)$ with $j \geq 1$.

**Subcase 2.1** $G \in B_{21}(n)$.

**Subcase 2.1.1** The common vertices of $C_3$ and $P_2$ (not in $C_3$) in $\widehat{G}$ are not the divarication-vertex. Then $d_1 = n - 4$, $d_2 = d_3 = 3$. By Lemma 2.5,
\[
\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq n - 1 < \mu(G_4).
\]

**Subcase 2.1.2** One of the common vertices of $C_3$ and $P_2$ (not in $C_3$) in $\widehat{G}$ is the divarication-vertex. By Lemmas 2.6 and 2.7, $\mu(G) \leq \mu(B^*(3, 1, 3))$ (see Fig. 5.1). Now we show that $\mu(B^*(3, 1, 3) < \mu(G_4))$. By Corollary 2.1, we have
\[
\Phi(B^*(3, 1, 3)) = x(x - 1)^{n - 7}[x^6 - (n + 9)x^5 + (11n + 22)x^4 + (10 - 44n)x^3
\]
\[
+ (76n - 87)x^2 + (63 - 51n)x + 9n].
\]

Then for $x = \mu(B^*(3, 1, 3)) > n - 1$ and $n \geq 5$, we have
\[
\Phi(B^*(3, 1, 3)) - \Phi(G_4) = x(x - 1)^{n - 6}(x - 2)[(n - 5)x^2 + (17 - 3n)x - n] > 0.
\]

It follows that $\Phi(G_4, \mu(B^*(3, 1, 3))) < 0$. So $\mu(G_4) > \mu(B^*(3, 1, 3))$.

**Subcase 2.2** $G \in B_{2j}(n)$ $(j \geq 2)$.

From Lemma 4.1, there exists $H \in B_{21}(n)$ with $\widehat{H} = \widehat{G}$ and $\mu(H) \geq \mu(G)$. So we have $\mu(G) \leq \mu(H) < \mu(G_4)$.

\[\square\]

6. The proof of (R6)

**Lemma 6.1.** Let $G \in B_3(n)$ with $\widehat{G} = B(P_k, P_l, P_m)$. If $k + l + m \geq 8$, then $\mu(G) < \mu(G_4)$.

**Proof.** Since $k + l + m \geq 8$, we have $n \geq 7$. If $G \in B_{30}(n)$, then $d_1 = d_2 = 3$, $d_3 = 2$. By Lemma 2.5, $\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} < 6 \leq n - 1 < \mu(G_4)$. So it suffices to consider the case $G \in B_3(n) \setminus B_{30}(n)$.

**Case 1.** $G \in B_{31}(n)$.

From Lemmas 2.6 and 2.7, we only need to consider the case that the tree attached to $\widehat{G}$ is a star, and the divarication-vertex is the center of the star. Then $d_1 \leq n - 4$, $d_3 \leq d_2 = 3$. By Lemma 2.5,
\[
\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq n - 1 < \mu(G_4).
\]

**Case 2.** $G \in B_{3j}(n)$ $(j \geq 2)$.

From Lemma 4.1, there exists $H \in B_{31}(n)$ with $\widehat{H} = \widehat{G}$ and $\mu(H) \geq \mu(G)$. Then we have $\mu(G) \leq \mu(H) < \mu(G_4)$.

\[\square\]

**Lemma 6.2.** Let $G \in B_3(n)$ $(n \geq 7)$ with $\widehat{G} = B(P_k, P_l, P_m)$. If $k + l + m = 7$, then $\mu(G) < \mu(G_4)$.
Proof. Since \( n \geq 7 \), we have \( G \in B_{3j}(n) \) with \( j \geq 1 \). We distinguish the following two cases.

Case 1. \( G \in B_{31}(n) \).

Since \( k + l + m = 7 \), then \( \hat{G} \) is one of the three graphs \( B(P_2, P_4, P_1), B(P_3, P_2, P_2) \) and \( B(P_3, P_3, P_1) \). By Lemmas 2.6 and 2.7, it suffices to consider the case that the tree attached to \( \hat{G} \) is a star and the divarication-vertex is the center of the star.

Subcase 1.1 The divarication-vertex is not a common vertex of the three cycles in \( \hat{G} \). Then \( d_1 \leq n - 4, \ d_2 = d_3 = 3 \). By Lemma 2.5,
\[
\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq n - 1 < \mu(G_4).
\]

Subcase 1.2 The divarication-vertex is a common vertex of the three cycles in \( \hat{G} \). We denote the following three graphs by \( H_1, H_2 \) and \( H_3 \) (shown in Fig. 6.1), respectively.

By direct calculations, we have
\[
\begin{align*}
\Phi(H_1) &= x(x - 1)^7[x^6 - (n + 9)x^5 + (11n + 22)x^4 + (8 - 44n)x^3 \\
&\quad + (78n - 81)x^2 + (65 - 59n)x + 14n], \\
\Phi(H_2) &= x(x - 1)^7[x^6 - (n + 9)x^5 + (11n + 22)x^4 + (6 - 44n)x^3 \\
&\quad + (79n - 74)x^2 + (60 - 62n)x + 16n], \\
\Phi(H_3) &= x(x - 1)^7[x^6 - (n + 9)x^5 + (11n + 22)x^4 + (6 - 44n)x^3 \\
&\quad + (78n - 71)x^2 + (51 - 59n)x + 15n].
\end{align*}
\]

Then for all \( x > n - 1 \) and \( n > 5 \), we have
\[
\Phi(H_2) - \Phi(H_3) = x(x - 1)^n - 7[x(n - 3)(x - 3) + n] > 0.
\]

It follows that \( \Phi(H_3, \mu(H_2)) < 0 \), which implies \( \mu(H_3) > \mu(H_2) \).

\[
\Phi(H_1) - \Phi(H_2) = x(x - 1)^n - 6[2x^3 - (n + 7)x^2 + (3n + 5)x - 2n].
\]

For all \( x > n - 1 \) and \( n > 5 \), \( \Phi(H_1) - \Phi(H_2) > 0 \), and thus \( \Phi(H_2, \mu(H_1)) < 0 \), which implies \( \mu(H_2) > \mu(H_1) \).

\[
\Phi(H_3) - \Phi(G_4) = x(x - 1)^n - 6(x - 3)[(n - 5)x^2 + (13 - 3n)x + n].
\]

For all \( x > 5, n \geq 6 \), \( \Phi(H_3) - \Phi(G_4) > 0 \), and thus \( \Phi(G_4, \mu(H_3)) < 0 \), so \( \mu(G_4) < \mu(H_3) \).

Thus, \( \mu(H_1) < \mu(H_2) < \mu(H_3) < \mu(G_4) \).

Case 2. \( G \in B_{3j}(n) (j \geq 2) \).

From Lemma 4.1, there exists \( H \in B_{31}(n) \) with \( \hat{H} = \hat{G} \) and \( \mu(H) \geq \mu(G) \). By Case 1, we have \( \mu(H) < \mu(G_4) \).

Thus, \( \mu(G) \leq \mu(H) < \mu(G_4) \).

Lemma 6.3. Let \( G \in B_3(n) \setminus \{G_2, G_3\} (n \geq 7) \) with \( \hat{G} = B(P_k, P_l, P_m) \). If \( k + l + m = 6 \), then \( \mu(G) < \mu(G_4) \).

Proof. Since \( k + l + m = 6 \), then \( \hat{G} \) is one of the two graphs \( B(P_2, P_3, P_1) \) and \( B(P_2, P_2, P_2) \). For \( n \geq 7 \), we have \( G \in B_{3j}(n) \) with \( j \geq 1 \).

Case 1. \( G \in B_{31}(n) \setminus \{G_2, G_3\} \).

Subcase 1.1 The divarication-vertex is a common vertex of the three cycles of \( \hat{G} \).

1. If \( \hat{G} = B(P_2, P_2, P_2) \). By Lemma 2.7, \( \mu(G) \leq \mu(B_5) \) (\( B_5 \) is shown in Fig. 6.2).
2. If \( \hat{G} = B(P_2, P_3, P_1) \). By Lemma 2.7, \( \mu(G) \leq \mu(B_5') \) (\( B_5' \) is shown in Fig. 6.2).
Corollary 2.1 and 2.7

Lemmas 2.6 and 2.7

If Fig. 6.2

Case 2. x

Then

Subcase 1.2

x

Then

Subcase 1.2

The divergence-vertex is not a common vertex of the three cycles of \( \hat{G} \).

1. If \( \hat{G} = B(P_2, P_2, P_2) \). By Lemmas 2.6 and 2.7, \( \mu(G) \leq \mu(B_6) \).

2. If \( \hat{G} = B(P_2, P_3, P_1) \). By Lemmas 2.6 and 2.7, \( \mu(G) \leq \mu(B'_6) \) or \( \mu(G) \leq \mu(B''_6) \).

Now we show that \( \mu(B'_6) < \mu(B_6) < \mu(G_4) \). (6) (6)

By Corollary 2.1, we have

\[
\phi(B_6) = x(x - 1)^{n-6}[x^5 - (n + 8)x^4 + (10n + 13)x^3 + (22 - 33n)x^2 + (40n - 48)x - 12n],
\]

\[
\phi(B'_6) = x(x - 1)^{n-6}[x^5 - (n + 8)x^4 + (10n + 13)x^3 + (24 - 33n)x^2 + (40n - 55)x - 11n],
\]

\[
\phi(B''_6) = x(x - 1)^{n-6}[x^5 - (n + 8)x^4 + (10n + 13)x^3 + (24 - 33n)x^2 + (39n - 50)x - 11n].
\]

Then

\[
\phi(B_6) - \phi(G_4) = (n - 6)x^2(x - 1)^{n-6}(x - 2)(x - 3).
\]

For \( x = \mu(B_6) \), we have \( \phi(B_6) - \phi(G_4) > 0 \), and thus \( \phi(G_4) < 0 \), which implies \( \mu(G_4) > \mu(B_6) \).

\[
\phi(B'_6) - \phi(B_6) = x(x - 1)^{n-6}(2x^2 - 7x + n).
\]

For all \( x > n - 1 \), we have \( \phi(B'_6) - \phi(B_6) > 0 \), and thus \( \phi(B_6, \mu(B'_6)) < 0 \), which implies \( \mu(B_6) > \mu(B'_6) \).

\[
\phi(B''_6) - \phi(B'_6) = (n - 5)x^2(x - 1)^{n-6}.
\]

For all \( x > 1 \) and \( n \geq 6 \), we have \( \phi(B''_6) - \phi(B'_6) > 0 \), and thus \( \phi(B'_6, \mu(B''_6)) < 0 \), which implies \( \mu(B''_6) > \mu(B'_6) \).

Case 2. \( G \in B_{32}(n) \).

By Lemmas 2.6 and 2.7, we only need to consider the case where the trees attached to \( \hat{G} \) are stars and the divergence-vertices are the centers of the stars.
Subcase 2.1 Neither of the two divarication-vertices is a common vertex of the three cycles of \( \hat{G} \). By Lemma 4.1, there exists a graph \( H \in B_{31}(n) \setminus \{G_2, G_3\} \) such that \( \mu(G) \leq \mu(H) \), by Subcase 1.2, \( \mu(H) < \mu(G_4) \), and then \( \mu(G) < \mu(G_4) \).

Subcase 2.2 Exactly one of the two divarication-vertices is a common vertex of the three cycles of \( \hat{G} \).

1. If \( \hat{G} = B(P_2, P_2, P_2) \). By Lemmas 2.3, 2.6 and 2.7,
   \[ \mu(G) \leq \max \{\mu(B_6), \mu(B_7)\} < \mu(G_4). \]

2. If \( \hat{G} = B(P_2, P_3, P_1) \). By Lemmas 2.3, 2.6 and 2.7,
   \[ \mu(G) \leq \max \{\mu(B_6'), \mu(B_6''), \mu(B_7'), \mu(B_7''), \mu(B_7''')\} < \mu(G_4). \]

Now we show that \( \mu(B_6'') < \mu(B_7') < \mu(B_7) < \mu(G_4) \) and \( \mu(B_7'') < \mu(B_7') \) (\( B_7, B_7', B_7'', B_7''' \) are shown in Fig. 6.2). By Corollary 2.1, we have

\[ \Phi(B_7) = x(x-1)^{n-7}[x^6 - (n+9)x^5 + (11n+21)x^4 + (9-43n)x^3 + (73n-70)x^2 + (48-52n)x + 12n] = \Phi(B_6), \]
\[ \Phi(B_7') = x(x-1)^{n-7}[x^6 - (n+9)x^5 + (11n+21)x^4 + (11-43n)x^3 + (72n-74)x^2 + (50-50n)x + 11n], \]
\[ \Phi(B_7'') = x(x-1)^{n-7}[x^6 - (n+9)x^5 + (11n+21)x^4 + (11-43n)x^3 + (73n-80)x^2 + (62-52n)x + 11n], \]
\[ \Phi(B_7''') = x(x-1)^{n-7}[x^6 - (n+9)x^5 + (11n+21)x^4 + (11-43n)x^3 + (72n-73)x^2 + (43-49n)x + 11n]. \]

Then
\[ \Phi(B_7') - \Phi(B_7) = x(x-1)^{n-5}(2x-n) > 0, \]
for \( x = \mu(B_7') > n-1, \Phi(B_7) < 0 \), which implies \( \mu(B_7) = \mu(B_6) > \mu(B_7') \).

\[ \Phi(B_7'') - \Phi(B_7') = x^2(x-1)^{n-7}(x-2)(n-6) > 0, \quad \text{for } n \geq 7. \]
\[ \Phi(B_7''') - \Phi(B_7'') = x^2(x-1)^{n-7}(x+n-7) > 0, \quad \text{for } n \geq 7. \]

So \( \mu(B_7') > \mu(B_7''), \mu(B_7') > \mu(B_7''') \) for \( n \geq 7 \).

Subcase 2.3 The two divarication-vertices are both common vertices of the three cycles of \( \hat{G} \).

1. \( \hat{G} = B(P_2, P_2, P_2) \). By Lemmas 2.3, 2.6 and 2.7, \( \mu(G) \leq \mu(B_8) \).

2. \( \hat{G} = B(P_2, P_3, P_1) \). By Lemmas 2.3, 2.6 and 2.7, \( \mu(G) \leq \mu(B_8') \).

Now we show that \( \mu(B_8') < \mu(B_8) < \mu(G_4) \) (\( B_8, B_8' \) are shown in Fig. 6.2). By Corollary 2.1, we have

\[ \Phi(B_8) = x(x-1)^{n-7}[x^6 - (n+9)x^5 + (11n+20)x^4 + (10-42n)x^3 + (71n-72)x^2 + (56-52n)x + 12n]. \]

Then
\[ \Phi(B_8) - \Phi(G_4) = x^2(x-1)^{n-7}(x-2)[(n-7)x^2 + (23-3n)x + 3n - 22]. \]

For all \( x > n-1 \) and \( n \geq 7 \), we have \( \Phi(B_8) - \Phi(G_4) > 0 \), and thus \( \Phi(G_4, \mu(B_8)) < 0 \), which implies \( \mu(G_4) > \mu(B_8) \).

\[ \Phi(B_8') = x(x-1)^{n-7}[x^6 - (n+9)x^5 + (11n+20)x^4 + (12-42n)x^3 + (69n-70)x^2 + (46-48n)x + 11n]. \]

Then
\[ \Phi(B_8') - \Phi(B_8) = x^2(x-1)^{n-7}[2x^3 + (2 - 2n)x^2 + (4n - 10)x - n]. \]
For all $x > n - 1$, we have $\Phi(B'_8) - \Phi(B_8) > 0$, and thus $\Phi(B_8, \mu(B'_8)) < 0$, which implies $\mu(B_8) > \mu(B'_8)$.

**Case 3.** $G \in \mathcal{B}_3(n)$ ($j \geq 3$).

From Lemma 4.1, there exists $H \in \mathcal{B}_3(n)$ with $\hat{H} = \hat{G}$ and $\mu(H) \geq \mu(G)$, by Case 2, we have $\mu(H) < \mu(G_4)$. Thus, $\mu(G) \leq \mu(H) < \mu(G_4)$. \[\square\]

**Lemma 6.4.** Let $G \in \mathcal{B}_3(n) \setminus \{G'_1, G'_4\}$ ($n \geq 7$) with $\hat{G} = B(P_k, P_l, P_m)$. If $k + l + m = 5$, then $\mu(G) < \mu(G_4)$.

**Proof.** Since $n \geq 7$, we have $G \in \mathcal{B}_3(n)$ with $j \geq 1$.

**Case 1.** $G \in \mathcal{B}_3(n)$.

**Subcase 1.1** The two divarication-vertices are both common vertices of the three cycles of $\hat{G}$.

By Lemmas 2.3, 2.6 and 2.7, we have $\mu(G) \leq \mu(B_9)$. Now we show that $\mu(B_9) < \mu(G_4)$ ($B_9$ is shown in Fig. 6.3). By Corollary 2.1, we have

$$\Phi(B_9) = \frac{x(x-1)^{n-6}[(x-2)(x^4-(n+6)x^3+(7n+4)x^2+(6-11n)x+4n]}}{\text{Def } x(x-1)^{n-6}(x-2)p(x)}.$$  

Combining the above equality with the equality (11) in Section 3, we have

$$p(x) = (x^2-3x-3)f(x) + q(x), \quad q(x) = -9x^2 + (10n-6)x - 6n,$$

and $q(x)$ is a strict decreasing function for all $x > n - 1$. $q(n) > 0$, so for $n - 1 < \mu(B_9) < n$, $q(\mu(B_9)) > 0$. So we have $f(\mu(B_9)) < 0$, which implies $\mu(G_4) > \mu(B_9)$.

**Subcase 1.2** Exactly one of the two divarication-vertices is a common vertex of the three cycles of $\hat{G}$.

By Lemmas 2.6 and 2.7, it suffices to consider the case that the trees attached to $\hat{G}$ are stars and the divarication-vertices are the centers of the stars. Let $v$ be the divarication-vertex which is a common vertex of the three cycles, $u$ be the other divarication-vertex.

1. If $|X_u| > |X_v|$. By Lemma 2.3, we have $\mu(G) < \mu(G_4)$.

2. If $|X_u| \leq |X_v|$. By Lemma 2.3, we have $\mu(G) \leq \mu(B_{10})$.

We now show that $\mu(B_{10}) < \mu(G_4)$ ($B_{10}$ is shown in Fig. 6.3). By Corollary 2.1,

$$\Phi(B_{10}) = \frac{x(x-1)^{n-6}[x^5-(n+8)x^4+(9n+17)x^3-(26n+2)x^2+(27n-13)x-8n]}{\text{Def } x(x-1)^{n-6}r(x)}.$$  

Combining the above equality with the equality (11) in Section 3, we have $r(x) = (x-1)(x-4)f(x) + (n-5)x$, for $x = \mu(B_{10})$ and $n \geq 6$, $f(x) < 0$, which implies $\mu(G_4) > \mu(B_{10})$.

**Subcase 1.3** Neither of the two divarication-vertices is a common vertex of the three cycles of $\hat{G}$.

By Lemma 2.3, $\mu(G) \leq \mu(B_{11})$. We now show that $\mu(B_{11}) < \mu(G_4)$ ($B_{11}$ is shown in Fig. 6.3). By Corollary 2.1, we have

$$\Phi(B_{11}) = \frac{x(x-1)^{n-6}[x^5-(n+8)x^4+(10n+12)x^3+(28-32n)x^2+(34n-48)x-8n]}{\text{Def } x(x-1)^{n-6}r(x)}.$$  

Then

$$\Phi(B_{11}) - \Phi(G_4) = \frac{x(x-1)^{n-5}[(n-7)x^3+(36-4n)x^2-36x+4n]}{\text{Def } x(x-1)^{n-5}r(x)}.$$  

For $x = \mu(B_{11}) > n - 1$ and $n \geq 7$, we have $\Phi(B_{11}) - \Phi(G_4) > 0$, and thus $\Phi(G_4) < 0$, which implies $\mu(G_4) > \mu(B_{11})$.

**Case 2.** $G \in \mathcal{B}_3(n) \setminus \{G'_1, G'_4\}$.
Subcase 2.1 The divarication-vertex is a common vertex of the three cycles of \( \hat{G} \).

By Lemma 2.7, \( \mu(G) \leq \mu(B_{12}) \) (\( B_{12} \) is shown in Fig. 6.3). We now show that \( \mu(B_{12}) < \mu(G) \). By Corollary 2.1, we have

\[
\Phi(B_{12}) = x(x-1)^{n-6}(x-2)[x^4 - (n + 6)x^3 + (7n + 6)x^2 + (8 - 13n)x + 4n].
\]

Then

\[
\Phi(B_{12}) - \Phi(B_{10}) = 2x^2(x-1)^6(x-3)(x-n+1).
\]

For \( x = \mu(B_{12}) > n - 1 \) and \( n \geq 5 \), \( \Phi(B_{12}) - \Phi(B_{10}) > 0 \), and thus \( \Phi(B_{10}) < 0 \), which implies \( \mu(B_{10}) > \mu(B_{12}) \), and then \( \mu(B_{12}) < \mu(G) \).

Subcase 2.2 The divarication-vertex is not a common vertex of the three cycles of \( \hat{G} \).

By Lemma 2.7, \( \mu(G) \leq \mu(B_{13}) \) (\( B_{13} \) is shown in Fig. 6.3). We now show that \( \mu(B_{13}) < \mu(G) \). By Corollary 2.1, we have

\[
\Phi(B_{13}) = x(x-1)^{n-7}(x-4)[x^5 - (n + 5)x^4 + (7n + 1)x^3 + (17 - 15n)x^2 + (10n - 8)x - 2n].
\]

Then

\[
\Phi(B_{13}) - \Phi(B_{4}) = x(x-1)^{n-7}[(n - 6)x^4 + (40 - 6n)x^3 + 8(n - 9)x^2 + 4(n + 5)x - 4n].
\]

For \( x = \mu(B_{13}) > n - 1 \) and \( n \geq 6 \), \( \Phi(B_{13}) - \Phi(B_{4}) > 0 \), and thus \( \Phi(B_{4}) < 0 \), which implies \( \mu(B_{4}) > \mu(B_{13}) \).

Case 3. \( G \in B_{3j}(n) \) \( (j \geq 3) \).

From Lemma 4.1, there exists \( H \in B_{32}(n) \) with \( \hat{H} = \hat{G} \) and \( \mu(H) \geq \mu(G) \), by Case 1, we have \( \mu(H) < \mu(G) \). Thus, \( \mu(G) \leq \mu(H) < \mu(G) \).

Combining the above lemmas, we obtain the result (R6). Namely,

**Theorem 6.1.** For any \( G \in B_{3j}(n) \) \( \setminus \{ G'_1, G_2, G_3, G'_4 \} \) \( (n \geq 7) \), \( \mu(G) < \mu(G) \).

7. The proof of (R1) and the main result

The proof of (R1). By Corollary 2.1, we have

\[
\Phi(G_2) = x(x-1)^{n-6}(x-2)[x^3 - (n + 4)x^2 + (5n - 2)x - 3n],
\]

\[
\Phi(G_3) = x(x-1)^{n-6}[x^5 - (n + 8)x^4 + (9n + 18)x^3 - (6 + 27n)x^2 + (31n - 10)x - 11n].
\]

Then

\[
\Phi(G_3) - \Phi(G_2) = x^2(x-1)^{n-5}(2x - n).
\]

For \( x = \mu(G_3) > n - 1 \), \( \Phi(G_3) - \Phi(G_2) > 0 \). So we have \( \Phi(G_2, \mu(G_3)) < 0 \), which implies \( \mu(G_2) > \mu(G_3) \).

From the results (R1) to (R6), we can obtain our main result.

**Theorem 7.1.** If \( G \) is a bicyclic graph of order \( n \geq 7 \), \( G_1, G'_1, G_2, G_3, G_4, G'_4 \) are graphs shown in Fig. 3.1, then

1. \( \mu(G_1) = \mu(G'_1) > \mu(G_2) > \mu(G_3) = \mu(G'_4) \).
2. \( \mu(G) < \mu(G_4) \) for \( G \not\in \{ G_1, G'_1, G_2, G_3, G_4, G'_4 \} \).
3. \( \mu(G_1) = \mu(G'_1) = n \).
4. \( \mu(G_2) \) is the largest root of the equation
   \[ x^3 - (n + 4)x^2 + (5n - 2)x - 3n = 0. \]
5. \( \mu(G_3) \) is the largest root of the equation
   \[ x^5 - (n + 8)x^4 + (9n + 18)x^3 - (6 + 27n)x^2 + (31n - 10)x - 11n = 0. \]
6. \( \mu(G_4) \) is the largest root of the equation
   \[ x^3 - (n + 3)x^2 + (4n - 2)x - 2n = 0. \]
References