Inequalities for $M$-matrices and inverse $M$-matrices

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Abstract

In this paper, we establish some determinantal inequalities concerning $M$-matrices and inverse $M$-matrices. The main results are as follows:

1. If $A = (a_{ij})$ is either an $n \times n$ $M$-matrix or inverse $M$-matrix, then for any permutation $i_1, i_2, \ldots, i_n$ of \{1, 2, ..., $n$\},
   
   (a) $\det A \leq \left(\prod_{i=1}^{n} a_{ii}\right) \prod_{s=2}^{n} \left(1 - \frac{|a_{i_1i_2} \cdots a_{i_{s-1}i_s} a_{i_si_1}|}{a_{i_1i_1} a_{i_2i_2} \cdots a_{i_si_s}}\right)$.
   
   (b) $\det A = \prod_{i=1}^{n} a_{ii}$ if and only if $A$ is essentially triangular.

2. If $A = (a_{ij})$ is an $n \times n$ $M$-matrix, $B = (b_{ij})$ is an $n \times n$ inverse $M$-matrix, $A \circ B$ denotes the Hadamard product of $A$ and $B$, then $A \circ B$ is an $M$-matrix, and for any permutation $i_1, i_2, \ldots, i_n$ of \{1, 2, ..., $n$\},
   
   $\det (A \circ B) \geq \det (AB) \prod_{s=2}^{n} \left( a_{i_si_s} \frac{\det A[i_1, i_2, \ldots, i_{s-1}] \text{det} B[i_1, i_2, \ldots, i_{s-1}, i_s]}{\text{det} A[i_1, \ldots, i_{s-1}, i_s]} + b_{i_i} \frac{\text{det} A[i_1, i_2, \ldots, i_{s-1}] \text{det} B[i_1, i_2, \ldots, i_{s-1}, i_s]}{\text{det} B[i_1, \ldots, i_{s-1}, i_s]} - 1\right)$.

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1. Introduction

A real matrix is called nonnegative if every entry is nonnegative. For two \(m \times n\) matrices \(A = (a_{ij})\) and \(B = (b_{ij})\), \(A \geq B\) (Perron–Frobenius order) means \(A - B\) is nonnegative, the Hadamard product of \(A\) and \(B\) is defined and denoted by \(A \odot B = (a_{ij} b_{ij})\).

For a positive integer \(n\), let \(N = \{1, 2, \ldots, n\}\) throughout. To avoid triviality we always assume that \(n > 1\).

Given an \(n \times n\) matrix \(A\) and a nonempty index set \(\alpha = \{i_1, i_2, \ldots, i_s\} \subseteq N\), we will write the principal submatrix of \(A\) in rows and columns \(i_1, i_2, \ldots, i_s\) as \(A[i_1, i_2, \ldots, i_s]\) or \(A[\alpha]\). In particular, we set \(A(k) = A[N \setminus \{k\}]\), \(A_k = A[1, 2, \ldots, k]\) for \(k \in N\). We of course adopt the convention that \(A[\emptyset] = I_n\).

Let us recall some definitions as follows.

A complex \(n \times n\) matrix \(A = (a_{ij})\) is called a \(W\)-matrix if for any indices \(i_1, i_2, \ldots, i_s (s \geq 2)\) different from each other in \(N\)

\[
|a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}| > |a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|,
\]

and is denoted by \(A \in W_n\). Notice that a matrix of order 1 is a \(W\)-matrix if and only if it is nonzero.

An \(n \times n\) real matrix \(A\) is called a \(Z\)-matrix if all of its off-diagonal entries are nonpositive, and is denoted by \(A \in Z_n\). A \(Z\)-matrix is called an \(M\)-matrix if it is nonsingular and its inverse is a nonnegative matrix, and denote by \(M_n\) the class of all \(n \times n\) \(M\)-matrices. The class of all matrices whose inverse belongs to \(M_n\), so-called inverse \(M\)-matrices, will be denoted by \(M^{-1}_n\).

For convenience, we introduce the following definition:

Definition 1.1. A complex \(n \times n\) matrix is called an \(HF\)-matrix if the following conditions are satisfied:

(1) All of the principal minors of \(A\) are positive.
(2) For arbitrary index sets \(\alpha, \beta \subseteq N\), the Hadamard–Fischer inequalities hold, that is

\[
\det A[\alpha \cup \beta] \leq \det A[\alpha] \det A[\beta] / \det A[\alpha \cap \beta].
\]

It is well known that the \(M\)-matrices, the inverse \(M\)-matrices, the positive definite Hermitian matrices, the totally positive matrices are all \(HF\)-matrices, which can be found in [1,2].

For an \(n \times n\) positive semi-definite Hermitian matrix \(A = (a_{ij})\), Hadamard’s inequality states that

\[
\det A \leq \prod_{i=1}^{n} a_{ii}.
\]

Furthermore, equality holds if and only if \(A\) is diagonal or \(A\) has a zero row or column.

In [3], Zhang and Yang have improved Hadamard’s inequality for totally nonnegative and totally positive matrices, and investigate the necessary and sufficient conditions for equality to hold.

Oppenheim’s inequality [4, p. 480]: If \(A = (a_{ij})\) and \(B = (b_{ij})\) are both positive semi-definite Hermitian matrices of order \(n\), then

\[
\det(A \odot B) \geq \left( \prod_{i=1}^{n} a_{ii} \right) \det B.
\]
Oppenheim’s inequality has been studied much in the literature. One of the most important results is of course that if $A = (a_{ij})$ and $B = (b_{ij})$ are both $M$-matrices of order $n$, then

$$\det(A \circ B) + \det A \cdot \det B \geq (\det A) \prod_{i=1}^{n} b_{ii} + (\det B) \prod_{i=1}^{n} a_{ii}, \quad (1)$$

which is attributed to Ando [5]. Notice that (1) also holds for the case that both $A$ and $B$ are positive definite Hermitian matrices of order $n$ [4, Problem 5, p. 483].

Besides there are other developments, for example, Liu and Zhu [6] have improved Oppenheim’s inequality for the case that $A$ is an $M$-matrix and $B$ is either an $M$-matrix or a positive definite real symmetric matrix; Yang and Liu [7] have strengthened Oppenheim’s inequality for the case that both $A$ and $B$ are $M$-matrices.

In [8], we have strengthened (1) as follows: if both $A = (a_{ij})$ and $B = (b_{ij})$ are $M$-matrices or positive definite real symmetric matrices of order $n$, $A_k$ and $B_k$ ($k = 1, 2, \ldots, n$) are the $k \times k$ leading principal submatrices of $A$ and $B$, respectively, then

$$\det(A \circ B) \geq \det(AB) \prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right). \quad (2)$$

For an $n \times n$ $M$-matrix $A = (a_{ij})$ and another $n \times n$ inverse $M$-matrix $B = (b_{ij})$, there exists an analog of Oppenheim’s inequality [9, Problem 5, p. 378]:

$$\det(A \circ B) \geq \left( \prod_{i=1}^{n} b_{ii} \right) \det A \geq \det(AB).$$

It is natural to ask whether Ando’s inequality (1) is also valid for $A \in M_n$ and $B \in M_n^{-1}$.

In this paper, we indicate that the answer is affirmative in a stronger form. This means that if $A = (a_{ij})$ is an $M$-matrix of order $n$, and $B = (b_{ij})$ is an inverse $M$-matrix of order $n$, then the inequality (2) is also valid. On the other hand, lower and upper bounds for the determinant of the Hadamard product of an $M$-matrix and another inverse $M$-matrix with the same size are derived.

2. Some lemmas

In this section, we give some lemmas which will be used in the proof of the main results.

$M$-matrices have important applications, for instance, in iterative methods in numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming.

$M$-matrices have many equivalent definitions and important properties, but for our purpose, we need only the following Lemmas 2.1 and 2.2, which can be found in [9,10].

**Lemma 2.1.** If $A \in Z_n$, then the following statements are equivalent.

(a) $A$ is an $M$-matrix.
(b) All of the principal minors of $A$ are positive.
(c) All the leading principal minors of $A$ are positive.

**Lemma 2.2.** If $A \in M_n$, $B \in Z_n$ and $B \succ A$, then

(a) $B$ is an $M$-matrix.
(b) $A^{-1} \succ B^{-1} \succ 0$.
(c) $\det B \succ \det A$. 
Lemma 2.3. If \( A = (a_{ij}) \in M_n, B = (b_{ij}) \in Z_n \) and \( B \succeq A \), then
\[
\frac{\det B}{b_{kk} \det B(k)} \geq \frac{\det A}{a_{kk} \det A(k)} \quad \forall k \in \mathbb{N}.
\]

**Proof.** Write \( A^{-1} = (\alpha_{ij}) \) and \( B^{-1} = (\beta_{ij}) \). By Lemma 2.2, \( A^{-1} \succeq B^{-1} \succeq 0 \), hence
\[
b_{kk}\beta_{kk} = \sum_{i \neq k} |b_{ki}|\beta_{ik} + 1 \leq \sum_{i \neq k} |a_{ki}|\alpha_{ik} + 1 = a_{kk}\alpha_{kk}.
\]
Since \( \beta_{kk} = \frac{\det B(k)}{\det B} \) and \( \alpha_{kk} = \frac{\det A(k)}{\det A} \), we have
\[
0 < \frac{b_{kk} \det B(k)}{\det B} \leq \frac{a_{kk} \det A(k)}{\det A}.
\]
Therefore
\[
\frac{\det B}{b_{kk} \det B(k)} \geq \frac{\det A}{a_{kk} \det A(k)}.
\]
\( \Box \)

Lemma 2.4. Suppose a real \( n \times n \) matrix \( A = (a_{ij}) \) is partitioned as \( A = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix} \). If \( \det A_{n-1} > 0 \), \( x \) is a real number, then
\[
\det \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & x \end{pmatrix} > 0 \text{ if and only if } x > a_{nn} - \det A/\det A_{n-1}.
\]

**Proof.** This can be found in [8, Lemma 2.1(a)]. \( \Box \)

Lemma 2.5. \( \forall \varepsilon > 0 \). If \( A = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix} \in M_n \), then
\[
B = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & a_{nn} - \det A/\det A_{n-1} + \varepsilon \end{pmatrix} \in M_n.
\]

**Proof.** Since \( A_{n-1} \in M_{n-1} \), \( \det B = \varepsilon \det A_{n-1} > 0 \), all the leading principal minors of \( B \) are positive by Lemma 2.1, thus \( B \in M_n \). \( \Box \)

Lemma 2.6. If \( A \in M_n, B \in W_n, \) and \( B \succeq 0 \), then \( A \circ B \in M_n \).

**Proof.** Observe that \( A \circ B \in Z_n \), this is a direct consequence of Theorem (3.1) of [11]. \( \Box \)

Lemma 2.7. If \( B = (b_{ij}) \in M_{n-1}^{\circ}(n \geq 3) \), then for any indices \( i, j, k \) in \( \mathbb{N} \)
\[
0 \leq b_{ik}b_{kj} \leq b_{kk}b_{ij}.
\]

**Proof.** This follows from [12, Lemma 2.2(ii)]. \( \Box \)

Lemma 2.8. Let \( A \in M_n, B \in M_n^{-1} \). If \( P \) is a permutation matrix of order \( n \), then \( P^{-1}AP \in M_n, P^{-1}BP \in M_n^{-1} \), and
\[
\det[(P^{-1}AP) \circ (P^{-1}BP)] = \det(A \circ B).
\]
Proof. It is quite evident that $P^{-1}AP \in M_n$ and $P^{-1}BP \in M_n^{-1}$.
Since $(P^{-1}AP) \circ (P^{-1}BP) = P^{-1}(A \circ B)P$, so (3) holds. □

3. Main results

In this section, we state and prove our main results.

**Theorem 3.1.** If $A = (a_{ij}) \in M_n \cup M_n^{-1}$, then for any $\alpha = \{i_1, i_2, \ldots, i_s\} \subseteq N$, where $i_1, i_2, \ldots, i_s$ are mutually distinct,

$$\frac{\det A}{a_{i_s i_s} \det A(i_s)} \leq 1 - \frac{|a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}}.$$  \hspace{1cm} (4)

**Proof.** Let us distinguish two cases:

**Case 1.** $A \in M_n$. Put $S = \{(i_1, i_2), \ldots, (i_{s-1}, i_s), (i_s, i_1), (1, 1), (2, 2), \ldots, (n, n)\}$.

We define an $n \times n$ matrix $B = (b_{ij})$ in the following manner:

$$b_{ij} = a_{ij} \text{ if } (i, j) \in S; b_{ij} = 0 \text{ if } (i, j) \notin S.$$  

Obviously, $B \succeq A$, by Lemma 2.3, we have

$$\frac{\det A}{a_{i_s i_s} \det A(i_s)} \leq \frac{\det B}{b_{i_s i_s} \det B(i_s)} \leq \left(\prod_{i \in \alpha} a_{ii} - |a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|\right) \cdot \prod_{i \notin \alpha} a_{ii} \bigg/ \prod_{i=1}^{n} a_{ii} = 1 - |a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}| \bigg/ \prod_{i \in \alpha} a_{ii}.$$

**Case 2.** $A \in M_n^{-1}$. First, let us prove the following inequality by induction on $n$, the order of matrices.

$$\frac{\det A}{a_{i_n i_n} \det A(i_n)} \leq 1 - \frac{a_{i_1 i_2} \cdots a_{i_{n-1} i_n} a_{i_n i_1}}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_n i_n}}.$$  \hspace{1cm} (5)

One can easily verify that (5) holds with equality for $n = 2$.

Now we assume that $n > 2$, and the inequality (5) is true in case the order of matrices is $n - 1$. Observe that $0 \leq a_{i_1 i_1} a_{i_1 i_2} \leq a_{i_1 i_1} a_{i_1 i_2}$ by Lemma 2.7, since $A$ is an HF-matrix, we have

$$\det A \leq \frac{\det A[i_1, i_2, \ldots, i_{n-1}] \det A[i_2, i_3, \ldots, i_n]}{\det A[i_2, i_3, \ldots, i_n]}$$

whence

$$\frac{\det A}{a_{i_n i_n} \det A(i_n)} \leq \frac{\det A[i_2, i_3, \ldots, i_n]}{a_{i_n i_n} \det A[i_2, i_3, \ldots, i_n]} \leq 1 - \frac{a_{i_2 i_3} \cdots a_{i_{n-1} i_n} a_{i_n i_1}}{a_{i_2 i_2} \cdots a_{i_{n-1} i_n} a_{i_n i_n}} \text{ (by the induction hypothesis)}.$$
Therefore, (5) is proved.
Again apply Hadamard–Fischer inequality, we can get
\[
\frac{\det A}{a_{i_i i_i} \det A(i_s)} \leq \frac{\det A[i_1, i_2, \ldots, i_{s-1}, i_s]}{a_{i_i i_i} \det A[i_1, i_2, \ldots, i_{s-1}]}
\leq 1 - \frac{a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_i i_i}}{a_{i_i i_i} a_{i_2 i_2} \cdots a_{i_s i_s}} (\text{by the inequality (5)}).
\]
This completes the proof. \(\square\)

**Corollary 3.1.** Let \(B = (b_{ij}) \in M_n^{-1}\), then

(a) All of the principal submatrices of \(B\) are \(W\)-matrices.
(b) \(\forall \varepsilon > 0.\) If \(B\) is partitioned as \(B = \begin{pmatrix} B_{n-1} & B_{12} \\ B_{12} & B_{nn} - \det B / \det B_{n-1} + \varepsilon \end{pmatrix} \in W_n.\)

**Proof.** Let \(i_1, i_2, \ldots, i_s \in N\) be \(s\) distinct indices, \(s \geq 2.\) Theorem 3.1 yields
\[
0 < \frac{\det B}{b_{i_i i_i} \det B(i_s)} \leq 1 - \frac{b_{i_1 i_2} \cdots b_{i_{s-1} i_s} b_{i_i i_i}}{b_{i_1 i_1} b_{i_2 i_2} \cdots b_{i_s i_s}}.
\]
Hence
\[
b_{i_1 i_1} b_{i_2 i_2} \cdots b_{i_s i_s} > b_{i_1 i_1} \cdots b_{i_{s-1} i_s} b_{i_i i_i} \geq 0,
\]
which proves (a).
Now we take \(i_s = n.\) According to (6), we can easily obtain
\[
b_{i_1 i_1} b_{i_2 i_2} \cdots b_{i_{s-1} i_s} (b_{nn} - \det B / \det B_{n-1}) \geq b_{i_1 i_1} \cdots b_{i_{s-1} i_s} b_{i_i i_i} \geq 0.
\]
Therefore
\[
b_{i_1 i_1} b_{i_2 i_2} \cdots b_{i_{s-1} i_s} (b_{nn} - \det B / \det B_{n-1} + \varepsilon) > b_{i_1 i_1} \cdots b_{i_{s-1} i_s} b_{i_i i_i} \geq 0.
\]
By the definition of \(W\)-matrix, we claim that (b) is valid. \(\square\)

**Theorem 3.2.** If \(A = (a_{ij}) \in M_n \cup M_n^{-1}\), then for any permutation \(i_1, i_2, \ldots, i_n\) of \(N,\)

(a) \(\det A \leq \left(\prod_{i=1}^n a_{ii}\right) \prod_{s=2}^n \left(1 - \frac{|a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_i i_i}|}{|a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}|}\right).\)
(b) \(\det A = \prod_{i=1}^n a_{ii}\) if and only if \(A\) is essentially triangular.

**Proof.** For \(s = 2, 3, \ldots, n,\) we deduce by Theorem 3.1 that
\[
\frac{\det A[i_1, \ldots, i_{s-1} i_s]}{a_{i_i i_i} \det A[i_1, \ldots, i_{s-1}]} \leq 1 - \frac{|a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_i i_i}|}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}}.
\]
Theorem 3.3. If

$$\prod_{s=2}^{n} \frac{\det A[i_1, \ldots, i_{s-1}i_s]}{a_{i_s} a_{i_s}} \leq \prod_{s=2}^{n} \left( 1 - \frac{|a_{i_2} \cdots a_{i_{s-1}i_s} a_{i_1}|}{a_{i_1} a_{i_2} \cdots a_{i_{s-1}i_s}} \right),$$

which yields

$$\det A \leq \left( \prod_{i=1}^{n} a_{ii} \right) \prod_{s=2}^{n} \left( 1 - \frac{|a_{i_2} \cdots a_{i_{s-1}i_s} a_{i_1}|}{a_{i_1} a_{i_2} \cdots a_{i_{s-1}i_s}} \right). \quad (7)$$

Recall that a square matrix $B$ is called essentially triangular if $PBP^{-1}$ is triangular for some permutation matrix $P$. Using Frobenius normal form of $A$ [13, Theorem 3.2.4], it follows that $A$ is essentially triangular if and only if for any indices $i_1, i_2, \ldots, i_s$ different from each other in $N$

$$a_{i_1 i_2} \cdots a_{i_{s-1}i_s} a_{i_1} = 0.$$ 

By (7), we claim that $\det A = \prod_{i=1}^{n} a_{ii}$ holds if and only if $A$ is essentially triangular. This completes the proof. □

Below we establish lower and upper bounds for the determinant of the Hadamard product of an $M$-matrix and another inverse $M$-matrix with the same size.

**Theorem 3.3.** If $A = (a_{ij}) \in M_n$, $B = (b_{ij}) \in M_n^{-1}$, then $A \circ B \in M_n$, and for any permutation $i_1, i_2, \ldots, i_n$ of $N$,

$$\det(A \circ B) \geq \det(AB) \prod_{s=2}^{n} \left( \frac{\det A[i_1, \ldots, i_{s-1}i_s]}{\det A[i_1, \ldots, i_{s-1}, i_s]} + \frac{b_{i_s} \det B[i_1, \ldots, i_{s-1}, i_s]}{\det B[i_1, \ldots, i_{s-1}, i_s]} - 1 \right). \quad (8)$$

and

$$\det(A \circ B) \leq \left( \prod_{i=1}^{n} a_{ii} b_{ii} \right) \prod_{s=2}^{n} \left( 1 - \frac{|a_{i_2} \cdots a_{i_{s-1}i_s} a_{i_1}|}{a_{i_1} a_{i_2} \cdots a_{i_{s-1}i_s}} \right) b_{i_1 i_2} \cdots b_{i_s} \cdots b_{i_s} \cdots b_{i_1} \cdots b_{i_s}. \quad (9)$$

**Proof.**  $\forall k \in N$, we have $A_k \in M_k$ and $B_k \in W_k$ by Corollary 3.1(a). Since $B_k \geq 0$, Lemma 2.6 yields that $A_k \circ B_k \in M_k$.

To prove (8), according to Lemma 2.8, we need only to prove the following inequality:

$$\det(A \circ B) \geq \det(AB) \prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right). \quad (10)$$

In fact, it is not difficult to verify that (10) holds with equality for $n = 2$. Now we assume that $n > 2$. For $k = 2, 3, \ldots, n$, we partition $A_k$ and $B_k$ as

$$A_k = \begin{pmatrix} A^{(k)}_{11} & A^{(k)}_{12} \\ A^{(k)}_{21} & a_{kk} \end{pmatrix}, \quad B_k = \begin{pmatrix} B^{(k)}_{11} & B^{(k)}_{12} \\ B^{(k)}_{21} & b_{kk} \end{pmatrix}.$$ 

$\forall \varepsilon > 0$, Lemma 2.5, Corollary 3.1(b) and Lemma 2.6 imply that

$$\begin{pmatrix} A^{(k)}_{11} - \varepsilon & A^{(k)}_{12} \\ A^{(k)}_{21} & a_{kk} - \det A_k \det A_{k-1} + \varepsilon \end{pmatrix} \circ \begin{pmatrix} B^{(k)}_{11} & B^{(k)}_{12} \\ B^{(k)}_{21} & b_{kk} - \det B_k \det B_{k-1} + \varepsilon \end{pmatrix} \in M_k.$$
By Lemma 2.4, we have

\[
(a_{kk} - \frac{\det A_k}{\det A_{k-1}} + \varepsilon) \left( b_{kk} - \frac{\det B_k}{\det B_{k-1}} + \varepsilon \right) > a_{kk}b_{kk} - \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})}.
\]

Letting \( \varepsilon \to 0 \), we obtain

\[
(a_{kk} - \frac{\det A_k}{\det A_{k-1}}) \left( b_{kk} - \frac{\det B_k}{\det B_{k-1}} \right) \geq a_{kk}b_{kk} - \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})}.
\]

From this we can get

\[
\frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})} \geq \frac{\det(A_k B_k)}{\det(A_{k-1} B_{k-1})} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right).
\]

Multiplying these inequalities

\[
\prod_{k=2}^{n} \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})} \geq \prod_{k=2}^{n} \frac{\det(A_k B_k)}{\det(A_{k-1} B_{k-1})} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right).
\]

This means that (10) is valid.

Taking into account that \( A \circ B \in M_n \), (9) is an immediate consequence of Theorem 3.2(a). The proof is complete. \( \square \)

**Corollary 3.2.** If \( A = (a_{ij}) \in M_n \), \( B = (b_{ij}) \in M_n^{-1} \), then

\[
\det(A \circ B) + \det(AB) \geq \det A \prod_{i=1}^{n} b_{ii} + \det B \prod_{i=1}^{n} a_{ii}. \tag{11}
\]

**Proof.** Obviously, (11) is equivalent to

\[
\det(A \circ B) \geq \det(AB) \left( \prod_{i=1}^{n} a_{ii} \frac{\det A}{\det B} + \prod_{i=1}^{n} b_{ii} \frac{\det B}{\det A} - 1 \right). \tag{12}
\]

By the inequality (8), (12) follows from the following inequality:

If both \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are HF-matrices of order \( n \), then

\[
\prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right) \geq \prod_{i=1}^{n} a_{ii} \frac{\det A}{\det B} + \prod_{i=1}^{n} b_{ii} \frac{\det B}{\det A} - 1 + \varepsilon_n(A, B), \tag{13}
\]

where

\[
\varepsilon_n(A, B) = \sum_{k=2}^{n} \left[ \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} - 1 \right) \left( \prod_{i=1}^{k-1} b_{ii} \frac{\det B_{k-1}}{\det B_k} - 1 \right) 
+ \left( \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right) \left( \prod_{i=1}^{k-1} a_{ii} \frac{\det A_{k-1}}{\det A_k} - 1 \right) \right] \geq 0.
\]

We prove it by induction on \( n \). It is easy to see that (13) is true with equality for \( n = 2 \).

Now assume that \( n > 2 \) and (13) is true for the case \( n - 1 \), then the induction hypothesis and our assumption yield the chain of inequalities
\[
\prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right) \\
\geq \left( \frac{a_{nn} \det A_{n-1}}{\det A} + \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right) \\
\times \left[ \left( \prod_{i=1}^{n-1} \frac{a_{ii} \det A_{i-1}}{\det A_{i-1}} + \prod_{i=1}^{n-1} \frac{b_{ii} \det B_{i-1}}{\det B_{i-1}} - 1 \right) + \varepsilon_{n-1}(A_{n-1}, B_{n-1}) \right] \\
\geq \left( \frac{a_{nn} \det A_{n-1}}{\det A} + \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right) \left( \prod_{i=1}^{n-1} \frac{a_{ii} \det A_{i-1}}{\det A_{i-1}} + \prod_{i=1}^{n-1} \frac{b_{ii} \det B_{i-1}}{\det B_{i-1}} - 1 \right) + \varepsilon_{n-1}(A_{n-1}, B_{n-1}) \\
= \prod_{i=1}^{n} a_{ii} + \prod_{i=1}^{n} b_{ii} + \prod_{i=1}^{n-1} \frac{a_{nn} \det A_{n-1}}{\det A} \left( \frac{a_{nn} \det A_{n-1}}{\det A} - 1 \right) + \prod_{i=1}^{n-1} \frac{b_{nn} \det B_{n-1}}{\det B} \left( \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right) \\
- \left[ \left( \frac{a_{nn} \det A_{n-1}}{\det A} - 1 \right) + \left( \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right) + 1 \right] + \varepsilon_{n-1}(A_{n-1}, B_{n-1}) \\
= \prod_{i=1}^{n} a_{ii} + \prod_{i=1}^{n} b_{ii} - 1 + \left( \frac{a_{nn} \det A_{n-1}}{\det A} - 1 \right) \left( \prod_{i=1}^{n-1} \frac{b_{ii} \det B_{i-1}}{\det B_{i-1}} - 1 \right) \\
+ \left( \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right) \left( \prod_{i=1}^{n-1} \frac{a_{ii} \det A_{i-1}}{\det A_{i-1}} - 1 \right) + \varepsilon_{n-1}(A_{n-1}, B_{n-1}) \\
= \prod_{i=1}^{n} a_{ii} + \prod_{i=1}^{n} b_{ii} - 1 + \varepsilon_{n}(A, B).
\]

This completes the induction. □

References