The Vertex Decomposable Property of Graphs Dual to $r$-Partite

Saba Yasmeen† and Tongsuo Wu‡

School of Mathematical Sciences, Shanghai Jiao Tong University

Abstract. Let $G$ be a non-complete graph such that its complement $\overline{G}$ is $r$-partite. In this paper, properties of the graph $G$, including the unmixed property and the sequentially Cohen-Macaulay property are studied. Some sufficient conditions for $G$ are given such that it is vertex decomposable, and connectedness of $\text{Ind}(G)$ is also discussed.

Key Words: vertex decomposable; Cohen-Macaulay; unmixedness; graphs dual to $r$-partite; connected simplicial complex

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1. Introduction

Simplicial complex $\Delta$ is a classical object in combinatorial commutative algebra. Every simplicial complex corresponds to monomial ideals, e.g., facet ideal $I(\Delta)$ and Stanley-Reisner ideal $I_\Delta$. If $\Delta$ is pure and vertex decomposable then $\Delta$ is pure and shellable, and $I_\Delta$ is Cohen-Macauley. Motivation of this research comes from the following results:

**Theorem 1.1.** Let $G$ be a bipartite graph with a vertex partition $V(G) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$. Then the following statements are equivalent:

1. $G$ is unmixed and vertex decomposable.

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†Corresponding author. saba@sjtu.edu.cn sabayasmin84@gmail.com
‡tswu@sjtu.edu.cn wutsc@online.sh.cn
(2) $G$ is unmixed and shellable.
(3) $G$ is unmixed and constructible.
(4) $G$ is Cohen-Macaulay.
(5) ([9]) $n = m$, and there is a labeling such that
(a) $\{x_i, y_i\} \in E(G)$ for each $i$;
(b) $\{x_i, y_j\} \in E(G)$ implies $i \leq j$; and
(c) for $i < j < k$, $\{x_i, y_j\} \in E(G)$ and $\{x_j, y_k\} \in E(G)$ imply $\{x_i, y_k\} \in E(G)$.

Note that the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are well-known; the equivalence of $(4)$ with $(5)$ is the classical [9, Theorem 3.4]. Our observation is $(5) \Rightarrow (1)$; note that condition (b) implies that $y_n$ is a weak shedding vertex, and one can use mathematical induction to conclude that both $G \setminus y_n$ and $G \setminus N_G[y_n]$ are vertex decomposable. Thus $G$ is unmixed and vertex decomposable.

**Proposition 1.2.** Let $\Delta$ be a simplicial complex of dimension 1. Then the following statements are equivalent:

1. $\Delta$ is pure and vertex decomposable.
2. $\Delta$ is pure and shellable.
3. $\Delta$ is pure constructible.
4. $\Delta$ is Cohen-Macaulay.
5. $\Delta$ is connected.

Again, the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are well-known. For the equivalence of $(4)$ and $(5)$, one refers to [13, Corollary 6.3.14]. For proving the implication $(5) \Rightarrow (1)$, one uses mathematical induction on the number of edges in $\Delta$.

As a corollary of Proposition 1.2 it is known that for a graph $G$ whose $\overline{G}$ is bipartite graph, $G$ is unmixed and vertex decomposable if and only if $\overline{G}$ is connected as a graph. This paper studies properties of graph $G$ whose $\overline{G}$ is $r$-partite for $r \geq 3$. This paper is organized as following. In section 2 some preliminaries on both graph theory and combinatorics are recalled and in section 3 the vertex decomposable property dual to the $r$-partite is discussed. In section 4 it is analyzed when $\text{Ind}(G)$ is unmixed and section 5 is devoted for the connectedness of $\text{Ind}(G)$.

2. Preliminaries

In this section, some relevant definitions and results on graphs and simplicial complexes are introduced that are commonly applied in combinatorial commutative algebra.
Recall that a simplicial complex $\Delta$ on the vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that if $F \in \Delta$ and $E \subseteq F$, then $E \in \Delta$. Each $F$ in $\Delta$ is called a face, and a facet $F$ is a maximal face with respect to inclusion. A simplicial complex $\Delta$ is called pure if all facets have the same cardinality. The set of all facets of $\Delta$ is denoted by $\mathcal{F}(\Delta)$, and if $\mathcal{F}(\Delta) = \{F_1, F_2, \ldots, F_t\}$, then $\Delta$ can be written as $\Delta = \langle F_1, F_2, \ldots, F_t \rangle$.

The dimension of a face $F$ is $\dim F = |F| - 1$, and the dimension of $\Delta$ is defined by $\dim \Delta = \max \{\dim F \mid F \in \Delta\}$. Recall that the Alexander dual $\Delta^\vee$ of a simplicial complex $\Delta$ is defined by $\Delta^\vee = \{[n] \setminus F \mid F \not\in \Delta\}$.

For a face $H$ of a simplicial complex $\Delta$, recall the following notations, namely deletion and link:

- $\Delta \setminus H =: \{F \in \Delta \mid H \cap F = \emptyset\}$,
- $\text{lk}_\Delta(H) =: \{F \in \Delta \mid H \cap F = \emptyset, F \cup H \in \Delta\}$.

Note that whenever $H = \{x\}$ is a vertex, the notations are usually written as $\Delta \setminus x$ and $\text{lk}_\Delta(x)$ respectively. Recall the following concept of a vertex decomposable simplicial complex, which is introduced by Provan and Billera [11] in pure case and extended to the nonpure case by Björner and Wachs [3]:

**Definition 2.1.** A simplicial complex $\Delta$ over $[n]$ is called vertex decomposable, if one of the following inductive conditions is satisfied:

1. $\Delta$ is a simplex, or
2. there is a vertex $x$ such that the following requirements are satisfied:
   - $(\alpha)$ Both $\Delta \setminus x$ and $\text{lk}_\Delta(x)$ are vertex decomposable, and
   - $(\beta)$ No facet of $\text{lk}_\Delta(x)$ is a facet of $\Delta \setminus x$, or equivalently, $\Delta \setminus x = \langle \{F \mid x \not\in F \in \mathcal{F}(\Delta)\} \rangle$.

A vertex $x$ satisfying conditions $(\alpha)$ and $(\beta)$ is called a shedding vertex of $\Delta$. If $x$ only satisfies condition $(\beta)$, then it is called a weak shedding vertex.

It is well known that a vertex cover of a graph $G$ is a subset $C$ of the vertex set $V(G)$ such that $C \cap \{i, j\} \neq \emptyset$ holds for all $\{i, j\} \in E(G)$. A vertex cover is called minimal if no proper subset of it is a vertex cover. An independent vertex set $I$ of graph $G$ is a subset of $V(G)$ such that there is no edge between any vertices in $I$. An independent vertex set is called maximal if there is no independent vertex set of $G$ containing it. Clearly, a subset $C$ of $V(G)$ is a minimal vertex cover of $G$ if and only if $V(G) \setminus C$ is a maximal independent vertex subset of $G$. Recall also that a graph $G$ is said to be unmixed (or alternatively, well-covered), if all minimal vertex covers of $G$ have the same cardinality. For a graph $G$, the cluster of all maximal independent vertex sets is a
simplicial complex, which is called the independent simplicial complex of graph $G$ and is denoted by $\text{Ind}(G)$.

A translation in the language of graphs is restated in the following.

**Definition 2.2.** A graph $G$ is called vertex decomposable if either it has no edges, or else has some vertex $x$ such that we have as follows:

1. Both graphs $G \setminus N_G[x]$ and $G \setminus x$ are vertex decomposable, where $N_G[x]$ is the union of the neighbourhood $N_G(x)$ together with $\{x\}$;
2. For every independent vertex set $S$ in $G \setminus N_G[x]$, there exists some $y \in N_G(x)$ such that $S \cup \{y\}$ is independent in $G \setminus x$.

A vertex $x$ with above properties is called a shedding vertex of $G$; a vertex with the second property is called a weak shedding vertex.

Vertex decomposable simplicial complexes and graphs are important in combinatorial commutative algebra and combinatorial topology because they provide examples of shellable simplicial complexes, while unmixed vertex decomposable graphs are further Cohen-Macaulay, refer to [7, 14, 8, 1, 5, 6, 10] for related studies. It is known that each matroid is unmixed and vertex decomposable ([12]); furthermore, each vertex is a shedding vertex. A chordal graph is also vertex decomposable. For each graph $G$, a class of constructed clique-whiskered graph $G^\pi$ is unmixed and vertex decomposable ([4]). As a generalization, a construction $\Delta_\chi$ is provided for each simplicial complex $\Delta$ and any $s$-coloring $\chi$ on $\Delta$, and it is proved that all $\Delta_\chi$ are unmixed and vertex decomposable simplicial complexes ([2]).

Here we recall the definition of connected simplicial complex from [8, pg 16]

**Definition 2.3.** A simplicial complex $\Delta$ is said to be connected if for any two facets $F$ and $H$ of $\Delta$, there exists a sequence of facets $F = F_0, F_1, \ldots, F_t = H$ such that $F_i \cap F_{i+1} \neq \emptyset$, for any $i \in \{0, 1, 2, \ldots, t - 1\}$. A disconnected simplicial complex is a complex which is not connected.

We begin with the following observation on graphs dual to bipartite:

**Lemma 3.1.** Let $G$ be a non-complete graph dual to bipartite. Then $G$ is not unmixed if and only if there is a clone vertex in $G$, i.e., a vertex which is adjacent to all other vertices.

**3. Vertex decomposable graphs dual to $r$-partite ($r = 2, 3$)**

We begin with the following observation on graphs dual to bipartite:

**Lemma 3.1.** Let $G$ be a non-complete graph dual to bipartite. Then $G$ is not unmixed if and only if there is a clone vertex in $G$, i.e., a vertex which is adjacent to all other vertices.
Proof. Note that \( \dim \Delta_G = 1 \) and that \( \overline{G} \) is a bipartite graph, thus \( G \) is not unmixed if and only if \( \Delta_G \) has isolated vertices, and the latter holds true if and only if there exists a clone vertex in \( G \).

**Lemma 3.2.** Let \( G \) be any vertex decomposable graph, and \( v \) a clone vertex in \( G \). Then \( v \) is a shelling vertex of \( G \).

**Proof.** Since \( v \) is a clone vertex, we have \( G \setminus N_G[v] = \emptyset \), thus \( v \) is a weak shelling vertex of \( G \); thus in order to show that \( v \) is a shelling vertex of \( G \), it is only necessary to show that \( G \setminus v \) is vertex decomposable. For this, we use induction on \( |V(G)| \).

Let \( w \) be any shelling vertex of \( G \) with \( w \neq v \); then both \( G_1 =: G \setminus N_G[w] \) and \( G_2 =: G \setminus w \) are vertex decomposable, and for any independent subset \( S \) of \( V(G_1) \), there exists a vertex \( y \in N_G(w) \) such that \( S \cup \{y\} \) is independent in \( G_2 \). Clearly, \( y \neq v \) whenever \( S \) is nonempty; hence it follows from

\[
(G \setminus v) \setminus w = (G \setminus w) \setminus v, \quad G \setminus N_G[w] = (G \setminus v) \setminus N_G \setminus v[w]
\]

and induction on the vertex decomposable graph \( G \setminus w \) that \( w \) is a shelling vertex of the graph \( G \setminus v \), hence \( G \setminus v \) is vertex decomposable.

**Theorem 3.3.** Let \( G \) be a non-complete graph dual to bipartite. Then the following statements are equivalent:

1. \( G \) is vertex decomposable;
2. The following bi-vd condition is fulfilled in the graph \( G \):

There exists an integer \( r \geq 0 \) and a possible sequence of vertices \( v_1, \ldots, v_r \), such that each \( v_i \) is a clone vertex of \( G \setminus \{v_1, \ldots, v_i-1\} \) for \( i \leq r \), and the complement graph of \( G \setminus \{v_1, \ldots, v_r\} \) is connected.

**Proof.** First, note that for a graph dual to bipartite, each clone vertex is a weak shedding vertex.

\( \Rightarrow \) Assume that \( G \) is vertex decomposable. If \( G \) has no clone vertex, then \( \overline{G} \) is connected by Lemma 3.1 and Proposition 1.2. If \( v_1 \) is a clone vertex, then by Lemma 3.2, \( G \setminus v_1 \) is vertex decomposable dual to bipartite; then we use induction on the number of vertices to complete the proof.

\( \Leftarrow \) By the assumption and Proposition 1.2, the graph \( G \setminus \{v_1, \ldots, v_r\} \) is vertex decomposable. Then \( v_r \) is a shedding vertex of \( G \setminus \{v_1, \ldots, v_{r-1}\} \), thus \( G \setminus \{v_1, \ldots, v_{r-1}\} \) is vertex decomposable. Hence, by induction, \( G \) is vertex decomposable.

In the following two corollaries, we assume that \( G \) consists of two cliques \( A \) and \( B \), with some possible edges in between.
Corollary 3.4. If there exists a vertex \( w \) in \( B \) such that \( w \) is not adjacent to any vertex of \( A \), then \( G \) is vertex decomposable.

Proof. We use induction on the number of clone vertices of \( G \). If \( G \) contains no clone vertex, then \( G \) is connected since \( A \subseteq N_G(w) \), thus \( G \) is vertex decomposable by Proposition 1.2. If \( G \) contains a clone vertex \( v_1 \), then \( v_1 \in B \) holds. Thus \( G \setminus v_1 \) satisfies the assumed condition and therefore, it is vertex decomposable by induction assumption. Thus \( G \) is vertex decomposable.

Corollary 3.5. Let \( G \) be a non-complete graph consisting of two cliques \( A \) and \( B \), with some additional edges in between, such that \( |B| \geq 3 \). Assume further that there exists a proper edge \( e \) such that \( V(e) \cap V(h) = \emptyset \) holds for other proper edge \( h \). If there exists a vertex \( w \in B \setminus V(e) \) such that \( w \) is a vertex of exactly one proper edge, then \( G \) is vertex decomposable and unmixed.

Proof. Clearly, there is no clone vertex in \( G \), thus \( G \) is unmixed. Assuming further \( e = \{x, y\}, h = \{w, u\} \), where \( \{x, w\} \subseteq B, \{y, u\} \subseteq A \). We claim that \( G \) is connected since in \( G \) we have the following edges:

\[
\{x, u\}, \{w, y\}, \{w, a\}, \{y, b\}, \forall a \in A \setminus \{u\}, \forall b \in B \setminus \{x\}.
\]

Theorem 3.3 should serve as the starting point to combine with the following general construction of vertex decomposable graphs:

Theorem 3.6. Let \( H_i \) be an induced subgraph of a graph \( G_i \), where \( V(G_1) \cap V(G_2) = \emptyset \); and let \( u, w_1, \ldots, w_r \) be some additional vertices. Let \( L \) be an enlarged graph such that each \( G_i \) is an induced subgraph of \( L \), and

\[
\{\{u, w_i\}, \{u, h\} \mid 1 \leq i \leq r, h \in V(G_i \setminus H_i) (i = 1, 2)\} \subseteq E(L).
\]

If each \( H_i \) and \( G_i \) are vertex decomposable, then \( L \) is also vertex decomposable.

Proof. Note that \( L \setminus N_L(u) = H_1 \cup H_2 \), and \( L \setminus u \) is identical with \( G_1 \cup G_2 \) together with some isolated vertices \( w_i \); and any independent subset \( S \) of \( V(H_1 \cup H_2) \) can be enlarged to an independent vertex set \( S \cup \{w_1\} \), thus \( u \) is a shedding vertex of \( L \), and \( L \) is vertex decomposable.

We illustrate this combination in the following simple example, in which the graph \( G_2 \) is empty.
Proof. Note that \( L \setminus N_L(u) = H \), and \( L \setminus u \) is identical with \( G \) together with some isolated vertices \( w_i \); and any independent subset \( S \) of \( V(H) \) can be enlarged to an independent vertex set \( S \cup \{w_1\} \), thus \( u \) is a shedding vertex of \( L \). \( \square \)

In the remaining part of this section, a graph \( G \) is always assumed to be dual to tripartite. In order to study such kind of graphs, we introduce the following:

**Definition 3.7.** Let \( G \) be a graph dual to 3-partite. Then \( G \) is called quasi-dual-connected if it satisfies the following inductive conditions: either \( G \) is a discrete graph or there is a vertex \( v \) such that the following four conditions are fulfilled:

1. The graph \( G \setminus N_G[v] \) satisfies the bi-vd condition;
2. The graph \( G \setminus v \) either is quasi-dual-connected or, is bipartite and satisfies the bi-vd condition, or it is a singleton;
3. For each vertex \( w \) in \( G \setminus N_G[v] \), there exists a vertex \( w' \in N_G(v) \) which is not adjacent to \( w \) in \( G \);
4. For each pair of vertices \( w_1, w_2 \) nonadjacent in \( G \setminus N_G[v] \), there exists a vertex \( w_3 \) in the third part, such that \( w_1, w_2, w_3 \) is independent in \( G \).

**Theorem 3.8.** Let \( G \) be a graph dual to 3-partite. Then \( G \) is vertex decomposable if and only if it is quasi-dual-connected.

Proof. Let \( G \) be a non-complete graph consisting of 3 cliques, together with some additional edges.

\( \implies \) If \( G \) is vertex decomposable, let \( v \) be a shedding vertex. If \( G \setminus v \) is 3-partite, then it is quasi-dual-connected; if it is 2-partite, then it satisfies the bi-vd condition by Theorem 3.3; or else, it is a single vertex and thus satisfies the bi-vd condition too. This shows that condition (2) in Definition 3.7 is fulfilled; while condition (1) is fulfilled for the same reason. The conditions (3) and (4) follows from the fact that each independent set \( S \) of \( G \setminus N_G[v] \) can be enlarged to \( G \setminus v \).

\( \iff \) Assume that \( G \) is quasi-dual-connected; then it follows from conditions 3 and 4 that for any independent set \( S \) of \( G \setminus N_G[v] \), there exists \( y \in N_G(v) \) such that \( S \cup \{y\} \) is
an independent set in \( G \setminus v \). From condition (1), it is known that \( G \setminus N_G[v] \) is either connected or a singleton, so it is vertex decomposable by Proposition 1.2. Also, if the graph \( G \setminus v \) is quasi-dual-connected, then \( G \setminus v \) is vertex decomposable by induction; if the graph \( G \setminus v \) is bipartite and satisfies the bi-vd condition, then it is vertex decomposable by Theorem 3.3; if the graph \( G \setminus v \) is a singleton, surely it is vertex decomposable. This shows that \( G \) is vertex decomposable and complete the verification. \( \square \)

In the end of this section, we use Theorem 3.8 to construct a series of vertex decomposable graphs dual to 3-partite.

**Example 3.9.** Let \( G_1 \) be any vertex decomposable graph dual to 2-partite, with parts \( A_1, B_1 \).

1. Take any additional \( u_{11}, w_{1i} \), where \( 1 \leq i \leq r_1 \). Let \( G_2 \) be a graph on the vertex set \( V(G_1) \cup \{u_{11}, w_{11}, \ldots, w_{1r_1}\} \). Let \( E(G_1) \cup \bigcup_{1 \leq j \leq r_1} \{u_{11}, w_{1j}\} \subseteq E(G_2) \), and add some edges between \( u_{11} \) and vertices of \( G_1 \), in such a way that makes \( G_2 \setminus N_{G_2}[u_{11}] \) a vertex decomposable graph. Then \( G_2 \) is a vertex decomposable graph dual to 3-partite, with parts
   \[
   A_2 = A_1 \cup \{w_{11}, \ldots, w_{1r_1}\}, B_2 = B_1, C_2 = \{u_{11}\}.
   \]

2. Take any additional \( u_{21}, w_{2i} \), where \( 1 \leq i \leq r_2 \). Let \( G_3 \) be a graph on the vertex set \( V(G_2) \cup \{u_{21}, w_{21}, \ldots, w_{2r_2}\} \). Let the edge sets be \( E(G_2) \cup \bigcup_{1 \leq j \leq r_2} \{u_{21}, w_{2j}\} \cup u_{21}, u_{11} \). Then \( G_3 \) is a vertex decomposable graph dual to 3-partite, with parts
   \[
   A_3 = A_2 \cup \{u_{21}\}, B_3 = B_2 \cup \{w_{21}, \ldots, w_{2r_2}\}, C_3 = \{u_{11}\}.
   \]

3. Take any additional \( u_{31}, w_{3i} \), where \( 1 \leq i \leq r_3 \). Let \( G_4 \) be a graph on the vertex set \( V(G_3) \cup \{u_{31}, w_{31}, \ldots, w_{3r_3}\} \). Let the edge sets be \( E(G_3) \cup \bigcup_{1 \leq j \leq r_3} \{u_{31}, w_{3j}\} \cup u_{31}, u_{21} \). Then \( G_4 \) is a vertex decomposable graph dual to 3-partite, with parts
   \[
   A_4 = A_3, B_4 = B_3 \cup \{u_{31}\}, C_3 = \{u_{11}, \ldots, w_{3r_3}\}.
   \]

\( \ldots \)

Refer to Figure 3.1 for an illustration.

### 4. Vertex Decomposable Graphs Dual to \( r \)-Partite

Throughout, all graphs are assumed to be non-complete, and consisting of several finite cliques with some additional edges in between. These additional edges are called proper edges, i.e., a proper edge means that the related two vertices belong to distinct cliques. If a vertex of a proper edge does not belong to another proper edge, then this vertex is called a rigid vertex. If a proper edge has both rigid vertices, then this edge
is called a rigid edge. If a vertex is not used in any proper edge, it is called an improper vertex. A clone vertex in a graph is a vertex which is adjacent to every other vertex.

**Proposition 4.1.** Let $G$ be a non-complete graph consisting of $r$ cliques $A_i (1 \leq i \leq r)$, together with some additional edges. Assume further that there exist $r-1$ cliques such that each has an improper vertex. Then the graph $G$ is vertex decomposable.

**Proof.** We use mathematical induction on $r$. For $r = 2$, it is vertex decomposable by Proposition 1.2. Now assume $r \geq 3$, and suppose it is true for the $r-1$ case. Assume further that each of $A_2, \ldots, A_r$ has an improper vertex. For $r$, take a vertex $v \in A_r$ such that $v$ is not an improper vertex. Note that $G \setminus N_G[v]$ is a non-complete graph consisting of $r-1$ cliques, having $r-2$ improper vertices in distinct cliques together with some additional edges, so it is vertex decomposable. While $G \setminus v$ consists of $r$ cliques with $r-1$ improper vertices. Furthermore, for any independent vertex set $M$ of the graph $G \setminus N_G[v]$, there is an improper vertex $w \in N(v)$ such that $M \cup \{w\}$ is an independent set of $G \setminus v$. Note that $G \setminus v$ shares the same property with $G$, thus mathematical induction can be applied to conclude that $G \setminus A_r$ is a non-complete graph consisting of $r-1$ cliques with some additional edges, of which $r-2$ cliques have improper vertex each, so it is vertex decomposable. □

**Theorem 4.2.** Let $G$ be a non-complete graph consisting of $r$ cliques $A_1, A_2, \ldots, A_r$ with some additional edges. Assume further that there is one improper vertex $w \in A_1$, and there exists a rigid edge from $A_1$ to each $A_i$ ($2 \leq i \leq r$). Then the graph $G$ is vertex decomposable.

**Proof.** Take any vertex $v \in A_1$ such that $v$ is a vertex of rigid edge. Clearly, $G \setminus N_G[v]$ is a non-complete graph consisting of $(r-1)$ cliques of which $(r-2)$ cliques have an improper vertex each, so it is vertex decomposable by proposition 4.1. While $G \setminus v$ consists of $r$ cliques with some additional edges, in which there are two cliques having improper vertex each. Furthermore, for any independent vertex set $M$ of $G \setminus N_G[v]$, there exists $w \in N_G(v)$ such that $M \cup \{w\}$ is an independent vertex set of $G \setminus v$. Then we use mathematical induction on $A_1$ to conclude that $G \setminus A_1$ is a graph consisting of $(r-1)$ cliques and each clique has an improper vertex, so it is vertex decomposable by [4, Lemma 3.2, Theorem 3.3]. □

**Proposition 4.3.** Let $G$ be a non-complete graph consisting of cliques $A_i (1 \leq i \leq r)$, with some additional edges in between. Set

$$I = \{A_i \mid 1 \leq i \leq r, and \ A_i \ contains \ improper \ vertices \ of \ G \} , \ P = \{A_1, \ldots, A_r\} \setminus I$$

and assume $|I| = m \geq 1, |P| = n \geq 1$. Then $G$ is vertex decomposable if one of the following conditions is satisfied:
(1) There exists an element \( A \in \mathcal{I} \) and a subset \( S \) of \( \mathcal{P} \) with \( |S| = n - 1 \), such that for each \( B \in S \), there is a rigid edge of \( G \) from \( A \) to \( B \), and there is a vertex \( v \) in \( A \) which is adjacent to a rigid vertex of \( B \) (\( \forall B \in S \)).

(2) There exist \( A \in \mathcal{I} \) and \( B \in \mathcal{P} \) and take \( v \in A, u \in B \) such that \( v \) is adjacent to \( u \), and every \( A_i \neq B \) from \( \mathcal{P} \) has two vertices rigid in \( G \), one of which is adjacent to \( v \) and the other is adjacent to \( u \).

**Proof.** For \( A \in \mathcal{I} \), there is an improper vertex \( w \in A \). Take a vertex \( v \in A \) and consider \( G \setminus N_G[v] \). Clearly, \( G \setminus N_G[v] \) is a non-complete graph consisting of \( r - 1 \) cliques with \( r - 2 \) improper vertices, so by Proposition 4.1 it is vertex decomposable. While \( G \setminus v \) consists of \( r \) cliques with \( r - 1 \) improper vertices, so it is vertex decomposable again by Proposition 4.1. For any independent vertex set \( M \) of \( G \setminus N_G[v] \), clearly there exists \( w \in N_G(v) \) such that \( M \cup \{w\} \) is an independent vertex set of \( G \setminus v \).

**Proposition 4.4.** Let \( G \) be a non-complete graph consisting of cliques \( A_i \) (\( 1 \leq i \leq r \)), with some additional edges in between. Assume that no \( A_i \) has an improper vertex in \( G \). Then

1. \( G \) is vertex decomposable if the following two conditions are fulfilled:
   a. There exist some clique \( A_1 \) and other \( r - 2 \) cliques \( A_j \) (\( 2 \leq j \leq r - 1 \)), such that there is a vertex \( v \) in \( A_1 \) which is adjacent to a rigid vertex of \( A_j \) (\( \forall 2 \leq j \leq r - 1 \)), and for each \( 2 \leq j \leq r - 1 \), there is an edge in \( E(A_1, A_j) \) which is rigid in the graph \( G \).
   b. There exist a number \( j \), where \( 2 \leq j \leq r - 1 \), such that \( E(A_1, A_j) \) contains one rigid edge in \( G \), and there exists a number \( t \), where \( t \neq j, 2 \leq t \leq r \), such that \( E(A_j, A_t) \) contains two edges rigid in \( G \).
   
   Or

2. \( G \) is vertex decomposable if the following two conditions are fulfilled:
   a. There exist some couple \( \{A_1, A_2\} \), and \( v \in A_1, u \in A_2 \) such that \( v \) is adjacent to \( u \), and every \( A_j(j \neq 1, 2) \) has two rigid vertices, one of which is adjacent to \( v \) and the other is adjacent to \( u \).
   b. There exists a number \( k \), where \( 2 \leq k \leq r \), such that \( E(A_1, A_k) \) contains three edges rigid in \( G \).

**Proof.** (1) Let \( A_j = A_2 \) and \( A_i = A_3 \); then \( e_1 = \{x_1, y_1\}, e_2 = \{x_2, y_2\} \) and \( e_3 = \{x_3, y_3\} \) are rigid edges, where \( x_1 \in A_1, \{y_1, x_2, x_3\} \subseteq A_2 \) and \( \{y_2, y_3\} \subseteq A_3 \). For the vertex \( y_1 \in A_2 \), the graph \( G \setminus N_G[y_1] \) consists of \( r - 1 \) cliques with an improper vertex in \( A_1 \), which satisfies condition (1) of Proposition 4.3, so it is vertex decomposable. While the graph \( G \setminus y_1 \) consists of \( r \) cliques with an improper vertex in \( A_1 \) that satisfies condition (1) of Proposition 4.3 also, so it is also vertex decomposable. For any independent vertex set \( M \) of \( G \setminus N_G[y_1] \), we have \( \{x_2, x_3\} \subseteq N_G(y_1) \), if \( M \cap V(e_2) = \emptyset \) holds. In this
case, \( M \cup \{x_2\} \) is an independent set of \( G \setminus y_1 \); if \( M \cap V(e_2) \neq \emptyset \) holds, then \( M \cup \{x_3\} \) is an independent vertex set of \( G \setminus y_1 \). This proves that \( G \) is vertex decomposable.

(2) Take \( A_k = A_2 \). Then there are three rigid edges, \( e_1 = \{x_1, y_1\}, e_2 = \{x_2, y_2\} \) and \( e_3 = \{x_3, y_3\} \), from \( A_1 \) to \( A_2 \), where \( x_i \in A_1 \) and \( y_i \in A_2 \) for \( i = \{1, 2, 3\} \). Take \( y_1 \in A_2 \) then \( G \setminus N_G[y_1] \) consists of \( r - 1 \) cliques and every clique has an improper vertex, so the graph \( G \setminus N_G[y_1] \) is vertex decomposable by [4, Lemma 3.2, Theorem 3.3]. The graph \( G \setminus y_1 \) consists of \( r \) cliques with an improper vertex \( x_1 \in A_1 \). So condition (2) of Proposition 4.3 is fulfilled and hence, \( G \setminus y_1 \) is vertex decomposable. Further, for any independent vertex set \( M \) of \( G \setminus N_G[y_1] \), we have \( \{y_2, y_3\} \subset N_G(y_1) \), if \( M \cap V(e_2) = \emptyset \) holds. Then \( M \cup \{y_2\} \) is an independent vertex set of \( G \setminus y_1 \); if \( M \cap V(e_2) \neq \emptyset \) holds, then \( M \cup \{y_3\} \) is an independent vertex set of \( G \setminus y_1 \). Note that when \( A_k = A_3 \), \( G \setminus N_G[y_1] \) consists of \( r - 1 \) cliques satisfying conditions (2) of Proposition 4.3, so it is vertex decomposable. This proves that \( G \) is vertex decomposable. This completes the proof. □

We have the following two remarks:

Remark 4.5. Condition \((b_1)\) can be replaced by any one of the following:

\((b_2)\) There exists a number \( j \), where \( 2 \leq j \leq r - 1 \), such that \( E(A_1, A_j) \) contains two edges rigid in \( G \).

\((b_3)\) There exist three edges in \( E(A_1, A_r) \), which are rigid in the graph \( G \).

\((b_4)\) There exist two rigid edges in \( E(A_1, A_r) \) and there exists a number \( t \), where \( 2 \leq t \leq r - 1 \), such that \( E(A_r, A_t) \) contains two rigid edges in \( G \).

Remark 4.6. Condition \((d_1)\) can be replaced by any one of the following:

\((d_2)\) There exists a number \( k \), where \( 3 \leq k \leq r \), such that \( E(A_1, A_k) \) contains two edges rigid in \( G \), and there exists a number \( x \) where \( x \neq k, 3 \leq x \leq r \), such that \( E(A_k, A_x) \) contains two edges rigid in \( G \).

\((d_3)\) There exists one rigid edge in \( E(A_1, A_2) \) and there exists a number \( k \), where \( 3 \leq k \leq r \), such that \( E(A_2, A_k) \) contains two edges rigid in \( G \).

In the following proposition, let \( (A_1, A_2, \ldots, A_r) \) be cyclic, i.e., we regard \( A_1 \) as \( A_{r+1} \).

Theorem 4.7. Let \( G \) be a non-complete graph with \( r \) cliques \( A_1, A_2, \ldots, A_r \) together with some additional edges in between. Then \( G \) is vertex decomposable if the following conditions are fulfilled:

1. There exist two disjoint cycles, each contains exactly one edge in \( E(A_i, A_{i+1}) \) (\( \forall 1 \leq i \leq r \)) which is rigid in \( G \).
(2) For each $1 \leq i \leq r$ and each $j \neq i - 1, i, i + 1$, $A_j$ has a rigid vertex adjacent to some vertex of $A_i$.

Proof. Let $G$ be a non-complete graph with above construction. There are two rigid edges $e_i = \{x_i, y_i\}$ from $A_r$ to $A_1$ ($i = 1, 2$), and two rigid edges $e_i = \{x_i, y_i\}$ for $i \in \{3, 4\}$ from $A_{r-1}$ to $A_r$. Assume $x_1 \in A_r$, and consider $G \setminus N_G[x_1]$ and $H =: G \setminus x_1$. Clearly, $G \setminus N_G[x_1]$ consists of $r - 1$ cliques, each of which has an improper vertex, so $G \setminus N_G[x_1]$ is vertex decomposable by $[4, \text{Lemma 3.2, Theorem 3.3}]$; while $G \setminus x_1$ consists of $r$ cliques with an improper vertex $y_1 \in A_1$. For any independent set $M$ of $G \setminus N_G[x_1]$, we have $\{y_3, y_4\} \subset N_G(x_1)$ such that $M \cup \{y_3\}$ is an independent set in $H$ if further $M \cap V(e_3) = \emptyset$ holds, and $M \cup \{y_1\}$ is an independent set of $G \setminus x_1$ if $M \cap V(e_3) \neq \emptyset$. This shows that $x_1$ is a weak shedding vertex of $G$, and we proceed to show that $G \setminus x_1$ is also vertex decomposable. For this, consider two rigid edges $e_i = \{x_i, y_i\}$ where $i = \{5, 6\}$ between $A_1$ to $A_2$, and assume $x_5 \in A_1$. Then $H \setminus N_H[x_5]$ consists of $r - 2$ cliques, each of which has an improper vertex, so it is vertex decomposable by $[4, \text{Lemma 3.2, Theorem 3.3}]$; while $H \setminus x_5$ consists of $r$ cliques with two improper vertices $y_1 \in A_1$ and $y_5 \in A_2$. For any independent set $M$ of $H \setminus N_H[x_5]$, there exists $y_1 \in N_H(x_5)$ such that $M \cup \{y_1\}$ is an independent set of $H \setminus x_5$.

Now apply mathematical induction on the path $A_2A_3, A_3A_4, \ldots, A_{r-1}A_r$. In the end, we get a graph consisting of $r$ cliques and each clique has improper vertex, so by $[4, \text{Lemma 3.2, Theorem 3.3}]$ it is vertex decomposable. □

Theorem 4.8. Let $G$ be a non-complete graph consisting of cliques $A_i$ $(1 \leq i \leq r)$, with some additional edges in between. Assume further that each $A_i$ contains no improper vertices in $G$. Then $G$ is vertex decomposable if the following conditions are fulfilled:

(1) For each $t$ where $3 \leq t \leq r$, there exists one rigid edge $e_i$ in $E(A_1, A_i)$ and two rigid edges $e_1, e_2$ in $E(A_1, A_2)$.

(2) There exist $a_i \in A_i$ $(i = 1, 2)$ such that neither is adjacent to a vertex of $A_j$ $(3 \leq j \leq r)$, and $a_i \cap V(e_j) = \emptyset$ holds, where $1 \leq j \leq r$ and $1 \leq i \leq 2$.

Proof. For $1 \leq j \leq r$, let $e_j = \{x_j, y_j\}$ be rigid edges in $G$, where $x_j \in A_1$ holds for all $1 \leq j \leq r$, $\{y_1, y_2\} \subset A_2$ and $y_i \in A_i$ holds for each $3 \leq t \leq r$. Consider the vertex $x_3 \in A_1$ from the rigid edge $e_3$, and note that $G \setminus N_G[x_3]$ consists of $r - 1$ cliques with some additional edges, of which $r - 2$ cliques have an improper vertex each. So it is vertex decomposable by Proposition 4.1. While $G \setminus x_3$ consists of $r$ cliques with some additional edges and one improper vertex $y_3 \in A_3$. For any independent vertex set $M$ of $G \setminus N_G[x_3]$, we have $\{x_1, x_2\} \subset N_G(x_3)$ and if $M \cap V(e_1) = \emptyset$ holds, then $M \cup \{x_1\}$ is an independent vertex set. But if $M \cap V(e_1) \neq \emptyset$ holds, then $M \cup \{x_2\}$ is an independent vertex set of $G \setminus x_3$. Next, apply the same procedure on sequence
of vertices $x_4, x_5, \ldots, x_r$. Let $H = G \setminus \{x_3, x_4, \ldots, x_r\}$ consists of $r$ cliques with some additional edges, of which $r - 2$ cliques have an improper vertex each. For $x_2 \in A_4$, the graph $H \setminus N_H[x_2]$ consists of $r - 1$ cliques and each clique has an improper vertex, so by [4, Lemma 3.2, Theorem 3.3] it is vertex decomposable. $H \setminus x_2$ consists of $r$ cliques with some additional edges, of which $r - 1$ cliques have an improper vertex each. So by Proposition 4.1, it is vertex decomposable. For any independent vertex set $M$ of $H \setminus N_H[x_2]$, we have $\{x_1, a_1\} \subset N_H(x_2)$ holds true if $M \cap V(e_i) = \emptyset$. In this case, $M \cup \{x_1\}$ is an independent vertex set of $H \setminus x_2$. But if $M \cap V(e_i) \neq \emptyset$ holds, then $M \cup \{a_1\}$ is an independent vertex set of $H \setminus x_2$. This shows that $G$ is vertex decomposable.

\[\square\]

5. Unmixed Property of Graphs Dual to $r$-Partite

It is known that Cohen-Macaulay condition implies unmixed condition, and the unmixed property of ideals of $k[x_1, \ldots, x_n]$ is important in commutative algebra. In this section, we provide some sufficient conditions for a graph dual to $r$-partite to be unmixed.

**Proposition 5.1.** Let $G$ be a non-complete graph with $r$ cliques $A_1, A_2, \ldots, A_r$ together with some additional edges. For each clique $A_i$, assume further that $A_i \not\subseteq \bigcup_{j \neq i} N_G[a_j]$ holds for each element $(a_j)_{j \neq i}$ of the Cartesian product $A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_r$. Then $G$ is unmixed, and $\dim \text{Ind}(G) = r - 1$.

**Proof.** First, note that each $A_i$ is a clique, thus each maximal independent vertex set of $G$ has at most $r$ elements. In the following, it is shown that every maximal independent vertex set of $G$ has $r$ elements, hence $G$ is unmixed.

Actually, for any vertex $a_1 \in A_1$, it is claimed that there exists a vertex $a_2 \in A_2$ such that $\{a_1, a_2\}$ is independent in $G$. In fact, if not, then $A_2 \subseteq N_G[a_1]$ holds, contradicting assumption. For the $\{a_1, a_2\}$, it is further claimed that there exists $a_3 \in A_3$ such that $\{a_1, a_2, a_3\}$ is independent in $G$; if not, then $A_3 \subseteq N_G[a_1] \cup N_G[a_2]$ holds, a contradiction. Continuing this discussion, we have got a maximal independent vertex set $M$ of $G$ consisting of $r$ elements, such that $a_1 \in M$.

Finally, it is claimed that each maximal independent vertex set of $G$ consists of $r$ elements. If not, let $B = \{b_1, b_2, \ldots, b_{r-1}\}$ be a maximal independent vertex set of $G$, where $b_i \in A_i$ holds. Then for any $b_r \in A_r$, such that $B \cup \{b_r\}$ is not independent in $G$, hence $b_r \in \bigcup_{i=1}^{r-1} N_G[b_i]$ holds. This shows $A_r \subseteq \bigcup_{i=1}^{r-1} N_G[b_i]$, contradicting assumption.

\[\square\]
Corollary 5.2. For a graph $G$ satisfying condition of Proposition 4.1, the edge ideal $I(G)$ is height-unmixed.

The conditions in Proposition 4.1 can be weakened somewhat to deduce the following:

Proposition 5.3. Let $G$ be a non-complete graph consisting of $r$ cliques $A_1, \ldots, A_r$ with some additional edges. Let $1 \leq d \leq r$. Then $G$ is unmixed and $\dim \text{Ind}(G) = d - 1$ holds, if the following conditions are satisfied:

1. There exists $1 \leq i \leq r$ such that for all $1 \leq j_1 < \cdots < j_d \leq r$ with $j_t \neq i$,
\[
(a_{j_1})_{t=1}^d \in A_{j_1} \times \cdots \times A_{j_d}
\]
implies $A_i \subseteq \cup_{t=1}^d N_G[a_{j_t}]$.

2. For any $1 \leq i \leq r$ and any $1 \leq j_1 < \cdots < j_{d-1} \leq r$ with $j_t \neq i$,
\[
(a_{j_1})_{t=1}^{d-1} \in A_{j_1} \times \cdots \times A_{j_{d-1}}
\]
implies $A_i \not\subseteq \cup_{t=1}^{d-1} N_G[a_{j_t}]$.

Proof. The proof of Proposition 4.1 together with condition (2) ensure that each independent vertex set of $G$ has at least $d$ elements. On the other hand, it is claimed that there is no independent vertex set of cardinality $d+1$; if not, let $C = \{a_1, a_2, \ldots, a_{d+1}\}$ be an independent vertex set. It implies that $a_i \not\in \cup_{j \neq i} N_G[a_j]$, where $j \in \{1, 2, \ldots, d+1\} \setminus \{i\}$. So $A_i \not\subseteq \cup_{j \neq i} N_G[a_j]$ for all $1 \leq i \leq r$, contradicting condition (1). The contradiction assures that there exists no independent vertex set containing $d+1$ elements. Thus $G$ is unmixed, and the dimension of the simplicial complex $\text{Ind}(G)$ is $d - 1$. \qed

Finally, this section is concluded with a different construction:

Proposition 5.4. Let $G$ be a non-complete graph with $r$ cliques $A_1, A_2, \ldots, A_r$ and some additional edges. Assume further that the following conditions are satisfied:

1. $V(A_1)$ can be decomposed into a disjoint union of $r - 1$ nonempty parts, $A_1 = \cup_{i=1}^{r-1} P_i$, such that $A_i \subseteq N_G[a]$ holds $\forall 2 \leq i \leq r$, $\forall a \in P_{i-1}$.

2. For any $a_1 \in A_1$, there exists $j \neq 1$ such that $A_j \subseteq N_G[a_1]$ holds, and there does not exist a sequence of elements $(a_i)_{i \neq 1, j}$ where $a_i \in A_i$ with $A_s \subseteq (\cup_{i \neq s} N_G[a_i]) \cup N_G[a_1]$, where $\{i, s\} \subset [r] \setminus \{1, j\}$.

3. For any clique $A_i$, there does not exist $a_j \in A_j$ such that $A_i \subseteq \cup_{i \neq j} N_G[a_j]$, where $\{i, j\} \subset \{2, 3, \ldots, r\}$.

Then $G$ is unmixed and every independent vertex set consists of $r - 1$ vertices.
Proof. Let $G$ be a graph with the above construction. From condition (1), the independent sets can be divided into two classes. First class consisting of independent sets containing a vertex from $A_1$ and second class that contains no vertex from $A_1$. For the first class, take a vertex $a_1 \in A_1$. By condition (2), we can find independent set $B$ of cardinality $r - 2$ using previous Proposition 5.1 so that $B \cup \{a_1\}$ is an independent set of $G$, of $r - 1$ cardinality. For the second class, by condition (3) and by using previous Proposition 5.1, independent subsets can be found of cardinality $r - 1$, which contains no vertex from $A_1$.

It is further claimed that there is no possibility for an independent subset of cardinality $r$. If not, let $\{v_1, v_2, \ldots, v_r\}$ be an independent set with cardinality $r$, then it contains a vertex from $A_1$, say $v_1 \in A_1$. Then there must be some clique, say $A_3$, such that $A_3 \subset N_G[v_1]$. This implies that $v_1$ is adjacent to $v_3$, a contradiction.

Also, there is no possibility for an independent set of cardinality less than $r - 1$, by conditions (2) and (3). This shows that $G$ is an unmixed graph. □

6. Connectedness of $\Delta G$

Proposition 1.2 implies that the connectedness of $\Delta G$ is important in some cases. So, we will discuss the connectedness of $\Delta G =: \text{Ind}(G)$ in the final section.

Proposition 6.1. Let $r \geq 2$. Let $G$ be a graph consisting of cliques $A_i$ ($1 \leq i \leq r$), with some additional edges in between. If $G$ has an improper vertex $w$ and there is no clone vertex in $G$ then $\Delta G$ is connected.

Proof. Let $w \in A_1$. Then in $\overline{G}$, $w$ is adjacent to each vertex in $V(A_2 \cup A_3 \cup \cdots \cup A_r)$. There is no clone vertex, so every vertex in $V(A_1 \setminus w)$ is adjacent to some vertex of $V(A_2 \cup A_3 \cup \cdots \cup A_r)$. So $\overline{G}$ is connected, which implies that $\text{Ind}(G)$ is connected. □

Proposition 6.2. Let $r \geq 3$. Let $G$ be a non-complete graph consisting of cliques $A_i$ ($1 \leq i \leq r$), with some additional edges in between. Then $\Delta G$ is connected if $G$ has two rigid edges from $A_1$ to $A_2$.

Proof. Let $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ be rigid edges from $A_1$ to $A_2$. Then in $\overline{G}$, $x_1$ is adjacent to each vertex in $V((A_2 \setminus y_1) \cup A_3 \cup \cdots \cup A_r)$, $x_2$ is adjacent to each vertex in $V((A_2 \setminus y_2) \cup A_3 \cup \cdots \cup A_r)$, $y_1$ is adjacent to each vertex in $V((A_1 \setminus x_1) \cup A_3 \cup A_4 \cup \cdots \cup A_r)$ and $y_2$ is adjacent to each vertex in $V((A_1 \setminus x_2) \cup A_3 \cup A_4 \cup \cdots \cup A_r)$. So $\overline{G}$ is connected, which implies that $\text{Ind}(G)$ is connected. □
Proposition 6.3. Let $G$ be a non-complete graph consisting of two cliques $A_i$ ($1 \leq i \leq 2$), $|A_i| \geq 3$ with some additional edges in between. Then $\Delta_G$ is connected if $G$ has one rigid edge from $A_1$ to $A_2$ and one rigid vertex $w \in A_2$.

Proof. Let $e_1 = \{x_1, y_1\}$ be a rigid edge from $A_1$ to $A_2$. Assume that $w$ is adjacent to $v \in A_1$. Then in $G$, $x_1$ is adjacent to each vertex in $V(A_2 \setminus y_1)$, $y_1$ is adjacent to each vertex in $V(A_1 \setminus x_1)$ and $w$ is adjacent to each vertex in $V(A_1 \setminus v)$. Note that $v \neq x_1$, so $G$ is connected which implies $\text{Ind}(G)$ is connected. □

References


