On Finitely Generated Projective Modules and Exchange Rings

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Abstract. In this paper, we study the relation between exchange rings $R$ and their $J$-semisimple indecomposable factor rings. In particular, we prove that for any exchange ring $R$ and any finitely generated projective $R$-modules $P$ and $Q$, $P$ is isomorphic to a direct summand of $Q$ if and only if for every ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple, the right $R/I$-module $P/P I$ is isomorphic to a direct summand of $Q/Q I$.

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1 Introduction

All rings in this paper are associative with identity and all modules are right unital. For any ring $R$, $FP(R)$ denotes the category of finitely generated projective $R$-modules and $J(R)$ denotes the Jacobson radical of $R$. For any module $P$, $P^n$ (or $nP$) denotes the direct sum of $n$ copies of $P$. For $P, Q \in FP(R)$, we write $P \lesssim Q$ to mean that $P$ is isomorphic to a direct summand of $Q$, and $P < \lesssim Q$ to mean that $P$ is a proper direct summand of $Q$. If for any $P, Q_1, Q_2 \in FP(R)$, $P \oplus Q_1 \cong P \oplus Q_2$ implies $Q_1 \cong Q_2$, then we say that the cancellation holds in $FP(R)$. If there exists a positive integer $n$ such that for any $P, Q \in FP(R)$, $P^n \cong Q^n$ implies $P \cong Q$, then we say that the uniqueness of $n$th roots holds in $FP(R)$.

In this paper, we study exchange rings and finitely generated projective modules over them. Our main idea comes from the observation that information about a ring $R$ and the category $FP(R)$ can be obtained by studying the $J$-semisimple indecomposable factor rings $R/I$ and the related
categories $FP(R/I)$. First, we study finitely generated projective modules over exchange rings, especially over exchange rings whose primitive factor rings are artinian. We prove that for any exchange ring $R$ and any $P, Q \in FP(R), P \cong Q$ if and only if for every ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple, the right $R/I$-module $P/PI$ is isomorphic to a direct summand of $Q/QI$. Next, we study ring-theoretic properties. In particular, we prove that a ring $R$ satisfies the unit 1-stable range condition if and only if all the indecomposable $J$-semisimple factor rings of $R$ satisfy the unit 1-stable range condition; and a ring $R$ has stable range at most $n$ if and only if all the indecomposable $J$-semisimple factor rings of $R$ have stable range at most $n$.

Recall that a ring $R$ is said to be an exchange ring if the right (or equivalently, the left) $R$-module $R$ has the exchange property (see [20]). By [18], a ring $R$ is an exchange ring if and only if idempotents lift modulo every right (left) ideal of $R$. The class of exchange rings is quite large. It includes the $\pi$-regular rings [19], clean rings [18], von Neumann regular rings, and $C^*$-algebras with real rank zero [2]. This class is closed under taking factor rings, matrix rings, and corners.

The class of exchange rings whose primitive factor rings are artinian includes the exchange rings $R$ such that $R$ or $R/J(R)$ has bounded index of nilpotency [24], the exchange rings $R$ such that all idempotents of $R$ (or more generally, all idempotents of $R/J(R)$) are central [23, Lemma 2.4], and abelian $\pi$-regular rings. For any exchange ring $R$, if all idempotents of $R$ are central in $R$, then all primitive factor rings of $R$ are artinian, but the converse is obviously not true. However, if all primitive factor rings of $R$ are artinian and $J(R) = 0$, then a theorem of Levitzki [15] says that in any non-zero ideal $I$ of $R$, there is a non-zero central idempotent $e$ such that $eRe$ is isomorphic to a full matrix ring over an exchange ring with idempotents central. That is to say, the class of exchange rings whose primitive factor rings are artinian is closely related to the exchange rings with idempotents central.

### 2 Preliminaries

In this section, we recall some crucial facts and notions which we need in the later sections.

**Lemma 2.1.** Let $R$ be an indecomposable $J$-semisimple exchange ring. If all primitive factor rings of $R$ are artinian, then $R$ is simple artinian.

**Proof.** This is a consequence of a theorem in [15].

**Lemma 2.2.** [18, Proposition 2.11] For any projective module $P$ with the finite exchange property, if $P = M_1 + M_2 + \cdots + M_n$, then there are submodules $P_i \subseteq M_i$ such that $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. 

Lemma 2.3. [2, Proposition 1.4] Let $R$ be an exchange ring, $P, Q \in FP(R)$, and $I$ a two-sided ideal of $R$.

1. If $P/PI \cong Q/QI$ as $R/I$-modules, then we have decompositions $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ such that $P_1 \cong Q_1$ and $P_2 = P_2I$.
2. If $P/PI \cong Q/QI$ as $R/I$-modules, then we have decompositions $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ such that $P_1 \cong Q_1$, $P_2 = P_2I$, and $Q_2 = Q_2I$.

The next lemma of Goodearl was originally expressed for regular rings (see Lemma 1.2 of [3]). It is actually true for all exchange rings, as can be seen from the proof given there.

Lemma 2.4. For any finitely generated projective modules $A$, $B$, and $C$ over an exchange ring $R$, if $A \oplus B \cong kC$ for some positive integer $k$, then there is a decomposition $C = C_0 \oplus C_1 \oplus \cdots \oplus C_n$ such that $A \cong C_1 \oplus 2C_2 \oplus \cdots \oplus kC_n$ and $B = C_0 \oplus (k - 1)C_1 \oplus \cdots \oplus C_{k-1}$.

Recall that a ring $R$ is said to have stable range at most $n$ if for any $a_1, \ldots, a_{n+1} \in R$ such that $a_1R + \cdots + a_{n+1}R = R$, there are $b_1, \ldots, b_n \in R$ such that $(a_1 + a_{n+1}b_1)R + \cdots + (a_n + a_{n+1}b_n)R = R$. When $n = 1$, the ring $R$ is said to have stable range one (or stable rank one).

3 On Finitely Generated Projective Modules

Theorem 3.1. Let $P$ and $Q$ be any finitely generated projective modules over an exchange ring $R$. We have:

1. $P \cong Q$ if and only if $P/PI \cong Q/QI$ for any ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple.
2. $P \cong Q$ if and only if $P/PI \cong Q/QI$ for any ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple.

Proof. (1) We only need to prove the sufficiency part. Suppose $P/PI \cong Q/QI$ for any ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple, but $P$ is not isomorphic to any summand of $Q$. Let $\Omega$ be the set of ideals $I$ of $R$ such that $P/PI \cong Q/QI$. Then $\Omega$ is not empty. For any chain $\{I_\alpha \mid \alpha \in \Lambda\}$ from $\Omega$, denote $I = \bigcup_{\alpha \in \Lambda} I_\alpha$. Then $I$ is an ideal of $R$ and we must have $I \in \Omega$. Otherwise, $P/PI \cong Q/QI$. By Lemma 2.3, we have decompositions

$$P = P_1 \oplus P_2 \quad \text{and} \quad Q = Q_1 \oplus Q_2$$

such that $P_1 \cong Q_1$ and $P_2 = P_2I$. Now assume $P_2$ is generated by $x_1, \ldots, x_n$. Since $P_2 = P_2I$, we have $x_i = \sum y_{ij}r_{ij}$, where $y_{ij} \in P_2$ and $r_{ij} \in I$. Then there exists $\beta \in \Lambda$ such that $I_\beta$ contains all the $r_{ij}$. Thus, we have $P_2 = P_2I_\beta$ and hence $P/PI_\beta \cong Q/QI_\beta$, which contradicts that $I_\beta \in \Omega$. This shows that every chain in the partially ordered set $\Omega$ has an upper bound in $\Omega$. By Zorn’s Lemma, $R$ has a two-sided ideal $M$ such that $M$ is maximal with the property of $P/PM \cong Q/QM$. Without loss of generality, we assume
$M = 0$, i.e., $P \ncong Q$, but for any non-zero two-sided ideal $K$ of $R$, we have $P/PK \cong Q/QK$. By assumption, $R$ is either decomposable or not $J$-semisimple.

If $R$ is decomposable, say $R = K \oplus L$ (as a ring direct sum). Then we have

$$P = PK \oplus PL \cong (P/PL) \oplus (P/PK) \cong (Q/QL) \oplus (Q/QK) \cong Q,$$

a contradiction.

Denote $J(R)$ by $J$. If $J \neq 0$, then $P/PJ \cong Q/QJ$ and so we have some $P' \in FP(R)$ such that $Q/QJ \cong (P \oplus P')/(P \oplus P')I$. Since $P \oplus P'$ and $Q$ are finitely generated projective modules, both $Q$ and $P \oplus P'$ are projective covers of $Q/QJ$. Thus, $P \cong Q$, a contradiction.

These contradictions show that the sufficiency part holds.

(2) It can be proved in a manner similar to that of (1) for exchange rings. Actually, (2) holds for any ring $R$. In fact, we only need to prove the sufficiency part. First, choose $n \geq 1$ and idempotents $e, f \in M_n(R)$ such that $P \cong e(nR)R$ and $Q \cong f(nR)R$. If $P \neq Q$, then there exist no $x, y \in M_n(R)$ such that

$$e = xy, \quad f = yx, \quad x = exf, \quad y = fye.$$

Obviously, they are equivalent to $4n^2$ equations in $2n^2$ unknown variables that have no simultaneous solutions in $R$. Denote these equations by $AX = 0$ and let $\Omega$ be the class of ideals $I$ of $R$ such that $\overline{A}X = 0$ has no solutions in $R/I$, where $\overline{A} = (\overline{a_{ij}})$. By Zorn’s Lemma, there is a maximal element in $\Omega$ with respect to the usual inclusion. Thus, $R$ has a two-sided ideal $I$ such that $I$ is maximal with the property of $P/PI \neq Q/QI$. Without loss of generality, we assume $I = 0$, i.e., $P \neq Q$, but for any non-zero two-sided ideal $K$ of $R$, we have $P/PK \cong Q/QK$. By assumption, $R$ is either decomposable or not $J$-semisimple. But in either case, we can deduce a contradiction. \hfill \Box

As an immediate application, we have the following result.

**Theorem 3.2.** For any finitely generated projective modules $P$ and $Q$ over a ring $R$, if $R/J(R)$ is an exchange ring and all the primitive factor rings of $R$ are artinian, then the following hold:

(1) If $P^n$ is isomorphic to a direct summand of $Q^n$, then $P$ is isomorphic to a direct summand of $Q$.

(2) If $P^n \cong Q^n$, then $P \cong Q$.

**Proof.** (1) Without loss of generality, we can assume $J(R) = 0$. For any ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple, by Lemma 2.1, $R/I$ is a simple artinian ring. Thus, if $P^n$ is isomorphic to a direct summand of $Q^n$, then $(P/PI)^n$ is isomorphic to a direct summand of the $R/I$-module $(Q/QI)^n$. Since $R/I$ is simple artinian, we have $P/PI \ncong Q/QI$.
for any ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple. By Theorem 3.1(1), we have $P \subseteq Q$.

(2) Similar to the proof of (1).

**Corollary 3.3.** Let $R$ be a ring with primitive factor rings artinian and suppose $R/J(R)$ is an exchange ring.

(1) For any ring $S$, if $M_n(R) \cong M_n(S)$, then $R \cong S$.

(2) For any ring $S$, if $M_n(S)$ is isomorphic to a ring direct summand of $M_n(R)$, then the ring $S$ is isomorphic to a ring direct summand of $R$.

**Proof.** (1) In $T = M_n(R)$, choose idempotents $e$ and $f$ such that $eTe \cong R$, $fTf \cong S$, and $(eT)^n \cong (fT)^n$. Since $T/J(T)$ is also an exchange ring with primitive factor rings artinian, by Theorem 3.2(1), we obtain $eT \cong fT$ from $(eT)^n \cong (fT)^n$. Then $R \cong S$ since $eT \cong fT$.

(2) If $M_n(S)$ is isomorphic to a ring direct summand of $M_n(R)$, then there is a non-zero central idempotent $h \in M_n(R)$ such that $hM_n(R)h \cong M_n(S)$. The element $h$ has the form of diag $\{e, \ldots, e\}$ for some central idempotent $e \in R$. Thus, $M_n(S) \cong M_n(eRe)$. Since $eRe/eJ(R)e$ is also an exchange ring and all primitive factor rings of $eRe$ are artinian, by (1), we have $S \cong eRe$. Thus, $S$ is isomorphic to a ring direct summand of $R$. □

We remark that Theorem 3.2 and Corollary 3.3 were first proved by Goodearl in the case of regular rings (see Proposition 6.11 and Corollary 6.12 in [11]). Note that an exchange ring $R$ has stable range one if all primitive factor rings of $R$ are artinian (see [24]). In contrast with Theorem 3.2, Goodearl [12] constructed simple regular rings $R$ with stable range one but the uniqueness of $n$th roots fails in $FP(R)$.

**Corollary 3.4.** Let $X$ and $Y$ be modules over a ring $R$, and $S = \text{End}_R(X)$. If all primitive factor rings of $S$ are artinian and $S/J(S)$ is an exchange ring, then for any positive integer $n$, $X^n \cong Y^n$ implies $X \cong Y$.

**Proof.** It follows from Theorem 3.2 of this paper and Theorem 4.7 in [8]. □

**Corollary 3.5.** [9, Proposition 2.1] Let $X$ and $Y$ be modules over a ring $R$. Suppose $\text{End}_R(X)$ and $\text{End}_R(Y)$ are semilocal. Then for any positive integers $n$, $X^n \cong Y^n$ implies $X \cong Y$.

**Proof.** All primitive factor rings of a semilocal ring $S$ are certainly artinian and $S/J(S)$ is an exchange ring. Now the result follows from Corollary 3.4. □

**Remark 1.** For any finitely generated projective modules $Q$ and $P$ over a ring $R$, if $R/J(R)$ is an exchange ring and $R$ has stable range one, then $P \cong Q$ if and only if $P/PI \cong Q/PI$ for all prime ideals $I$ of $R$; $P \subseteq Q$ if and only if $P/PI \subseteq Q/PI$ for all prime ideals $I$ of $R$. The proof is similar to that of Theorem 4.19 in [11] with the aid of Lemma 2.3.
Remark 2. For any prime ideal $I$ of a ring $R$, $R/I$ is obviously indecomposable. But $R/I$ need not be $J$-semisimple. On the other hand, for an ideal $I$ of $R$, $R/I$ being indecomposable (and $J$-semisimple) does not imply the primeness of $I$.

Remark 3. Recently, Lam [14] gave a survey on modules with isomorphic multiples and rings with isomorphic matrix rings.

Theorem 3.6. Let $R$ be a ring such that $R/J(R)$ is an exchange ring.

(1) The cancellation holds in $FP(R)$ if and only if for any ideal $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple, the cancellation holds in $FP(R/I)$.

(2) If for all ideals $I$ of $R$ such that $R/I$ is indecomposable and $J$-semisimple, the uniqueness of $n$th roots holds in $FP(R/I)$, then the uniqueness of $n$th roots holds in $FP(R)$.

Proof. The proof for (2) and the sufficiency part of (1) is similar to that of Theorem 3.2 and Corollary 3.3. We now prove the necessity part of (1). In fact, for any exchange ring $R$, the cancellation holds in $FP(R)$ if and only if $R$ has stable range one by [6] and [24]. If the cancellation holds in $FP(R)$, then it also holds in $FP(R/J(R))$, and hence, $R$ has stable range one. Thus, $R/I$ has stable range one and so the cancellation holds in $FP(R/I)$. 

We conjecture that the converse of Theorem 3.6(2) holds for any ring $R$ and any ideal of $R$. Although we are still unable to prove this conjecture right now, we have a partial answer for some class of exchange rings. A ring $R$ is said to satisfy the DCC on $n$-copied direct summands if there exists no infinite properly descending chain of $R$-modules $R^n >_\oplus P_1 >_\oplus P_2 >_\oplus \cdots$, where each $P_i \cong Q_i^n$ for some $Q_i$.

Proposition 3.7. Let $R$ be an exchange ring and $n$ a positive integer. If the uniqueness of $n$th roots holds in $FP(R)$, and either $R$ satisfies the DCC on $n$-copied direct summands or $R^n$ satisfies the ACC on direct summands, then the uniqueness of $n$th roots holds in $FP(R/I)$ for any ideal $I$ of $R$.

Proof. For any $P, Q \in FP(R/I)$, there exist $P, Q \in FP(R)$ such that $P \cong P/P I$ and $Q \cong Q/Q I$. Assume $P^n/P^n I \cong Q^n/Q^n I$. By Lemma 2.3, we have decompositions $P^n = P' \oplus P''$ and $Q^n = Q' \oplus Q''$ such that $P' \cong Q'$, $P'' = P^n I$, and $Q'' = Q^n I$. By Lemma 2.4, we have decompositions

\[ P = P_0 \oplus P_1 \oplus \cdots \oplus P_n \quad \text{and} \quad Q = Q_0 \oplus Q_1 \oplus \cdots \oplus Q_n \]

such that

\[ P' \cong P_1 \oplus 2P_2 \oplus \cdots \oplus nP_n, \]
\[ Q' \cong Q_1 \oplus 2Q_2 \oplus \cdots \oplus nQ_n, \]
\[ P'' \cong nP_0 \oplus (n-1)P_1 \oplus \cdots \oplus P_{n-1}, \]
\[ Q'' \cong nQ_0 \oplus (n-1)Q_1 \oplus \cdots \oplus Q_{n-1}. \]
Since \( P'' = P'''I \), we have
\[
nP_0 \oplus (n-1)P_1 \oplus \cdots \oplus P_{n-1} = nP_0I \oplus (n-1)P_1I \oplus \cdots \oplus P_{n-1}I.
\]
Thus, \( P_i = P_iI \) for \( i = 0, 1, \ldots, n-1 \). Similarly, we can obtain \( Q_i = Q_iI \) for all \( i = 0, 1, \ldots, n-1 \). Therefore,
\[
\begin{align*}
P_n^n/P_n^nI &= P^n/P^nI \cong Q^n/Q^nI = Q^n/Q^2nI, \\
P/PI &= P_n/P_nI, \\
Q/QI &= Q_n/Q_nI.
\end{align*}
\]
Let \( K = P_1 \oplus 2P_2 \oplus \cdots \oplus (n-1)P_{n-1} \) and \( L = Q_1 \oplus 2Q_2 \oplus \cdots \oplus (n-1)Q_{n-1} \). If \( K = 0 \) or \( L = 0 \), then \( nP_n \cong nQ_n \). Thus, \( P_n \cong Q_n \) and hence \( P/PI \cong Q/QI \) since the uniqueness of \( n \)th roots holds in \( FP(R) \). If \( K \neq 0 \) or \( L \neq 0 \), then either \( nR \supset K \) and \( 0 \prec K \) for some \( K_1 \cong nP_n \) or \( nR \prec L \) for some \( L_1 \cong nQ_n \). We can repeat the above discussion for \( P^n/P^nI \cong Q^n/Q^nI \). Since either \( R \) satisfies the ACC on \( n \)-copied direct summands or \( R^n \) satisfies the ACC on direct summands, this process stops after a finite number of steps, and finally, we can obtain
\[
P/PI \cong P_n/P_nI \cong \cdots \cong Q_n/Q_nI \cong Q/QI.
\]
This shows that the uniqueness of \( n \)th roots holds in \( FP(R/I) \).

According to [2], a ring \( R \) is said to be separative if for any \( P, Q \in FP(R) \), \( 2P \cong 2Q \) and \( 3P \cong 3Q \) imply \( P \cong Q \). In [2], it is proved that this is equivalent to the following nice separative cancellation property: For any \( A, B, C \in FP(R) \), if \( A \oplus C \cong B \oplus C \) with \( C \leq_{\oplus} mA \) and \( C \leq_{\oplus} nB \) for some positive integers \( m \) and \( n \), then \( A \cong B \). As another application of Theorem 3.1, we have the following result.

**Corollary 3.8.** A ring \( R \) is separative if and only if all indecomposable \( J \)-semisimple homomorphic images of \( R \) are separative.

### 4 Some Results on Exchange Rings

According to [17], a ring \( R \) is said to satisfy the **unit 1-stable range condition** if for any \( a, b, c \in R \) with \( ab+c = 1 \), there is an invertible element \( u \in R \) such that \( au + c \) is invertible in \( R \). It is obvious that every ring \( R \) satisfying the unit 1-stable range condition must have stable range one, but the converse is not true as shown by the ring \( Z/2Z \). By [22], this condition is left-right symmetric, and a semilocal ring \( R \) satisfies the unit 1-stable range condition if and only if \( Z/2Z \) is not a homomorphic image of \( R \).

Recall that a ring \( R \) is said to be a **\( \pi \)-regular ring** if for any \( x \in R \), there exist \( y \in R \) and a positive integer \( n \) such that \( x^n = x^n y x^n \). Obviously, every von Neumann regular ring is \( \pi \)-regular. In [16], Menal proved that if \( R \) is
a $\pi$-regular ring with all primitive factor rings artinian, then $R$ has stable range one. This result is a generalization of Theorem 1 in [10]. We observe that the proof of Lemma 6 in [16] really says that a ring $R$ has stable range one if and only if all the indecomposable $J$-semisimple factor rings of $R$ satisfy the stable range one condition. For the unit 1-stable range condition and the general $n$-stable range condition, we have similar conclusions.

**Proposition 4.1.**

1. A ring $R$ satisfies the unit 1-stable range condition if and only if all the indecomposable $J$-semisimple factor rings of $R$ satisfy the unit 1-stable range condition.

2. A ring $R$ has stable range at most $n$ if and only if all the indecomposable $J$-semisimple factor rings of $R$ have stable range at most $n$.

**Proof.** (1) We only need to prove the “if” part. Suppose $R$ does not satisfy the unit 1-stable range condition. Then there exist elements $a, b, c$ in $R$ such that $ab + c = 1$, but $aU(R) + c$ does not contain units of $R$, where $U(R)$ is the set of invertible elements of $R$. By Zorn’s Lemma, there is an ideal $K$ of $R$ maximal with the property that $aU(R) + c$ does not contain units of $R$, where $R = R/K$ and $\bar{a} = a + K$. Without loss of generality, we can assume $K = 0$, i.e., for any non-zero two-sided ideal $I$ of $R$, $aU(R/I) + c$ contains a unit of $R/I$. By assumption, $R$ is either decomposable or not $J$-semisimple. But in either case, there is a contradiction.

(2) Similar to the proof of (1). \[\Box\]

**Corollary 4.2.** For any exchange ring $R$ with primitive factor rings artinian, the following statements are equivalent:

1. $R$ satisfies the unit 1-stable range condition.

2. Every element of $R$ is a sum of two invertible elements in $R$.

3. The identity element of $R$ is a sum of two invertible elements in $R$.

4. For any factor ring $R_1$ of $R$, every element of $R_1$ is a sum of two invertible elements of $R_1$.

5. $Z/2Z$ is not a homomorphic image of $R$.

**Proof.** (1)$\Rightarrow$(2). If $R$ satisfies the unit 1-stable range condition, then for any $x \in R$, since $x0 + 1 = 1$, there is a unit $u \in R$ such that $xu + 1$ is invertible. Thus, $x$ is a sum of two invertible elements of $R$.

(2)$\Rightarrow$(3)$\Rightarrow$(5) and (2)$\Rightarrow$(4)$\Rightarrow$(5). Obvious.

(5)$\Rightarrow$(1). By [21], $M_n(D)$ satisfies the unit 1-stable range condition for any $n \geq 2$ and any division ring $D$. Thus, if all primitive factor rings of $R$ are artinian, then by Lemma 2.1, every indecomposable $J$-semisimple factor ring of $R$ is simple and hence artinian. By assumption, $Z/2Z$ is not a homomorphic image of $R$. Thus, every indecomposable $J$-semisimple factor ring of $R$ satisfies the unit 1-stable range condition. Finally, by Proposition 4.1, $R$ satisfies the unit 1-stable range condition. \[\Box\]
We remark that the main result of Corollary 4.2 was also proved independently in Theorem 2 of [7].

According to [5], a \( \pi \)-regular ring \( R \) is said to be abelian if all idempotents of \( R \) are in the center of \( R \). Thus, an abelian \( \pi \)-regular ring \( R \) is an exchange ring with idempotents central, and hence, all prime factor rings of \( R \) are local, and primitive factor rings of \( R \) are artinian. Thus, we have the following.

**Corollary 4.3.** [5, Theorem 6] For any abelian \( \pi \)-regular ring \( R \), the following statements are equivalent:

1. Every element of \( R \) is a sum of two invertible elements of \( R \).
2. The identity element of \( R \) is a sum of two invertible elements of \( R \).
3. \( \mathbb{Z}/2\mathbb{Z} \) is not a homomorphic image of \( R \).

For any \( \pi \)-regular ring \( R \), it is easy to see that every element in \( J(R) \) is nilpotent. If \( R \) is an abelian \( \pi \)-regular ring, then any prime factor ring \( R_1 \) of \( R \) is a \( \pi \)-regular local ring. Thus, each element of \( R_1 \) is either nilpotent or invertible in \( R_1 \). This gives an alternative proof of the following interesting result of Badawi: If a ring \( R \) is abelian \( \pi \)-regular, then for any prime ideal \( P \) of \( R \), each element of \( R/P \) is either nilpotent or invertible in \( R/P \) (see Theorem 5 of [5]). Since every local ring is an exchange ring with idempotents central, this result could not be extended to the exchange rings. However, for any exchange ring \( R \) with idempotents central, if every prime factor ring of \( R \) is artinian, then for any prime ideal \( P \) of \( R \), each element of \( R/P \) is either nilpotent or invertible in \( R/P \). In fact, this is a special case of Proposition 4.4.

A module \( M \) is said to be sum-indecomposable if \( M \) is not a sum of two non-zero proper submodules. For example, let \( p \) be a prime integer and \( N = \{ a/p^n | a \in \mathbb{Z}, n \in \mathbb{N} \} \). Then \( N/\mathbb{Z} \) is a sum-indecomposable noetherian \( \mathbb{Z} \)-module.

**Proposition 4.4.** Let \( R \) be an exchange ring with idempotents central in \( R \). Suppose for any prime ideal \( P \) of \( R \), \( R/P \) is a noetherian ring. Let \( M \) be a non-zero sum-indecomposable artinian module and \( S = \text{End}_R(M) \). Then \( S \) is a local ring and \( J(S) \) is a nilpotent ideal of \( S \). Hence, each element of \( S \) is either nilpotent or invertible in \( S \).

**Proof.** For any \( f \in S \), there is \( n \geq 1 \) such that \( M = \text{Im}(f^n) + \text{Ker}(f^n) \) since \( M \) is artinian. Thus, \( f \) is either nilpotent or epimorphic because \( M \) is sum-indecomposable. Let \( T \) be the set of all nilpotent elements of \( S \). For any \( f, g \in T \) and \( h \in S \), we must have \( f + g \in T \) and \( fh \in T \). Indeed, if \( f + g \notin T \), then we have \( \text{Im}(f) + \text{Im}(g) = M \) since \( f + g \) is an epimorphism. But both \( \text{Im}(f) \) and \( \text{Im}(g) \) are non-zero proper submodules of \( M \), which is a contradiction. If \( fh \notin T \), then \( fh(M) = M \). Assume the nilpotency index of \( f \) is \( n \). Then we would have \( f^{n-1}(M) = f^n h(M) = 0 \) and so \( f^{n-1} = 0 \), a contradiction. Thus, \( fh \) is not an epimorphism of \( M \), and hence, \( fh \in T \).
This shows that $T$ is a nil right ideal of $S$. If $f \in S$ is an epimorphism, by an argument similar to the proof of Theorem 2.5 in [4], one can see that the conditions ensure that $f$ is an isomorphism. Finally, $T = J(S)$ and $S$ is a local ring. □

In the final part of this paper, we characterize $J$-semisimple exchange rings $R$ with all idempotents central in $R$. We need the following lemma.

**Lemma 4.5.** For any exchange ring $R$ with $J(R) = 0$ and any right ideals $I, K$ of $R$, if $\text{Hom}_R(I, K) \neq 0$, then $I$ and $K$ have non-zero isomorphic submodules.

**Proof.** For any non-zero $f \in \text{Hom}_R(I, K)$, there is a non-zero idempotent $e$ in $\text{Im}(f)$. Let $f(a) = e$ for some $a \in I$. Then $f$ induces an epimorphism $f : aR \to eR$. Since $eR$ is projective, this epimorphism is splitting. Thus, $I$ and $K$ have non-zero isomorphic submodules. □

**Proposition 4.6.** For any exchange ring $R$ with $J(R) = 0$, the following statements are equivalent:

1. All idempotents of $R$ are central in $R$.
2. Any two isomorphic non-zero right (left) ideals of $R$ have a non-zero intersection.
3. For any right (left) ideal $I$ of $R$, if $2I \leq R$, then $I = 0$.
4. For any right (left) ideals $I, K$ of $R$, if $I \cap K = 0$, then $\text{Hom}_R(I, K) = 0$.
5. For any idempotents $e, f$ of $R$, if $eR \cap fR = 0$, then $\text{Hom}_R(eR, fR) = 0$.

**Proof.** (1)⇒(2). For any non-zero isomorphic right ideals $I$ and $K$ of $R$ with $\phi : I \to K$ isomorphic, there is a non-zero idempotent $e \in I$. Let $a = \phi(e)$. Then we have $a = ae = ea$ and so $aR \subseteq eR$. Hence, $0 \neq aR \subseteq K \cap I$.

(2)⇒(3). For any $I, K \leq R$ and $I \cap K = 0$ and $\text{Hom}_R(I, K) \neq 0$, by Lemma 4.5, $I$ and $K$ have non-zero isomorphic submodules. In this case, we obtain a non-zero right ideal $L$ of $R$ such that $2L \leq R$.

(3)⇒(4). For any $I, K \leq R$ and $I \cap K = 0$ and $\text{Hom}_R(I, K) \neq 0$, by Lemma 4.5, $I$ and $K$ have non-zero isomorphic submodules. In this case, we obtain a non-zero right ideal $L$ of $R$ such that $2L \leq R$.

(5)⇒(1). If $R$ has a non-central idempotent $e$, then $eR(1 - e) \neq 0$. Hence, $\text{Hom}_R(eR, (1 - e)R) \cong eR(1 - e) \neq 0$. On the other hand, we have $eR \cap (1 - e)R = 0$. There is a contradiction with the assumption of (5). □

**Corollary 4.7.** For any exchange ring $R$ with $J(R) = 0$ and any finitely generated projective $R$-module $P$, let $S = \text{End}_R(P)$. Then the following statements are equivalent:

1. All idempotents of $S$ are central in $S$.
2. Any two isomorphic submodules of $P$ have a non-zero intersection.
3. $P$ is square-free, i.e., for any submodule $K$ of $P$, if $2K \leq P$, then $K = 0$.
4. For any submodule $K$ of $P$, if $2K \leq P$, then $K = 0$. 
(5) For any submodules $I$, $K$ of $P$, if $I \cap K = 0$, then $\text{Hom}_R(I, K) = 0$.

Corollary 4.8. For any exchange ring $R$ with $J(R) = 0$, if all idempotents of $R$ are central in $R$, then for any non-zero isomorphic right (left) ideals $I$ and $K$ of $R$, there is an idempotent-generated right (left) ideal $L$ such that $L$ is essential in both $I$ and $K$. In this case, $I \cap K$ is essential in both $I$ and $K$.

Proof. Let $\phi : I \to K$ be any right module isomorphism. Let $I_1 = \sum eR$, where $e$ ranges through all the idempotents of $R$ in $I$. Then $I_1$ is an essential right submodule of $I$. Let $K_1 = \sum \phi(e)R$. Then $K_1$ is an essential submodule of $K$ and we have $K_1 = \sum \phi(e)eR = \sum e\phi(e)R \subseteq I_1$. Repeating the same discussion for $K_1$, we have an essential submodule $K_2$ of $K_1$, which is generated by all the idempotents of $R$ in $K_1$ and hence essential in $K_1$. Thus, $K_2 \subseteq I \cap K$ and $K_2$ is essential in $K$. If we begin the above discussion from $K$, then we obtain an idempotent-generated essential submodule $I_2$ of $I$ such that $I_2 \subseteq K$. Finally, $L = I_2 + K_2$ is idempotent-generated and it is an essential submodule of both $I$ and $K$.\qed

References


