THE POWER-SUBSTITUTION CONDITION OF
ENDOMORPHISM RINGS OF QUASI-PROJECTIVE MODULES

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Dedicated to Professor Zhou Boxun for his 80’th birthday.

Abstract. In the first part of this paper, we give necessary and sufficient conditions for a quasi-projective module whose endomorphism ring has the power-substitution property, by way of completion of diagrams. In the second part, we study exchange rings with the right power-substitution property. We prove that for any module $M$ with the finite exchange property, the ring $\text{End}_R(M)$ has the right power-substitution property if (and only if) $M$ has the power cancellation property, if and only if for any regular element $a \in \text{End}_R(M)$, there exists an integer $n \geq 1$ such that $aI_n$ is unit-regular in $M_n(\text{End}_R(M))$.

1. Introduction

All rings in this paper are associative with identity and all modules right unital. Let $M_n(R)$ be the ring of all $n \times n$ matrices over $R$, let $I$ be the identity matrix in $M_n(R)$. Recall that a ring $R$ is said to have the right power-substitution property, if for any $a, b$ and $c$ in $R$ satisfying $ab + c = 1$, there exist a positive integer $n$ and a matrix $Q \in M_n(R)$ such that $aI + cQ$ is invertible in $M_n(R)$. This definition is left-right symmetric([7]) and for any module $M$, $\text{End}_R(M)$

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has the power-substitution property if and only if the module $M$ has the power-substitution property, only if $M$ has the power cancellation property (i.e., for any module isomorphism $M \oplus A \cong M \oplus B$, there exists an integer $n \geq 1$ such that $A^n \cong B^n$). Goodearl in [6] also proved, among other things, that the power-substitution property of rings is preserved under taking corners and fractions; but it is not Morita invariant ([3]). This property and the related power cancellation of modules have also been studied by several other authors (see, e.g., [9], [2] and [5]).

In the first part of this paper, motivated by the work of Canfell [4], we give necessary and sufficient conditions for a quasi-projective module whose endomorphism ring has the power-substitution property, by way of completion of diagrams. In the second part, we study exchange rings with the right power-substitution property. We prove that for any module $M$ with the finite exchange property, the ring $S$ has the right power-substitution property if (and only if) $M$ has the power cancellation property. This is the converse to one of the above mentioned results in the regular case. We also prove that an exchange ring $R$ has the power-substitution property, if and only if for any regular element $a \in R$, there exists an integer $n \geq 1$ such that $aI_n$ is unit-regular in $M_n(R)$.

Throughout, $P^n$ represents direct sums of $n$ copies of a module $P$, $J(R)$ the Jacobson radical of a ring $R$. For other notations and results used in this paper without mention, please refer to the books [1], [8] and [10].

2. The power-substitution condition versus completion of diagrams

**Definition 2.1.** For any module $P$, $P$ is said to be power-epi-projective if for any epimorphisms $f : P \to M$ and $g : P \to M$, there exist a positive integer $n$ and an isomorphism $h : P^n \to P^n$ such that the following diagram commutes:

\[
\begin{array}{ccc}
P^n & \xrightarrow{\oplus f} & M^n \\
\downarrow & & \\
\oplus g & \rightarrow & \\
\end{array}
\]

that is, $\oplus f = (\oplus g)h$.

We first record an easy but useful result:
Proposition 2.2. For any power-epi-projective module $P$, any epimorphism of $P$ is an isomorphism.

Proof. For any epimorphism $f : P \to P$, there is an integer $n$ and an isomorphism $h : P^n \to P^n$ such that the following diagram commutes:

$$
\begin{array}{c}
P^n \\
\downarrow 1 \\
P^n \oplus f \to P^n,
\end{array}
$$

that is, $1 = (\oplus f)h$. Thus $\oplus f$ and hence $f$, are isomorphisms. QED

Recall that a module $M$ is said to be $N$-projective, if for any epimorphism $f : N \to K$ and any $g : M \to K$, there exists an $h : M \to N$ such that $g = fh$; a module $M$ is said to be quasi-projective, if $M$ is $M$-projective. It is well-known that $P^r$ is quasi-projective for any positive integer $r$ if $P$ is quasi-projective. Obviously, every projective module is quasi-projective. We have

Theorem 2.3. For any right module $P$, let $S = \text{End}_R(P)$. If $P$ is quasi-projective, then the following statements are equivalent:

1. The ring $S$ has the right power-substitution property;
2. The ring $S$ has the left power-substitution property;
3. The module $P$ is power-epi-projective;
4. $S$ is power-epi-projective as a right(or left) $S$-module.

Proof. (1)$\iff$(2). see [7].

(1)$\implies$(3). For any given epimorphisms $f : P \to M$ and $g : P \to M$, since $P$ is quasi-projective, we have an $h : P \to P$ such that $f = gh$. In this case, we have $P = \text{im}(h) + \text{ker}(g)$. Now consider the following diagram

$$
\begin{array}{c}
P \\
\downarrow \pi \\
\pi h : P \to P/\text{ker}(g).
\end{array}
$$

Since both $\pi$ and $\pi h$ are epimorphisms, there exists an $\alpha : P \to P$ such that $\pi h \alpha = \pi$. In this case, $\text{im}(1 - h \alpha) \subseteq \text{ker}(g)$. Since $h \alpha + (1 - h \alpha) = 1$, where
$h \in S$, $\alpha \in S$, by assumption there exist an integer $n \geq 1$ and an $Q \in M_n(S)$ such that $hI + (1 - h\alpha)Q$ is invertible in $M_n(S)$. In this case, denotes

$$\phi = hI + (1 - h\alpha)Q : P^n \rightarrow P^n.$$ 

Then $\phi$ is an isomorphism such that the following diagram commutes

$$
\begin{array}{ccc}
P^n & \xrightarrow{\phi} & P^n \\
\downarrow^{\oplus f} & & \downarrow^{\oplus g} \\
P^n & \xrightarrow{\oplus} & M^n.
\end{array}
$$

Thus $P$ is power-epi-projective.

(3) $\Rightarrow$ (1). Suppose that $P$ is a power-epi-projective module and let $b + cd = 1$, where $b \in S$, $c \in S$ and $d \in S$. Consider the following diagram:

$$
\begin{array}{ccc}
P & \xrightarrow{\pi} & P/im(b) \\
\downarrow^{\pi c} & & \downarrow^{\pi} \\
P & \xrightarrow{\pi} & P/im(b).
\end{array}
$$

Since both $\pi c$ and $\pi$ are epimorphisms, there exists a natural number $n$ and an isomorphism $\alpha : P^n \rightarrow P^n$ such that the following diagram commutes:

$$
\begin{array}{ccc}
P^n & \xrightarrow{\oplus (c)} & \oplus(P/im(b)) \\
\downarrow^{\oplus (\pi c)} & & \downarrow^{\oplus \pi} \\
P^n & \xrightarrow{\oplus (\pi c)} & \oplus(P/im(b)).
\end{array}
$$

that is, $(\oplus \pi)\alpha = \oplus (\pi c)$. Now that $im(\alpha - \oplus c) \subseteq \oplus im(b)$, We have got an $Q : P^n \rightarrow P^n$ (or equivalently, $Q \in M_n(S)$) such that the following diagram commutes:

$$
\begin{array}{ccc}
P^n & \xrightarrow{\alpha - \oplus c} & \oplus im(b), \\
\downarrow^{\alpha - \oplus c} & & \downarrow^{\oplus b} \\
P^n & \xrightarrow{\oplus (\pi c)} & \oplus(P/im(b)).
\end{array}
$$

i.e., $\alpha = \oplus c + (\oplus b)Q$. Thus we have an $n \times n$ matrix $Q \in M_n(S)$ such that $cI_n + bQ$ is invertible in $M_n(S)$. 
(1)⇐⇒(4). This is a special case of the "(1)⇐⇒(3)". QED

[Remark] Consider the following weak right power-substitution of rings \( S \): For any \( f, g \) and \( h \) in \( S \) with \( fg + h = 1 \), there exist a positive integer \( n \) and an \( Q \in M_n(S) \) such that \( fI + hQ \) is right invertible in \( M_n(S) \). In Definition 2.1., simply replacing the condition "there exist a positive integer \( n \) and an isomorphism \( h : P^n \to P^n \)" by the condition "there exist a positive integer \( n \) and a splitting epimorphism \( h : P^n \to P^n \)" we obtain a weak version of Theorem 2.3.

Corollary 2.4. For any ring \( R \), the following statements are equivalent:

1. The ring \( R \) has the right power-substitution property;
2. \( R \) is power-epi-projective as a right \( R \)-module;
3. \( R \) is power-epi-projective as a left \( R \)-module.

Corollary 2.5. For any quasi-projective module \( P \), let \( S = \text{End}_R(P) \). If \( P \) is power-epi-projective, then for any \( f_i \in S \) \((i=1,2,3)\) with \( f_1(P) + f_2(P) = f_3(P) \), there exist a positive integer \( n, k_1, k_2 \in M_n(S) \) and invertible matrices \( h_1, h_2 \in M_n(S) \) such that \( f_1h_1 + f_2k_1 = f_3I_n, f_1k_2 + f_2h_2 = f_3I_n \).

Proof. Similar to the Proof of the (3)⇒(1) part of Theorem 2.3. QED

Definition 2.6. The principal right ideals of a ring \( R \) are said to be powerly uniquely generated, if for any \( a, b \in R \) such that \( aR = bR \), there exist an integer \( n \geq 1 \) and an invertible matrix \( Q \in M_n(R) \) such that \( (a, a, \ldots, a) = (b, b, \ldots, b)Q \). As we shall see, this condition is weaker than the right power-substitution property for any ring \( R \).

Proposition 2.7. For any quasi-projective module \( P \), let \( S = \text{End}_R(P) \). Then the following statements are equivalent:

1. The principal right ideals of \( S \) are powerly uniquely generated;
2. For any endomorphic image \( M \) of \( P \) and any epimorphisms \( f : P \to M \) and \( g : P \to M \), there exists an integer \( n \geq 1 \) and an isomorphism \( Q : P^n \to P^n \) such that the following diagram commutes:
Proof. (1)⇒ (2). For any endomorphic image $M$ of $P$ and any epimorphisms $f : P \rightarrow M$ and $g : P \rightarrow M$, we have $fS = gS$. By assumption, there exist an integer $n \geq 1$ and an invertible matrix $Q \in M_n(S)$ such that $\oplus g = (\oplus f)Q$. Thus we have got the required commutative diagram.

(2)⇒ (1). Let $fS = gS$ for some $f, g \in S$. Assume $f = gk, g = fh$. Then $fP = gP$. By assumption, there exist an integer $n \geq 1$ and an isomorphism $Q : P^n \rightarrow P^n$ such that $\oplus g = (\oplus f)Q$. Now recognizing $Q$ as the corresponding invertible $n \times n$ matrix, we finally have $(g, g, \cdots, g) = (f, f, \cdots, f)Q$. Thus the principal right ideals of $S$ are powerly uniquely generated.

QED

Corollary 2.8. If a ring $R$ has the right power-substitution property, then the principal right(left) ideals of $R$ are powerly uniquely generated.

Proposition 2.9. For any $R$, the following statements are equivalent:

(1) The principal right ideals of $R$ are powerly uniquely generated;

(2) For any $a, b \in R$, $aR + rann(b) = R$ implies that there exist an integer $n \geq 1$ and a matrix $Q \in M_n(rann(b))$ such that $aI_n + Q$ is invertible in $M_n(R)$.

Proof. First assume that $aR + rann(b) = R$ for some $a, b \in R$, then $baR = bR$. If the principal right ideals of $R$ are powerly uniquely generated, then there exist an integer $n \geq 1$ and an invertible matrix $Q \in M_n(R)$ such that $b(a, a, \cdots, a) = (b, b, \cdots, b)Q$. Let $U = Q - aI_n$, then we have $(b, b, \cdots, b)U = 0$. Thus $U \in M_n(rann(b))$ and $aI_n + U$ is invertible.

Conversely, assume $aR = bR$ for some $a, b \in R$. Then we have $a(1 - cd) = 0$, where $a = bd, b = ac$. Since $1 - cd \in rann(a)$, by assumption, there exist an integer $n \geq 1$ and a matrix $Q \in M_n(rann(a))$ such that $cI_n + Q$ is invertible in $M_n(R)$. And we have

$$(a, a, \cdots, a)(cI_n + Q) = (b, b, \cdots, b),$$
i.e., the principal right ideals of $R$ are powerly uniquely generated. QED

Dualizing the concept of power-epi-projectivity, we now define:

**Definition 2.10.** For any module $P$, $P$ is said to be power-mono-injective if for any monomorphisms $f : M \to P$ and $g : M \to P$ there is a positive $n$ and an isomorphism $h : P^n \to P^n$ such that the following diagram commutes:

\[
\begin{array}{c}
P^n \\
\uparrow^{\oplus f} \\
\oplus g M^n.
\end{array}
\]

The proof of the following results are dual to the corresponding results in projective cases, so we omit all the proofs:

**Theorem 2.11.** For any right module $P$, let $S = \text{End}_R(P)$. If $P$ is quasi-injective, then the following statements are equivalent:

1. The ring $S$ has the right power-substitution property;
2. The ring $S$ has the left power-substitution property;
3. $P$ is power-mono-injective;
4. $S$ is power-mono-injective as a right(left) $S$-module.

[Remark] By Theorem 2.3. and [6, Proposition 2.6.], any direct summand of power-epi-projective module is power-epi-projective; but direct sums of power-epi-projective modules need not to be power-epi-projective. In fact, by [3], there exists a power-epi-projective module $P$ and a positive integer $n$, such that $P^n$ is not power-epi-projective.

3. **Power-substitution property of exchange rings**

Recall that a ring $R$ is an exchange ring, if the right $R$-module $R$ has the finite exchange property. Examples of exchange rings include von Neumann regular rings, $\pi$-regular rings, right semi-artinian rings, etc. In this section, we study the right power-substitution property of exchange rings. Our results are analogous to those on the stable range one condition of regular rings.
Theorem 3.1. For any exchange ring $R$, $R$ has the right power-substitution property if and only if for any regular element $a \in R$, there exists a positive integer $n$ such that $aI_n$ is unit-regular in $M_n(R)$.

Proof. For any regular element $a \in R$, let $a = axa$. If $R$ has the right power-substitution property, then there exist an integer $n$ and a matrix $Q \in M_n(R)$ such that $U = aI + (1 - ax)Q$ is invertible in $M_n(R)$. We have $axI_n = aU^{-1}$.

Thus we have

$$aI_n = axaI_n = axI_nax = axI_n = aU^{-1}a,$$

and we known that $aI_n$ is unit-regular in $M_n(R)$.

Conversely, for any $a, b', c' \in R$ with $ab' + c' = 1$, there exists an idempotent $c \in c'R$ such that $ab + c = 1$. In this case, if $a$ is regular in $R$, then there exist a positive integer $r$ and invertible matrices $U, W \in M_r(R)$ such that

$$aI_r = axaI_r = axI_r = axI_r = aU^{-1}a,$$

and we known that $aI_r$ is unit-regular in $M_r(R)$.

Denote $e_1 = cW$, $e_2 = aU$. Then both $e_i$ are idempotent matrices of the ring $M_r(R)$, and $M_r(R) = aM_r(R) + cM_r(R) = e_2M_r(R) + e_1M_r(R)$. Let $(1 - e_2) = (e_2)t + e_1s$, then we have $1 - e_2 = (1 - e_2)e_1s$. Thus if we let $f = (1 - e_2)e_1s(1 - e_2)$, then $f$ is idempotent and orthogonal with $e_2$, and $(1 - e_2)e_1 = fe_1$. Thus we have $M_r(R) = e_2M_r(R) + fM_r(R)$. Then $e_2 + f = 1$, that is, $aU[1 - e_2e_1d(1 - e_2)] + cWd(1 - e_2) = 1$. Finally, we have got a positive integer $r$ and a matrix $Q \in M_r(R)$ such that $aI_r + c'Q$ is invertible in $M_r(R)$.

If $a$ is not regular in $R$, then we already have the idempotent $c \in c'R$ and $b \in R$ such that $1 = c + ab$. Then $aba$ is regular in $R$ and $1 = c + (aba)b$. In this case, there exist a positive integer $r$ and a matrix $Q \in M_r(R)$ such that $(aba)I_r + cQ$ is invertible in $M_r(R)$. Since $aba - a \in cR \subseteq c'R$, we have $(aba)I_r + cQ = aI + c'Q'$ which is invertible in $M_r(R)$ and this shows that the ring $R$ has the right power-substitution property.

Since the left version of Theorem 3.1. is obviously also true, this provides a proof of the left-right symmetry of the power-substitution property for exchange rings.

Recall that a module $M$ is said to have the (finite) exchange property, if for any (finite) index set $I$ and any module decompositions $E = M_1 \oplus N = \bigoplus_{i \in I} N_i$ with
$M_1 \cong M$, there exist submodules $M_i$ of $N_i$ such that $E = M_1 \oplus (\oplus_{i \in I} M_i)$; a right module $M$ is said to satisfy power-substitution, if for any module decompositions $E = M_1 \oplus N_1 = M_2 \oplus N_2$, where $M_i \cong M$, there exist a positive integer $n$ and a submodule $C$ such that $E = C \oplus N_1^n = C \oplus N_2^n$ ([6]). Goodearl in [6] proved that this condition is equivalent to the right power-substitution property of the ring $\text{End}_R(M)$, and that it implies the following power cancellation property of $M$: for any isomorphism $M \oplus A \cong M \oplus B$, there exists an integer $n$ such that $A^n \cong B^n$. Now we will prove the converse under the assumption that the module $M$ has the finite exchange property:

**Theorem 3.2.** For any right module $M$, let $S = \text{End}_R(M)$. If $M$ has finite exchange property, then the following statements are equivalent:

1. The ring $S$ has the right power-substitution property;
2. The module $M$ has the right power-substitution property;
3. The module $M$ has the power cancellation property;
4. The module $M$ has the internal power cancellation property, i.e., for any decompositions $M = N_1 \oplus A = N_2 \oplus B$ with $N_1 \cong N_2$, there is a positive integer $n$ such that $A^n \cong B^n$.

**Proof.**

$(1) \implies (2)$. This was proved in [6].

$(2) \implies (3) \implies (4)$. These implications hold obviously for any module $M$.

$(4) \implies (1)$. For any regular element $f \in S$, let $f = fgf$. Then by

$$M = fM \oplus (1 - fg)M = \text{ker}(g) \oplus gfM,$$

and the assumption, there is a positive integer $n$ such that

$$[(1 - fg)M]^n \cong \text{ker}(f)^n.$$

Let $t : \text{ker}(f)^n \to [(1 - fg)M]^n$ be any isomorphism. Then we have got an isomorphism:

$$U = (t \oplus f^n) : M^n = [\text{ker}(f) \oplus gfM]^n \to M^n = [(1 - fg)M \oplus fM]^n.$$

Then $U \in \text{GL}_n(S)$ and it is routine to verify that $fI_n = fUf$. Finally, by Theorem 3.1., we conclude that the ring $S$ has the right power-substitution property. QED
**Corollary 3.3.** For any regular projective module $P$, let $S = \text{End}_R(P)$. Then the following statements are equivalent:

1. The ring $S$ has the right(left) power-substitution property;
2. The module $P$ has the power cancellation property;
3. The module $P$ has the internal power cancellation property;
4. For any $a \in S$, there exist a positive integer $n$ such that $aI_n$ is unit-regular in $M_n(S)$;
5. The module $P$ is power-epi-projective;
6. $S$ is power-epi-projective as a right(left) $S$-module.

**Proof.** Since regular projective modules have the exchange property, the result follows from Theorem 3.1., Theorem 3.2. and Theorem 2.3.

**Corollary 3.4.** For any quasi-injective module $M$, let $S = \text{End}_R(M)$. Then the following statements are equivalent:

1. The ring $S$ has the right(left) power-substitution property;
2. The module $M$ has the power cancellation property;
3. The module $M$ has the internal power cancellation property;
4. The regular ring $S/J(S)$ has the right power-substitution property;
5. For any $a \in S/J(S)$, there exist a positive integer $n$ such that $aI_n$ is unit-regular in $M_n(S/J(S))$;
6. The module $M$ is power-mono-injective;
7. $S$ is power-mono-injective as a right(left) $S$-module.

**Corollary 3.5.** For any exchange ring $R$, the following statements are equivalent:

1. The ring $R$ has the right power-substitution property;
2. $R \oplus A \cong R \oplus B$ implies $A^n \cong B^n$ for some $n \geq 1$;
3. For any $P, Q \in p(R)$, the category of all finitely generated projective right $R$-modules, $R \oplus P \cong R \oplus Q$ implies $P^n \cong Q^n$ for some $n \geq 1$;
4. For any idempotents $e, f \in R$, $eR \cong fR$ implies that $[(1 - e)R]^n \cong [(1 - f)R]^n$ for some $n \geq 1$.

**Proof.** This follows from Theorem 3.2. QED
In [12, Theorem 12], we proved that a right module $M$ satisfies the internal $n$-weak cancellation (i.e., for any decompositions $M = A_1 \oplus A_2$ and $M^n = B_1 \oplus B_2$, where $A_1 \cong B_1$, $A_2$ is isomorphic to a direct summand of $B_2$), if and only if for any regular element $b \in S^n$, there exists a unimodular column $u \in S^n$ such that $b = b u b$, where $S = \text{End}_R(M)$. Now for any module $M$, we characterize the internal power cancellation property as follows:

**Theorem 3.6.** For any module $M$, let $S = \text{End}_R(M)$. Then the following statements are equivalent:

1. $M$ satisfies the internal power cancellation property, i.e., for any decompositions $M = A_1 \oplus N_1 = A_2 \oplus N_2$, where $A_1 \cong A_2$, there exists an integer $n \geq 1$ such that $N_1^n \cong N_2^n$;
2. For any regular element $a \in S$, there exists an integer $n \geq 1$ such that $a I_n$ is unit-regular in $M_n(S)$.

**Proof.** (1)$\Rightarrow$(2). For any regular element $a$ of $S$, let $a = axa$ for some $x \in S$. Let $e = ax$ and $f = xa$. Then both $e$ and $f$ are idempotent elements of $S$. We have a right $R$-module decomposition

$$M = (I - f)(M) \oplus f(M).$$

Also, $x : eM \longrightarrow f(M)$ is a $R$-module isomorphism. By assumption, there exist a positive integer $n$ and an isomorphism

$$((I - e)M)^n \longrightarrow ((I - f)(M))^n.$$

Thus there is an isomorphism $\sigma : M^n \longrightarrow M^n$ such that the restriction of $\sigma$ on $(eM)^n$ is the same as the left multiplication by $x I^n$. Then it is routine to verify that $(a I_n)\sigma(a I_n) = a I_n$, where $I_n$ is the identity $n \times n$ matrix of $M_n(S)$. Thus $a I_n$ is unit-regular in $M_n(S)$.

(2)$\Rightarrow$(1). Let $M = A_1 \oplus B_1 = A \oplus B$ such that $A \cong A_1$. Let

$$x : M \longrightarrow M : B_1 \longmapsto 0, A_1 \longmapsto A,$$

$$y : M \longrightarrow M : B \longmapsto 0, A \longmapsto A_1.$$

Then $y = yxy$. Then by assumption, there exist a positive integer $n$ and a unit $u \in M_n(S)$ such that $y I_n = (y I_n)u(y I_n)$. Finally, from

$$u((A_1)^n) \oplus B^n = M^n = u((A_1)^n) \oplus u((B_1)^n),$$
we obtain $B^n \cong (B_1)^n$. QED

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