ON THE STABLE RANGE OF ENDOMORPHISM RINGS OF QUASI-PROJECTIVE MODULES

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1. Introduction

All rings in this paper are associative with identity and all modules right unital. For any ring $R$, let $R^n = R \times \cdots \times R$ and $^{n}R = \{b^t, \text{ all } b \in R^n\}$. Recall that $b \in R^n$ is said to be a unimodular row, if $b^t(R) = R$; A ring $R$ is said to have stable range at most $n$, if for any unimodular row $b = (b_1, b') \in R \times R^n$, there exists $x \in R^n$ such that $b_1x + b'$ is a unimodular row of $R^n$. In this paper, we introduced the concepts of $n$-epi-projectivity and its dual. For any quasi-projective module $P$, we proved that the stable range of $End_R(P)$ is at most $n$ if and only if $P$ is an $n$-epi-projective module. This generalizes a recent result of Canfell [3] and certainly, it is true particularly for projective modules. In section 3, we will study the properties of $n$-epi-projective modules.

The stable range condition was introduced by Bass to study the stability of the $K_1$-groups ([2]). In [9], Warfield proved that for any module $M$, $End_R(M)$ has stable range at most $n$, if and only if $M$ satisfies the $n$-substitution condition (see [9] for the detailed definition.). When $End_R(M)$ is von Neumann regular, Menal and Moncasi [5] proved that this condition can be replaced by a somewhat weaker condition, i.e., the $n$-weak cancellation. Recently, we studied
the general stable range condition of the endomorphism rings of modules with the finite exchange property ([10]). We proved that for any module $M$ with the finite exchange property, $S$ has stable range at most $n$ if and only if $M$ satisfies the (internal) $n$-weak cancellation, if and only if for any von Neumann regular element $b \in S^n$, there exists a unimodular column $u \in {}^nS$ such that $b = bub$, where $S = \text{End}_R(M)$. On the other hand, Canfell in [3] characterized quasi-projective modules whose endomorphism rings have stable range one, by way of the completions of diagrams, a fundamental technique widely used in homological algebra.

For definitions and results used in this paper without mention, one can refer to [1] and [6]

2. The Stable Range Condition

**Definition 2.1.** For any module $P$ and any positive integer $n$, $P$ is said to be $n$-epi-projective if for any epimorphism
\[
P^n \xrightarrow{f} P \xrightarrow{g} M,
\]
there is an epimorphism $h : P^n \to P$ such that $f = gh$.

Recall that a module $M$ is said to be $N$-projective, if for any epimorphism $f : N \to K$ and any $g : M \to K$, there exists an $h : M \to N$ such that $g = fh$; a module $M$ is said to be quasi-projective, if $M$ is $M$-projective. Obviously, every projective module is quasi-projective. We have

**Proposition 2.2.** (1) A module $P$ is quasi-projective if and only if for any $n \geq 1$, $P$ is $P^n$-projective and $P^n$ is $P$-projective;

(2) If $P$ is quasi-projective, then any epimorphism $P^n \to P$ is splitting.

**Proof.** (1) By [1, P186], $\oplus P_i$ is $Q$-projective if and only if each $P_i$ is $Q$-projective; $P$ is $\oplus Q_i$-projective if and only if $P$ is $Q_i$-projective for all $i$. The result follows from these facts.

(2) If $P$ is quasi-projective, then $P$ is $P^n$-projective by (1). Thus for any epimorphism $f : P^n \to P$, there exists an $g : P \to P^n$ such that $1_P = fg$. QED
Theorem 2.3. For any right module $P$, let $S = \text{End}_R(P)$. If $P$ is quasi-projective, then the following statements are equivalent:

1. The stable range of $S$ is at most $n$;
2. $P$ is $n$-epi-projective;
3. $S$ is $n$-epi-projective as a right $S$-module.

Proof. (1)$\Rightarrow$(2). Suppose that the stable range of $S$ is at most $n$. For any given epimorphism $f : P^n \to M$ and $g : P \to M$, since $P$ is quasi-projective, we have an $h : P^n \to P$ such that $f = gh$. In this case, we have

$$P = \text{im}(h) + \ker(g).$$

Now consider the following diagram

$$\begin{array}{ccc}
P & \xrightarrow{\pi} & P/\ker(g) \\
\downarrow{\pi h} & & \\
P^n & \to & P/\ker(g).
\end{array}$$

Since both $\pi$ and $\pi h$ are epimorphism, by Proposition 2.2, there exists $\alpha : P \to P^n$ such that $\pi h \alpha = \pi$. In this case, $\text{im}(1 - h \alpha) \subseteq \ker(g)$. Since $h \alpha + (1 - h \alpha) = 1$, where $h \in S^n$, $\alpha \in nS$, by assumption there exists an $u \in S^n$ such that $h + (1 - h \alpha)u$ is a unimodular row of $S^n$. Let $\phi = h + (1 - h \alpha)u$. Then $\phi : P^n \to P$ is a splitting epimorphism such that $g \phi = f$. This proves that $P$ is $n$-epi-projective.

(2)$\Rightarrow$(1). Suppose that $P$ is an $n$-epi-projective module and let $b + cd = 1$, where $b \in S$, $c \in S^n$ and $d \in nS$. Consider the following diagram:

$$\begin{array}{ccc}
P^n & \xrightarrow{\pi c} & P/im(b) \\
\downarrow{\pi c} & & \\
P & \xrightarrow{\pi} & P/im(b).
\end{array}$$

Since both $\pi c$ and $\pi$ are epimorphism, there exists an epimorphism $\alpha : P^n \to P$ such that $\pi \alpha = \pi c$. By Proposition 2.2, $\alpha$ is splitting. Now $\text{im}(c - \alpha) \subseteq \text{im}(b)$. We have

$$\begin{array}{ccc}
P^n & \xrightarrow{\alpha - c} & P/im(b) \\
\downarrow{\alpha - c} & & \\
P & \xrightarrow{b} & \text{im}(b).
\end{array}$$
Since $P$ is quasi-projective, we have got an $\beta : P^n \to P$ such that $b \beta = \alpha - c$. Hence $b \beta + c$ is a unimodular row of $S^n$.

(1)$\Leftrightarrow$(3). This is a special case of (1)$\Leftrightarrow$(2). QED

**Corollary 2.4.** For any ring $R$, the following statements are equivalent:

1. The stable range of $R$ is at most $n$;
2. $R$ is $n$-epi-projective as a right $R$-module;
3. $R$ is $n$-epi-projective as a left $R$-module.

*Proof.* This follows from the fact that the stable range condition is left-right symmetric. QED

**Theorem 2.5.** For any quasi-projective module $P$ with the finite exchange property, let $S = \text{End}_R(P)$. Then the following statements are equivalent:

1. The stable range of $S$ is at most $n$;
2. $P$ is $n$-epi-projective;
3. $P^n \oplus A \cong P \oplus B$ implies that $A$ is isomorphic to a direct summand of $B$;
4. For any decompositions $P = A_1 \oplus B_1$, $P^n = A_2 \oplus B_2$, where $A_1 \cong A_2$, $B_1$ is isomorphic to a direct summand of $B_2$;
5. $S$ is $n$-epi-projective as a right $S$-module;
6. Any regular element $b \in S^n$ can be written as $b = bu$ for some unimodular column $u \in nS$.

*Proof.* (1)$\Leftrightarrow$(2)$\Leftrightarrow$(5). By Theorem 2.3.

(1)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Leftrightarrow$(6). By [Theorem13 of 10].

By [7], *every regular projective module has the exchange property*, hence Theorem 2.5. holds for each regular projective module.

Dualizing the concept of $n$-epi-projectivity, we now define:

**Definition 2.6.** For any module $P$ and any positive integer $n$, $P$ is said to be
n-mono-injective if for any monomorphism

\[
\begin{array}{c}
P^n \\
\uparrow f \\
P \xleftarrow{g} M,
\end{array}
\]

there is a monomorphism \( h : P \to P^n \) such that \( f = hg \).

The proof of the following results are dual to the corresponding results in projective cases, so we omit all the proofs:

**Proposition 2.7.** (1) A module \( E \) is quasi-injective if and only if for any \( n \geq 1 \), \( E \) is \( E^n \)-injective and \( E^n \) is \( E \)-injective;

(2) If \( E \) is quasi-injective, then any monomorphism \( f : E \to E^n \) is splitting.

**Theorem 2.8.** For any right module \( P \), let \( S = \text{End}_R(P) \). If \( P \) is quasi-injective, then the following statements are equivalent:

(1) The stable range of \( S \) is at most \( n \);

(2) \( P \) is \( n \)-mono-injective;

(3) \( S \) is \( n \)-mono-injective as a right \( S \)-module.

It is well-known that quasi-injective modules have the finite exchange property. So we have

**Theorem 2.9.** For any quasi-injective module \( P \), let \( S = \text{End}_R(P) \). Then the following statements are equivalent:

(1) The stable range of \( S \) is at most \( n \);

(2) \( P \) is \( n \)-mono-injective;

(3) \( P^n \oplus A \cong P \oplus B \) implies that \( A \) is isomorphic to a direct summand of \( B \);

(4) For any decompositions \( P = A_1 \oplus B_1 \), \( P^n = A_2 \oplus B_2 \), where \( A_1 \cong A_2 \), \( B_1 \) is isomorphic to a direct summand of \( B_2 \);

(5) \( S \) is \( n \)-mono-injective as a right \( S \)-module;

(6) Any regular element \( b \in S^n \) can be written as \( b = b u b \) for some unimodular column \( u \in \mathbb{N} S \).

**Proof.** The result follows from Theorem 2.8 and [Theorem 13 of 10]. QED
Proposition 3.1. For any \( n \)-epi-projective module \( P \), every epic endomorphism of \( P \) is splitting.

Proof. For any epimorphism \( g \) of \( P \), let \( \pi : P^n \to P \) be the projection from the first component. Then there exists an epimorphism \( h : P^n \to P \) such that \( \pi = gh \). Let \( i : P \to P^n \) be the injection. Then we have \( g(hi) = 1 \). Thus \( g \) is splitting. QED

Theorem 3.2. (1) For any projective module \( P \) and \( Q \), if both \( P \) and \( Q \) are \( n \)-epi-projective, then \( P \oplus Q \) is also \( n \)-epi-projective;

(2) For any quasi-projective module \( P \) and any integers \( r \) and \( n \), if \( P \) is \( n \)-epi-projective, then \( P^r \) is \( n \)-epi-projective.

Proof. (1) If \( P \) and \( Q \) are \( n \)-epi-projective, by Theorem 2.3, both \( \text{End}_R(P) \) and \( \text{End}_R(Q) \) have stable ranges at most \( n \). By [9, Theorem 1.6.], both \( P \) and \( Q \) have the \( n \)-substitution properties. Thus by [9, Theorem 1.9.], \( P \oplus Q \) has the \( n \)-substitution property. Thus \( \text{End}_R(P \oplus Q) \) has stable range at most \( n \). Again by Theorem 2.3., \( P \oplus Q \) is \( n \)-epi-projective.

(2) If \( P \) is quasi-projective and \( n \)-epi-projective, then by theorem 2.3, \( \text{End}_R(P) \) has stable range at most \( n \). Since \( P^r \) is also quasi-projective and \( \text{End}_R(P^r) \) has \( n \) in its stable range, again by theorem 2.3, \( P^r \) is \( n \)-epi-projective. QED

A natural question one would like to ask is the following: If \( P \) is \( n \)-epi-projective, is the direct summand of \( P \) also \( n \)-epi-projective? For \( n \geq 2 \), this need not to be true, even for projective modules. In fact, by a formula given by Vaserstein [8], the stable range of \( M_n(R) \) is at most 2 if the stable range of \( R \) is \( n \) (\( > 2 \)). Thus there exists a ring \( R \) whose stable range is \( n \) (\( > 2 \)), but the endomorphism ring of \( R^{n-1} \) is at most 2. Thus by Theorem 2.3, the right \( R \)-module \( R^{n-1} \) is 2-epi-projective, but its direct summand \( R \) is not 2-epi-projective. When \( n = 2 \), we have the following example which is essentially taken from [5, P38]:

Example 3.3. Let \( R = \text{End}_K(V) \), where \( V \) is a countably infinitely generated vector space over a field \( K \). Then \( R \) is a regular ring whose stable range is not
finite. Choose any cardinal $\aleph > \aleph_0$ and let $W$ be the $K$-vector space of dimension $\aleph$. Let

$$M = \{\phi \in \text{End}_K(W) | \dim \text{Im}(\phi) < \aleph\}.$$ 

Let $S = M + K$. Then by [5, Example 3], $S$ is a regular ring with stable range 2, and $R \cong eSe$ for some idempotent element $e \in S$. By Theorem 2.3, $S_S$ is 2-epi-projective, but its direct summand $eS$ is not $n$-epi-projective for any $n \geq 1$.

Inspite of the above remark, we have the following

**Proposition 3.4.** For any quasi-projective module $P$, if $P$ is 1-epi-projective, then any direct summand of $P$ is also 1-epi-projective.

*Proof.* By Theorem 2.3, $\text{End}_R(P)$ has one in its stable range. For any direct summand $Q$ of $P$, $\text{End}_R(Q)$ is a corner of $\text{End}_R(P)$. By [8], the stable range one property is preserved under taking corners. Thus $\text{End}_R(Q)$ also has stable range one. Since $Q$ is also quasi-projective, thus $Q$ is 1-epi-projective by Theorem 2.3. QED

We remark that Proposition 3.4 was proved in [3] without the condition of quasi-projectivity. But there is a serious gap in the proof there that could not be filled.

It would be interesting to give a direct module-theoretic proof of Theorem 3.2. and Proposition 3.4.

**Proposition 3.5.** Let $Q$ be a direct summand of an $n$-epi-projective module. If $\text{Hom}_R(P/Q, Q) = 0$, then $Q$ is also $n$-epi-projective.

*Proof.* Let $P = Q \oplus K$, and $\pi : K^n \rightarrow K$ be any projection. For any epimorphism $f : Q \rightarrow M$ and $g : Q^n \rightarrow M$, we have epimorphism

$$f' : Q \oplus K \rightarrow M \oplus K, \text{ and } g' : Q^n \oplus K^n \rightarrow M \oplus K,$$

where $f'$ is determined by $f$ and $1_K$, $g'$ is determined by $g$ and $\pi$. Since $P$ is $n$-epi-projective, there exists an epimorphism $h : P^n \rightarrow P$ such that $g' = f'h$.

Thus we have $h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ such that

$$\begin{pmatrix} g & 0 \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$
Since $\text{Hom}_R(P/Q, Q) = 0$, $h_{11} : Q^n \to Q$ is epimorphic and $g = fh_{11}$. This shows that $Q$ is also $n$-epi-projective. QED

The proof of the following results are dual to the proof of the above results:

**Proposition 3.6.** For any $n$-mono-injective module $P$, every monic endomorphism of $P$ is splitting.

**Theorem 3.7.** (1) For any injective module $P$ and $Q$, if both $P$ and $Q$ are $n$-mono-injective, then $P \oplus Q$ is also $n$-mono-injective;

(2) For any quasi-injective module $P$, if $P$ is $n$-mono-injective, then $P^r$ is also $n$-mono-injective.

The direct summand of an $n$-mono-injective module need not to be $n$-mono-injective.

**Proposition 3.8.** For any quasi-injective module $P$, if $P$ is 1-mono-injective, then any direct summand of $P$ is also 1-mono-injective.

**Proposition 3.9.** Let $Q$ be a direct summand of an $n$-mono-injective module $P$. If $\text{Hom}_R(Q, P/Q) = 0$, then $Q$ is also $n$-mono-injective.

We end this paper with the following example

**Example 3.10.** Any semisimple artinian module is quasi-projective and quasi-injective. It is also $n$-epi-projective and $n$-mono-injective for all $n$. It needs not to be projective.

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**REFERENCES**