On a relative calculus for Chow rings of projective bundles

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Abstract

For a vector bundle $E \to X$ with $X$ a smooth variety over a field, we prove a formula for the proper pushforward $\pi_* : A^* \mathbb{P}(E) \to A^* X$, where $\pi : \mathbb{P}(E) \to X$ denotes the associated projective bundle of lines in $E$ and $A^*$ denotes the Chow ring functor. Our formula is relative in the sense that it depends on the rank of $E$ while being independent of the dimension of $X$, and is given in terms of a differential operator on the Chow ring of $X$. As applications, we apply the formula to compute the relative Chern class of a fibration $Y \to X$ (whose fibers are projective hypersurfaces of arbitrary degree), and then derive a ‘relative Bézout formula’ for the intersection of $\operatorname{dim}(\mathbb{P}(E))$ divisors in $\mathbb{P}(E)$ in terms of invariants of $X$.

1 Introduction

Let $X$ be a smooth variety over field $\mathbb{K}$ endowed with a vector bundle $E \to X$, and denote by $\pi : \mathbb{P}(E) \to X$ the associated projective bundle of lines in $E$. A fundamental task in the investigation of $\mathbb{P}(E)$ is to relate its invariants to invariants of $X$, the prototypical example of which being Grothendieck’s formula for the Chow ring $A^* \mathbb{P}(E)$ in terms of $A^* X$, namely

$$A^* \mathbb{P}(E) \cong A^* X \left[ \zeta \right] / \left( \zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_r(E) \right),$$

(1.1)

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$, $r = \operatorname{rk}(E)$ and $c_i(E)$ denotes the $i$th Chern class of $E$. Other than its simplicity, an attractive feature of formula (1.1) is that the right-hand side only depends on the rank of $E$, thus being independent of the dimension of $X$.

In this note we consider the map

$$\pi_* : A^* \mathbb{P}(E) \to A^* X,$$
i.e., the proper pushforward of algebraic cycles modulo rational equivalence associated with the bundle projection \( \pi \), and derive a formula for \( \pi_* \), which (in a similar vein) only depends on the rank of \( \mathcal{E} \). As the pushforward operation \( \pi_* \) may be thought of as ‘integration along the fiber’, we view our formula (and its applications) as providing a ‘relative calculus’ for Chow rings of projective bundles. Our motivation for such a formula comes primarily from the fact that

\[
\int_{\mathbb{P}(\mathcal{E})} \alpha = \int_X \pi_* \alpha \quad \forall \alpha \in A^*(\mathbb{P}(\mathcal{E})),
\]  

(1.2)

so that invariants of \( \mathbb{P}(\mathcal{E}) \) (and of its closed subschemes) captured by the Chow ring may be expressed in terms of invariants of \( X \), which are naturally much simpler. To state our formula we first introduce some notation.

Denote by \( L_1, \ldots, L_m \) the distinct non-trivial Chern roots of \( \mathcal{E} \) and let \( k_i \) be such that the multiplicity of \( L_i \) is \( k_i + 1 \), so that

\[
c(\mathcal{E}) = (1 + L_1)^{k_1+1} \cdots (1 + L_m)^{k_m+1} \in A^*X.
\]

We then consider the ring \( A^*X[x_1, \ldots, x_m] \) where the \( x_i \) are formal variables, and let \( D \) be the differential operator on this ring given by

\[
g \mapsto 1 \prod_{i=1}^{k_m} \frac{\partial^{k_i+\cdots+k_m}}{k! \cdots k_m! \partial x_1^{k_1} \cdots \partial x_m^{k_m}}(x_1^{k_1} \cdots x_m^{k_m} \cdot g).
\]

Now let \( \alpha \in A^*\mathbb{P}(\mathcal{E}) \) be arbitrary, which by formula (1.1) may be written as

\[
\alpha = \alpha_0 + \alpha_1 \zeta + \alpha_2 \zeta^2 + \cdots,
\]

where \( \alpha_j \zeta^j \) denotes \( \pi^* \alpha_j \cdot (c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cap [\mathbb{P}(\mathcal{E})])^j \) with \( \alpha_j \in A^*X \). We then associate with \( \alpha \) the elements

\[
g_\alpha(L_i) = \frac{\alpha - (\alpha_0 + \alpha_1 \zeta + \cdots + \alpha_{n-1} \zeta^{n-1})}{\zeta^{n-m+1}} \bigg|_{\zeta = L_i} \in A^*X[L_i],
\]

where \( n = \text{rk}(\mathcal{E}) - 1 \). Our formula is then given by the following expression, which will be formally stated and proved as Theorem 2.1 below.

**Theorem (Main Theorem).** With notations as above, the map \( \pi_* : A^*\mathbb{P}(\mathcal{E}) \to A^*X \) is given by

\[
\pi_* \alpha = D \cdot \sum_{i=1}^{m} \frac{g_\alpha(L_i)}{\prod_{l=1, l \neq i}^{m} (L_i - L_l)} \in A^*X.
\]  

(1.3)

\(^1\)The integral sign in (1.2) is notation for the the proper pushforward associated with the corresponding structure morphism to \( \text{spec}(\mathcal{R}) \).
Clearly formula (1.3) is symmetric in the distinct Chern roots of \( \mathcal{E} \) so that the end result may be expressed in terms of the Chern classes \( c_i(\mathcal{E}) \) and the \( \alpha_j \) for \( j \geq n \). As we view the operation \( \pi_* \) as a relative form of integration, it is interesting to note that the appearance of the differential operator on the right-hand side of formula (1.3) gives it the flavor of a ‘Stokes-type formula’. The (non-invertible) terms \( \prod_{l=1, l \neq i}^{m} (L_i - L_l) \) appearing in the denominator of the formula are formal expressions that end up canceling when the above expression is put over a common denominator. When \( \mathcal{E} \) is the direct sum of tensor powers of a fixed line bundle on \( X \) we recover the main formula derived in [7].

For projective bundles of small rank, formula (1.3) is reasonable to apply by hand, though in any case a computer implementation is straightforward and is of negligible cost. We note that such an implementation may be of utility for software packages computing characteristic classes of varieties. In particular an implementation of (1.3) allows the applicability setting of algorithms for computing singular Chern classes and Euler characteristics of (possibly singular) projective varieties and schemes (such as [11, 10]) to be broadened to the setting of projective bundles. To this end, and to simplify computations for the interested reader, we provide a test implementation in the form of a Macaulay2 [9] package which can be downloaded from https://github.com/Martin-Helmer/PBF.

For an explicit example of an application of the Main Theorem (proved as Theorem 2.1 below), let \( X \) be a smooth Fano variety over \( \mathbb{C} \),

\[
\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}(-2K_X) \oplus \mathcal{O}(-3K_X),
\]

and let \( \iota : Y \hookrightarrow \mathbb{P}(\mathcal{E}) \) be a smooth anti-canonical hypersurface in \( \mathbb{P}(\mathcal{E}) \) given by the Weierstraß equation

\[
Y : (y^2 = x^3 + f x z^2 + g z^3) \subset \mathbb{P}(\mathcal{E}),
\]

where the coefficients \( f \) and \( g \) are sections of \( \mathcal{O}(-4K_X) \) and \( \mathcal{O}(-6K_X) \) respectively. The map \( \varphi = \pi \circ \iota \) then endows \( Y \) with the structure of an elliptic fibration \( \varphi : Y \to X \). Motivated by physical considerations, in [13] Sethi, Vafa and Witten derived a formula for the Euler characteristic of \( Y \) in terms of the Chern classes of \( X \) for the case \( \dim(X) = 3 \), namely

\[
\chi(Y) = \int_X 12c_1(X)c_2(X) + 360c_1(X)^3.
\]

(1.5)

We note that the above formula may be simplified further since we necessarily have \( c_1(X)c_2(X) = 24 \) for \( X \) Fano. It immediately follows via Theorem 2.1 that formula (1.5) is simply a manifestation of formula (1.2) for \( \alpha = \iota_* c(Y) \), i.e.,

\[
\int_{\mathbb{P}(\mathcal{E})} \iota_* c(Y) = \int_X \pi_* (\iota_* c(Y)),
\]

(1.6)

as the left-hand side of equation (1.6) coincides with \( \chi(Y) \) by the Gauß-Bonnet theorem, and Theorem 2.1 immediately yields the right-hand side to be \( \int_X 12c_1(X)c_2(X) + 360c_1(X)^3 \).
More precisely, by the adjunction formula and standard exact sequences of vector bundles we have

\[ \iota^* c(Y) = \frac{(1 + \zeta)(1 + \zeta - 2K_X)(1 + \zeta - 3K_X)(3\zeta - 6K_X)}{(1 + 3\zeta - 6K_X)} \cdot \pi^* c(X), \]

where again \( \zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \cap [\mathbb{P}(E)] \). Thus

\[ \varphi^* c(Y) = \pi^*(\iota^* c(Y)) = \pi^* \left( \frac{(1 + \zeta)(1 + \zeta - 2K_X)(1 + \zeta - 3K_X)(3\zeta - 6K_X)}{(1 + 3\zeta - 6K_X)} \right) c(X), \]

where the second equality follows from the projection formula. We then take

\[ \alpha = \frac{(1 + \zeta)(1 + \zeta - 2K_X)(1 + \zeta - 3K_X)(3\zeta - 6K_X)}{(1 + 3\zeta - 6K_X)} = \alpha_0 + \alpha_1 \zeta + \cdots \in A^* \mathbb{P}(E), \]

where the \( \alpha_i \) are the Taylor coefficients of \( \alpha \) when expanded with respect to \( \zeta \). By Theorem 2.1 we then have

\[ \pi^* \alpha = \left( \alpha - \left( \frac{\alpha_0 + \alpha_1 \zeta}{\zeta(-3K_X + 2K_X)} \right)_{\zeta=2K_X} \right) \left( \frac{\alpha - (\alpha_0 + \alpha_1 \zeta)}{\zeta(-2K_X + 3K_X)} \right)_{\zeta=3K_X} = \frac{-12K_X}{1 - 6K_X}, \]

thus

\[ \varphi^* c(Y) = \frac{12c_1(X)}{1 + 6c_1(X)} c(X). \quad (1.7) \]

Such a formula is often referred to in the literature as a Sethi-Vafa-Witten formula, see \[7, 1, 5\]. We note that one of the advantages of our methods is the dimension-independent nature of Theorem 2.1, so that equation (1.7) is valid for \( X \) of any dimension. One may then obtain an analogue of formula (1.5) for \( \chi(Y) \) with \( \dim(X) \) arbitrary simply by reading off the term of codimension \( \dim(X) \) on the right-hand side of equation (1.7), namely

\[ \chi(Y) = \int_X 12c_1(X) \sum_{i=0}^{\dim(X)-1} c_i(X) (-6c_1(X))^{\dim(X)-1-i}. \]

In an analogous fashion one may use Theorem 2.1 to realize other invariants of closed subschemes of a projective bundle \( \mathbb{P}(E) \) – such as intersection numbers, Chern numbers, Hirzebruch genera, etc. – in terms of invariants of its base, which as noted earlier are often much simpler. We devote §2 to the statement and proof of Theorem 2.1 (referred to as the Main Theorem in the introduction); in the remaining sections we derive some
applications of this theorem. In particular, we use Theorem 2.1 to derive a relative Bézout formula for the intersection of \( \dim(\mathbb{P}(\mathcal{E})) \) divisors in \( \mathbb{P}(\mathcal{E}) \) in terms of the Chern classes of \( \mathcal{E} \) and \( TX \), and we generalize formula (1.7) to the case where \( Y \) is a general hypersurface in an arbitrary projective bundle \( \mathbb{P}(\mathcal{E}) \to X \), so that \( Y \) is a general fibration of projective hypersurfaces. In §3 we give explicit embeddings of such fibrations in a \( \mathbb{P}^n \)-bundle (for every \( n > 0 \)) with the fiber of arbitrary degree.

Notation and conventions. The assumptions made at the outset of this section will hold throughout the rest of the paper. In particular, we will work over an arbitrary field \( \mathbb{K} \), except in §3 where we assume that \( \mathbb{K} \) is an algebraically closed field of characteristic zero. We denote by \( X \) an arbitrary smooth \( \mathbb{K} \)-variety, which will always come endowed with a vector bundle \( \mathcal{E} \to X \). We denote by \( \pi : \mathbb{P}(\mathcal{E}) \to X \) the associated projective bundle of lines in \( \mathcal{E} \). The distinct non-trivial Chern roots of \( \mathcal{E} \) will be denoted by \( L_1, \ldots, L_m \), and \( k_i \in \mathbb{Z} \) will be such that the multiplicity of \( L_i \) is \( k_i + 1 \). We will always assume \( m \leq \text{rk}(\mathcal{E}) - 1, \) as \( \mathbb{P}(\mathcal{E}) \) isomorphic to \( \mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \) for any line bundle \( \mathcal{L} \) implies without loss of generality that we can set one of the Chern roots of \( \mathcal{E} \) equal to zero.

2 Proof of Main Theorem

Let \( \alpha \in A^* \mathbb{P}(\mathcal{E}) \) be arbitrary, which by the isomorphism

\[
A^* \mathbb{P}(\mathcal{E}) \cong A^* X[\zeta]/(\zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_r(\mathcal{E}))
\]

may be written as \( \alpha = \alpha(\zeta) = \alpha_0 + \alpha_1 \zeta + \alpha_2 \zeta^2 + \cdots \), where we recall \( \alpha_j \zeta^j \) denotes \( \pi^* \zeta^j \cdot \left( c_1(\mathcal{E}_\mathbb{P}(\mathcal{E})(1)) \cap \left[ \mathbb{P}(\mathcal{E}) \right] \right)^j \).

Theorem 2.1 (Main Theorem). With notations as above we have that the map \( \pi_* : A^* \mathbb{P}(\mathcal{E}) \to A^* X \) is given by

\[
\pi_* \alpha = \left( D \cdot \sum_{i=1}^m \frac{g_\alpha(x_i)}{\prod_{l=1, l \neq i}^m (x_i - x_l)} \right) \bigg|_{x_1 = -L_1, \ldots, x_m = -L_m} \in A^* X,
\]

where

\[
g_\alpha(x_i) = \frac{\alpha(x_i) - (\alpha_0 + \alpha_1 x_i + \cdots + \alpha_{\text{rk}(\mathcal{E}) - 2} x_i^{\text{rk}(\mathcal{E}) - 2})}{x_i^{\text{rk}(\mathcal{E}) - m} \prod_{l=1, l \neq i}^m (x_i - x_l)},
\]

and \( D \) is the differential operator on \( A^* X[x_1, \ldots, x_m] \) given by

\[
g \mapsto \frac{1}{k_1! \cdots k_m!} \frac{\partial^{k_1 + \cdots + k_m}}{\partial x_{i_1}^{k_1} \cdots \partial x_{i_m}^{k_m}} (x_1^{k_1} \cdots x_m^{k_m} \cdot g).
\]
Proof. As $\alpha \in A^*\mathbb{P}(\mathcal{E})$ may be written as $\alpha = \sum_{j=0}^{\text{rk}(\mathcal{E})} \pi^* \alpha_j \cdot \zeta^j$, by the projection formula for intersection products we have

$$\pi_* \alpha = \sum_{j=0}^{\text{rk}(\mathcal{E})} \alpha_j \cdot \pi_* (\zeta^j),$$

thus $\pi_* \alpha$ is completely determined by the $\pi_* (\zeta^j)$. For this, we use the fact that by Fulton’s definition of Chern class ([8], §3.2) we have

$$\pi_* (1 + \zeta + \zeta^2 + \cdots) = c(\mathcal{E})^{-1} = \frac{1}{(1 + L_1)^{k_1+1} \cdots (1 + L_m)^{k_m+1}} \in A^* X, \quad (2.1)$$

where we recall that the $L_i$ are the distinct non-trivial Chern roots of $\mathcal{E}$ and the $k_i$ are uniquely determined non-negative integers. As the proper pushforward of a pure dimensional cycle (not in the kernel of $\pi_*$) preserves its dimension, it follows that $\pi_* (\zeta^j)$ coincides with the term of dimension $\dim(\mathbb{P}(\mathcal{E})) - j$ in the formal series expansion of \[\left((1 + L_1)^{k_1+1} \cdots (1 + L_m)^{k_m+1}\right)^{-1} \text{ (so that } \pi_* (\zeta^j) = 0 \text{ for } j < \text{rk}(\mathcal{E}) - 1).\] Now since

$$\frac{1}{(1 + L_i)^{k_i+1}} = \frac{1}{k_i!} \frac{\partial^{k_i}}{\partial x_i^{k_i}} \left(\frac{1}{1 - x_i}\right)\bigg|_{x_i = -L_i} = \left(\sum_{j=0}^{\sum_{i_1 + \cdots + i_m = d} \beta_{i_1}^{(k_1)} (-L_1)^{i_1} \cdot \beta_{i_2}^{(k_2)} (-L_2)^{i_2} \cdots \beta_{i_m}^{(k_m)} (-L_m)^{i_m}}\right),$$

the term of codimension $d$ in $X$ of $c(\mathcal{E})^{-1}$ is precisely

$$\sum_{i_1 + \cdots + i_m = d} \beta_{i_1}^{(k_1)} (-L_1)^{i_1} \cdot \beta_{i_2}^{(k_2)} (-L_2)^{i_2} \cdots \beta_{i_m}^{(k_m)} (-L_m)^{i_m},$$

where

$$\beta_{i_t}^{(k_t)} = \begin{cases} \frac{(k_t + i_t)!}{k_t! i_t!} & \text{for } i_t \neq 0 \\ 1 & \text{for } i_t = 0. \end{cases}$$

Now let $n = \text{rk}(\mathcal{E}) - 1$. It then follows that $\dim(\zeta^n) = \dim(X)$, so that the term $\pi_* (\zeta^{n+j})$ is of codimension $j$ in $X$. We then have

$$\pi_* (\zeta^{n+j}) = \begin{cases} 0 & \text{for } j < 0 \\ \sum_{i_1 + \cdots + i_m = j} \beta_{i_1}^{(k_1)} (-L_1)^{i_1} \cdot \beta_{i_2}^{(k_2)} (-L_2)^{i_2} \cdots \beta_{i_m}^{(k_m)} (-L_m)^{i_m} & \text{for } j \geq 0. \end{cases} \quad (2.2)$$
Thus

\[\pi_* \alpha = \pi_* \left( \sum_{j=-n}^{n} \alpha_{n+j} s^{n+j} \right)\]

\[= \sum_{j=-n}^{n} \alpha_{n+j} \cdot \pi_* (s^{n+j}) \quad \text{(By the projection formula)}\]

\[= \sum_{j=0}^{2} \alpha_{n+j} \left( \sum_{i_1 + \cdots + i_m = j} \beta_{i_1}^{(k_1)} (-L_1)^{i_1} \cdot \beta_{i_2}^{(k_2)} (-L_2)^{i_2} \cdots \beta_{i_m}^{(k_m)} (-L_m)^{i_m} \right)\]

\[= \sum_{j=0}^{2} \alpha_{n+j} \left( \sum_{i_1 + \cdots + i_m = j} \frac{1}{k_1!} \frac{\partial^{k_1}}{\partial x_1^{i_1+k_1}} \left( x_1^{i_1+k_1} \right) \right) \left( \sum_{i_1 + \cdots + i_m = j} \frac{1}{k_m!} \frac{\partial^{k_m}}{\partial x_m^{i_m+k_m}} \left( x_m^{i_m+k_m} \right) \right)\]

\[= \sum_{j=0}^{2} \alpha_{n+j} \left( \frac{1}{k_1!} \cdots \frac{1}{k_m!} \frac{\partial^{k_1+\cdots+k_m}}{\partial x_1^{i_1+k_1} \cdots \partial x_m^{i_m+k_m}} \left( x_1^{i_1+k_1} \cdots x_m^{i_m+k_m} \right) \right)\]

\[= \sum_{j=0}^{2} \alpha_{n+j} \left( D \cdot \sum_{i_1 + \cdots + i_m = j} \prod_{l=1}^{m} \frac{x_i^{m+j-1}}{x_l^m (x_i - x_l)} \right)\]

\[= \left( D \cdot \sum_{i_1 + \cdots + i_m = j} \prod_{l=1}^{m} \frac{x_i^{m+j-1}}{x_l^m (x_i - x_l)} \right)\]

The conclusion follows from the last expression; note that in the seventh equality we used the fact that (see [4])

\[\sum_{i_1 + \cdots + i_m = j} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} = \sum_{i_1 + \cdots + i_m = j} \prod_{l=1}^{m} \frac{x_i^{m+j-1}}{x_l^m (x_i - x_l)}. \quad (2.3)\]

\[\square\]

**Remark 2.1.** It follows from formula (2.3) (along with degree considerations) that terms of the form \(L_i^{\text{rank}(\mathcal{E})-m} \prod_{l=1, l \neq i}^{m} (L_i - L_l)\) appearing in final expression for \(\pi_* \alpha\) in fact cancel once put over a common denominator. As such, in practice one may in fact avoid the use of the differential operator \(D\) by first assuming all the non-trivial Chern roots are distinct, and then evaluating the final simplified rational expression at the possibly non-distinct total number of non-trivial Chern roots of \(\mathcal{E}\).
3 Chern Class of a Fibration

We first review the theory of Chern classes from a functorial perspective, for more details one may consult [3, 14]. In this section we impose the constraint that \( \mathfrak{K} \) denotes an algebraically closed field of characteristic zero. For \( V \) a \( \mathfrak{K} \)-variety denote by \( K_0(\text{Var}_V) \) the free \( \mathbb{Z} \)-module generated by isomorphism classes of proper morphisms to \( V \), modulo the relation

\[
[Y \xrightarrow{\varphi} V] = [Z \xrightarrow{\varphi|_Z} V] + [U \xrightarrow{\varphi|_U} V],
\]

whenever \( Z \) is a closed subvariety of \( Y \) with open complement \( U \) in \( Y \). We may endow \( K_0(\text{Var}_V) \) with a ring structure by setting

\[
[Y \xrightarrow{\varphi} V] \cdot [W \xrightarrow{\psi} V] = [Y \times_V W \rightarrow V],
\]

where \( Y \times_V W \rightarrow V \) denotes the fibered product of \( Y \) with \( W \) over \( V \) with respect to the morphisms \( \varphi \) and \( \psi \). We refer to this ring \( K_0(\text{Var}_V) \) as the relative Grothendieck ring of varieties (over \( V \)). When \( V = \text{spec}(\mathfrak{K}) \) the relative Grothendieck ring is often denoted \( K_0(\text{Var}_\mathfrak{K}) \), and referred to simply as the Grothendieck ring of varieties. Moreover, we may view \( K_0(\text{Var}_\mathfrak{K}) \) as a covariant functor from varieties to groups with respect to proper morphisms.

For this, given a proper morphism \( V \xrightarrow{f} S \) we define a homomorphism

\[
K_0(f) : K_0(\text{Var}_V) \longrightarrow K_0(\text{Var}_S)
\]

given by

\[
[Y \rightarrow V] \mapsto [Y \rightarrow V \rightarrow S].
\]

With these prescriptions it is straightforward to show that indeed \( K_0(\text{Var}_\mathfrak{K}) \) is a covariant functor with respect to proper morphisms, which we refer to as the relative Grothendieck functor.

Now denote by \( A_* \) the Chow group functor, which takes a variety \( V \) to its group of algebraic cycles modulo rational equivalence \( A_* V \), and acts on proper morphisms via proper pushforward of algebraic cycles. For \( V \) smooth the Chow group may be given a ring structure via Fulton’s intersection products [8, §6]. In such a case we will denote \( A_* V \) by \( A^* V \), which we refer to as the Chow ring of \( V \), see [8, §1.4]. For a proper morphism \( f : V \rightarrow S \) the associated pushforward is denoted \( f_* : A_* V \rightarrow A_* S \). The Chow group functor is also covariant with respect to proper morphisms, and there exists a unique natural transformation

\[
c_* : K_0(\text{Var}_\mathfrak{K}) \longrightarrow A_*
\]

such that if \( V \) is smooth then

\[
c_*(V)([V \xrightarrow{\text{id}} V]) = c(TV) \cap [V] \in A^* V,
\]
where \( c_*(V) \) denotes the homomorphism \( c_*(V) : K_0(\text{Var}_V) \to A_* V \) induced by the natural transformation \( c_* \), and \( c(TV) \cap [V] \) denotes the total homological Chern class of \( V \), details of this construction are given in [3, 14]. For \( V \) possibly singular the class \( c_*(V)([V \xrightarrow{\text{id}} V]) \) will be denoted by \( c_{\text{SM}}(V) \), which is referred to in the literature as the Chern-Schwartz-MacPherson class of \( V \). It follows from functoriality that for a more general morphism \( \varphi : Y \to V \) we have
\[
c_*(V)([Y \xrightarrow{\varphi} V]) = \varphi_! c_{\text{SM}}(Y),
\]
which from here on will be denoted by \( c_{\text{SM}}(\varphi) \), and referred to as the Chern-Schwartz-MacPherson class of the morphism \( \varphi \). If \( Y \) and \( V \) are both smooth then we will simply refer to \( c_{\text{SM}}(\varphi) \) as the Chern class of \( \varphi \), written \( c(\varphi) \), so that the usual Chern class of a smooth variety may be referred to as the Chern class of the identity morphism.

From this perspective we may reformulate the Sethi-Vafa-Witten formula (1.7) from §1 as
\[
c(\varphi) = \frac{12c_1(X)}{1 + 6c_1(X)} \cdot c(\text{id}_X) \in A^* X,
\]
where \( \varphi : Y \to X \) is the elliptic fibration defined by (1.4). We now generalize this formula to the case where \( \varphi : Y \to X \) is a fibration of degree \( d \geq 1 \) hypersurfaces in \( \mathbb{P}^n \).

Let \( X \) be a smooth \( \mathbb{R} \)-variety endowed with a vector bundle \( \mathcal{E} \to X \) (as in the Notation and Conventions stated in §1). Assume \( Y \) is the smooth zero-scheme of a non-trivial section of a line bundle \( \mathcal{M} \to \mathbb{P}(\mathcal{E}) \), so that \( Y \) is an embedded smooth hypersurface in \( \mathbb{P}(\mathcal{E}) \). Denote by \( \iota : Y \hookrightarrow \mathbb{P}(\mathcal{E}) \) the natural inclusion, so that \( \varphi = \pi \circ \iota : Y \to X \) endows \( Y \) with the structure of a fibration of degree \( d \) hypersurfaces in \( \mathbb{P}^{\text{rk}(\mathcal{E}) - 1} \), where \( d > 0 \) is such that \([Y] = d \zeta + \pi^* \beta \) with \( \beta \) a divisor in \( X \), and \( \zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cap [\mathbb{P}(\mathcal{E})] \) ([\( Y \)] is of this form by Theorem 3.3 of [8]). We then associate with \( Y \) a rational expression \( Q_Y \) in the distinct non-trivial Chern roots \( L_i \) of \( \mathcal{E} \) and the class \( \beta \in A^1(\mathbb{X}) \) (the expression for \( Q_Y \) will be seen to arise from the application of Theorem 2.1 to the current setting in proof of Theorem 3.1 below), which is given by
\[
Q_Y = \sum_{i=1}^n \left( \frac{(1 - L_i) \prod_{j=1}^n (1 + L_j - L_i)}{L_i \prod_{j=1, j \neq i}^n (L_i - L_j)} \cdot \frac{\beta - d \cdot L_i}{1 + \beta - d \cdot L_i} - \frac{\prod_{j=1}^n (1 + L_j)}{L_i \prod_{j=1, j \neq i}^n (L_i - L_j)} \cdot \frac{\beta}{1 + \beta} \right),
\]
where \( n = \text{rk}(\mathcal{E}) - 1 \), which coincides with the total number of non-trivial Chern roots of \( \mathcal{E} \) by assumption. After putting the above sum over a common denominator, canceling common factors from the numerator and denominator of the form \( L_i \prod_{j=1, j \neq i}^r (L_i - L_j) \) (which one may in fact do as mentioned in Remark 2.1), we arrive at an expression of the form
\[
P_Y = \frac{1}{(1 + \beta) \cdot (1 + \beta - dL_1)^{k_1+1} \cdots (1 + \beta - dL_m)^{k_m+1}},
\]
where we recall $k_i + 1$ is the multiplicity of $L_i$, and $P_Y$ is a polynomial in the non-trivial Chern roots of $\mathcal{E}$ and $\beta \in A^1 X$. For example if $Y$ is the elliptic fibration in Weierstrass form as defined in §1 we have

$$Q_Y = \frac{12c_1(X)(1 - 3c_1(X))}{(1 + 6c_1(X))(1 + 6c_1(X) - 6c_1(X))(1 - 3c_1(X))} = \frac{12c_1(X)}{1 + 6c_1(X)} \in A^* X.$$ 

We note that if $E^\vee d$ denotes a vector bundle whose distinct non-trivial Chern roots are $-dL_1, \ldots, -dL_m$, where $-dL_i$ has the same multiplicity as $L_i$ (i.e. $k_i + 1$), then the denominator of $Q_Y$ may be associated with the Chern class of the vector bundle $E^\vee d \otimes O(\beta)$, so that $Q_Y$ may be written as

$$Q_Y = c(E^\vee d \otimes O(\beta))^{-1} \cap P_Y \in A^* X.$$ 

The Chern class of $\varphi$ is then given by the following

**Theorem 3.1.** Let $\varphi : Y \to X$ be a fibration of degree $d$ projective hypersurfaces as defined above. Then

$$c(\varphi) = Q_Y \cdot c(id_X) \in A^* X.$$ 

(3.2)

**Remark 3.1.** If $X$ in the above theorem is in fact a point, then $Q_Y$ coincides with the Euler characteristic of $Y$. In light of this – coupled with the fact that $Q_Y$ is independent of $X$– we may think of $Q_Y$ as a relative Euler characteristic for the morphism $\varphi : Y \to X$.

We first prove a lemma which will be useful not only for the proof of Theorem 3.1 but also in the next section for the proof of Theorem 4.1.

**Lemma 3.2.** Let $R$ be a commutative ring with unity and let $m \in \mathbb{N}$ be a positive integer. Then for any choice of $\alpha_0, \alpha_1, \ldots, \alpha_{m-1} \in R$ we have

$$\sum_{i=1}^{m} \frac{\alpha_0 + \alpha_1 x_i + \cdots + \alpha_{m-1} x_i^{m-1}}{x_i \prod_{j=1, j \neq i}^{m} (x_i - x_j)} = \sum_{i=1}^{m} \frac{\alpha_0}{x_i \prod_{j=1, j \neq i}^{m} (x_i - x_j)} \in R(x_1, \ldots, x_m).$$ 

(3.3)

**Proof.** The left-hand side of equation (3.3) may be expanded as

$$\sum_{i=1}^{m} \frac{\alpha_0}{x_i \prod_{j=1, j \neq i}^{m} (x_i - x_j)} + \alpha_1 \sum_{i=1}^{m} \frac{1}{x_i \prod_{j=1, j \neq i}^{m} (x_i - x_j)} + \cdots + \alpha_{m-1} \sum_{i=1}^{m} \frac{x_i^{m-2}}{x_i \prod_{j=1, j \neq i}^{m} (x_i - x_j)},$$

and in (i) of the proof of Theorem 3.2 of [4] it is shown that for $0 \leq p \leq m - 2$ we have

$$\sum_{i=1}^{m} \frac{x_i^p}{\prod_{j=1, j \neq i}^{m} (x_i - x_j)} = 0,$$

from which the lemma follows. 

\[\square\]
Proof of Theorem 3.1. By Remark 2.1 it suffices to prove the case where the non-trivial Chern roots of $\mathcal{E}$ are all distinct, i.e. when $m = n = \text{rk}(\mathcal{E}) - 1$. Let $\mathcal{O}(Y) \to \mathbb{P}(\mathcal{E})$ denote the line bundle corresponding to the divisor class of $Y$ (whose restriction to $Y$ yields the normal bundle to $Y$ in $\mathbb{P}(\mathcal{E})$). In order to write an explicit expression for $c(\varphi)$, we first write an expression for $c(\iota)$, where we recall $\iota : Y \to \mathbb{P}(\mathcal{E})$ is the natural inclusion. For this, we consider the following exact sequences of vector bundles (as Chern classes are multiplicative)

$$0 \to TY \to \iota^*T\mathbb{P}(\mathcal{E}) \to \iota^*\mathcal{O}(Y) \to 0$$

$$0 \to T_{\mathbb{P}(\mathcal{E})/X} \to T\mathbb{P}(\mathcal{E}) \to \pi^*TX \to 0$$

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to T_{\mathbb{P}(\mathcal{E})/X} \to 0.$$

The first sequence (from the top) is standard, the second one may take as the definition of the relative tangent bundle $T_{\mathbb{P}(\mathcal{E})/X}$, and the third is given in B.5.8 of [8]. We then have

$$c(\iota) = \iota_*\left(\frac{c(\iota^*T\mathbb{P}(\mathcal{E}))}{c(\iota^*\mathcal{O}(Y))} \cap [Y]\right)$$

$$= \iota_*\left(\frac{c(\pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cdot c(\pi^*TX)}{c(\mathcal{O}(Y))} \cap \iota_*[Y]\right)$$

$$= \frac{c(\pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cdot c(\pi^*TX)}{c(\mathcal{O}(Y))} \cap (c_1(\mathcal{O}(Y)) \cap \pi^*[X])$$

$$= \frac{c(\pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cdot c_1(\mathcal{O}(Y))}{c(\mathcal{O}(Y))} \cap (c(\pi^*TX) \cap \pi^*[X])$$

$$= \frac{c(\pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cdot c_1(\mathcal{O}(Y))}{c(\mathcal{O}(Y))} \cap \pi^*c(\text{id}_X),$$

where the third equality follows from the projection formula, the fourth from the fact that $\pi^*[X] = [\mathbb{P}(\mathcal{E})]$ (since $\iota_*[Y] = c_1(\mathcal{O}(Y)) \cap [\mathbb{P}(\mathcal{E})]$), the fifth from the commutativity of Chern classes, and the sixth by the behavior of Chern classes with respect to pullback.

Now since $(1, L_1, \ldots, L_n)$ are the Chern roots of $\mathcal{E}$ we have that the Chern roots of $\pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ are then $(\zeta, L_1 + \zeta, \ldots, L_n + \zeta)$, with $\zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cap [\mathbb{P}(\mathcal{E})]$ (note that technically, the Chern roots are of the form $\pi^*L_i + \zeta$ but we have omitted the pullbacks; we shall also slightly abuse notation and let $1 + Y$ denote $1 + c_1(Y)$). Thus

$$c(\iota) = \frac{(1 + \zeta)(1 + \zeta + L_1) \cdots (1 + \zeta + L_n) \cdot [Y]}{1 + Y} \cdot \pi^*c(\text{id}_X).$$
so that

\[ c(\varphi) = \varphi_\ast c(id_Y) \]

\[ = \pi_\ast \circ \iota_\ast c(id_Y) \]

\[ = \pi_\ast \left( \frac{(1 + \zeta)(1 + \zeta + L_1) \cdots (1 + \zeta + L_n) \cdot [Y]}{1 + Y} \cdot \pi_\ast c(id_X) \right) \]

\[ = \pi_\ast \left( \frac{(1 + \zeta)(1 + \zeta + L_1) \cdots (1 + \zeta + L_n) \cdot [Y]}{1 + Y} \right) \cdot c(id_X), \]

where the second equality follows from functoriality and the fifth follows via the projection formula. The theorem then follows once we show

\[ \pi_\ast \left( \frac{(1 + \zeta)(1 + \zeta + L_1) \cdots (1 + \zeta + L_n) \cdot [Y]}{1 + Y} \right) = Q_Y. \]

Let \( \alpha = \alpha(\zeta) = \frac{(1 + \zeta) \cdot \prod_{i=1}^{n} (1 + \zeta + L_i) \cdot [Y]}{1 + Y} \), applying Theorem 2.1 along with Lemma 3.2 to the class \( \alpha \) then yields

\[ \pi_\ast \alpha = \sum_{i=1}^{m} \left( \frac{\alpha(x_i) - \alpha_0}{x_i \prod_{j=1, j \neq i}^{m} (x_i - x_j)} \right) \bigg|_{x_1 = -L_1, \ldots, x_m = -L_m} \]

\[ = \sum_{i=1}^{m} \left( \frac{(1 - L_i) \prod_{j=1}^{m} (1 + L_j - L_i)}{L_i \prod_{j=1, j \neq i}^{m} (L_j - L_i)} \cdot \frac{\beta - dL_i}{1 + \beta - dL_i} - \frac{\prod_{j=1}^{m} (1 + L_j) \cdot \beta}{L_i \prod_{j=1, j \neq i}^{m} (L_j - L_i) \cdot \frac{1 + \beta}{1 + \beta}} \right), \]

which coincides with \( Q_Y \) by definition, thus concluding the proof.

In Table 3.1 we list \( Q_Y \) for various explicit examples. The first three examples recover results previously appearing in [1, 2, 6], while (to the best of our knowledge) the other examples are new. For an explicit embedding of hypersurface fibrations with fiber of arbitrary degree and dimension, let \( n > 0 \) and \( \mathcal{E} = \mathcal{O} \oplus \bigoplus_{i=1}^{n} \mathcal{L} \) with \( \mathcal{L} \to X \) a (suitably ample) line bundle, so that \( \mathbb{P}(\mathcal{E}) \) is then a \( \mathbb{P}^n \)-bundle over \( X \). A fibration \( Y_d \to X \) of degree \( d \) hypersurfaces in \( \mathbb{P}^n \) may then be given by the equation

\[ Y_d : (X_1^d + \cdots + X_n^d + f X_1 X_{d-1}^d + g X_0^d = 0) \subset \mathbb{P}(\mathcal{E}), \]

where \( X_1, \ldots, X_n \in H^0(\mathcal{O}(1) \otimes \mathcal{L}), X_0 \in H^0(\mathcal{O}(1)), f \in H^0(\mathcal{L}^{d-1}) \) and \( g \in H^0(\mathcal{L}^d) \). The class of \( Y_d \) is then \( dH + dL \in A^\ast \mathbb{P}(\mathcal{E}) \), and the discriminant of the fibration is then given by

\[ \Delta_d : ((d-1)^{d-1} f^d + d^d g^{d-1} = 0) \subset X. \]
Non-trivial Chern roots of $\mathcal{E}$  

<table>
<thead>
<tr>
<th>$[Y]$</th>
<th>$Q_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2L,3L)$</td>
<td>$3\zeta + 6L$ Elliptic curve $\frac{12L}{1+6L}$</td>
</tr>
<tr>
<td>$(L,S)$</td>
<td>$3\zeta + 2L + S$ Elliptic curve $\frac{6(2L+2L^2-LS-S^2)}{(1+2L-2S)(1+2L+3S)}$</td>
</tr>
<tr>
<td>$(L,2L)$</td>
<td>$3\zeta + 4L$ Elliptic curve $\frac{12L}{1+4L}$</td>
</tr>
<tr>
<td>$(L,L,L)$</td>
<td>$3\zeta + 3L$ del Pezzo surface $\frac{9+3L}{1+3L}$</td>
</tr>
<tr>
<td>$(L,L,L)$</td>
<td>$4\zeta + 4L$ K3 surface $\frac{24-12L}{1+4L}$</td>
</tr>
<tr>
<td>$(L,L,L)$</td>
<td>$5\zeta + 5L$ Surface of general-type $\frac{55-45L}{1+5L}$</td>
</tr>
<tr>
<td>$(L,L,L,L)$</td>
<td>$5\zeta + 5L$ Calabi-Yau threefold $\frac{280L-200}{1+5L}$</td>
</tr>
</tbody>
</table>

Table 3.1: Explicit examples of $Q_Y$

## 4 A relative Bézout Formula

An immediate corollary of Theorem 2.1 yields a formula for the intersection of $\dim(\mathbb{P}(\mathcal{E}))$ divisors in $\mathbb{P}(\mathcal{E})$ in terms of invariants of its base $X$, which we think of as a ‘relative Bézout formula’, since it recovers the classical Bézout formula for $X$ a point. The formula is given in terms of elementary symmetric polynomials and complete homogeneous polynomials, which we now define.

We denote the $j^{th}$ elementary symmetric polynomial in the $p$ variables $y_1,\ldots,y_p$ by $e_j(y_1,\ldots,y_p)$, which is given by

$$e_j(y_1,\ldots,y_p) = \sum_{l_1 \leq l_2 \leq \cdots \leq l_j \leq p} y_{l_1} \cdots y_{l_j}. \quad (4.1)$$

The complete homogeneous symmetric polynomial of degree $d$ in the $q$ variables $y_1,\ldots,y_q$ will be denoted by $h_d(y_1,\ldots,y_q)$, and is given by

$$h_d(y_1,\ldots,y_q) = \sum_{\substack{l_1+\cdots+l_q=d \\ l_i \geq 0}} y_1^{l_1} \cdots y_q^{l_q}. \quad (4.2)$$

For more on the theory of such polynomials see for example [12] or [4].

We denote by $h_d(\mathcal{E})$ the $d^{th}$ homogeneous symmetric polynomial in the negative distinct non-trivial Chern roots of $\mathcal{E}$, i.e.

$$h_d(\mathcal{E}) = h_d(-L_1,\ldots,-L_m).$$

Before stating and proving our relative Bézout formula we recall that by Theorem 3.3 of [8] a divisor $D \in A^1\mathbb{P}(\mathcal{E})$ is of the form

$$D = \pi^*D + d\zeta,$$
where \( D \) is a divisor in \( X \), \( \zeta = c_1(\mathcal{O}_\mathbb{P}(\mathcal{E})(1)) \cap [\mathbb{P}(\mathcal{E})] \) and \( d \) is an integer. The relative Bézout-type formula is then given by the following

**Theorem 4.1.** Let \( N = \dim(\mathbb{P}(\mathcal{E})) \), \( D_1, \ldots, D_N \) be \( N \) distinct divisors in \( \mathbb{P}(\mathcal{E}) \), and let \( \mathcal{D}_i \) and \( \mathcal{d}_i \) be such that \( D_i = \pi^* \mathcal{D}_i + \mathcal{d}_i \zeta \). Denote by \( \alpha \) the intersection product

\[
\alpha = D_1 \cdots D_N \in A^*(\mathcal{E}),
\]

and denote by \( e_j(\alpha) \) the \( j \)-th elementary symmetric polynomial in \( \mathcal{D}_i / \mathcal{d}_i \), so that

\[
e_j(\alpha) = e_j(\mathcal{D}_1 / \mathcal{d}_1, \ldots, \mathcal{D}_N / \mathcal{d}_N).
\]

Then

\[
\pi_* \alpha = d_1 \cdots d_N \cdot \sum_{j=0}^{\dim(X)} h_{\dim(X)-j}(\mathcal{E}) e_j(\alpha) \in A^*X.
\]

(4.3)

**Proof.** As in the proof of Theorem 3.1 we first assume all the non-trivial Chern roots of \( \mathcal{E} \) are distinct, which by Remark 2.1 we may do without any loss of generality. Expanding \( \alpha \) we obtain

\[
\alpha = \alpha(\zeta) = d_1 \cdots d_N \sum_{j=0}^N \zeta^{N-j} e_j(\alpha).
\]

Now let \( n = \text{rk}(\mathcal{E}) - 1 \). Theorem 2.1 along with Lemma 3.2 then yields

\[
\pi_* \alpha = d_1 \cdots d_N \sum_{i=1}^n \left( \frac{\alpha(x_i) - d_1 \cdots d_N \cdot e_N(\alpha)}{x_1 \prod_{j=1, j \neq i}^n (x_i - x_j)} \right) \bigg|_{x_1=-L_1, \ldots, x_n=-L_n}
\]

\[
= d_1 \cdots d_N \sum_{i=1}^n \left( \frac{\sum_{j=0}^{N-1} x_i^{N-j} e_j(\alpha)}{\prod_{j=1, j \neq i}^n (x_i - x_j)} \right) \bigg|_{x_1=-L_1, \ldots, x_n=-L_n}.
\]

Expanding the summation, applying again Lemma 3.2 and recalling that \( N = \dim(X) + n \) we obtain

\[
\pi_* \alpha = d_1 \cdots d_N \cdot \sum_{j=0}^{\dim(X)} h_{\dim(X)-j}(\mathcal{E}) e_j(\alpha).
\]

Acknowledgements. Martin Helmer was supported by an NSERC (Natural Sciences and Engineering Research Council of Canada) postdoctoral fellowship during the preparation of this work. We are grateful to the Institute of Mathematical Research at the University of Hong Kong for hosting Martin Helmer’s visit there, where a substantial part of this work was completed.
References


