Reduction of singularities of holomorphic maps of $\mathbb{C}^2$ tangent to the identity

By FENG RONG (Shanghai)

Abstract. We study the singularities of holomorphic maps of $\mathbb{C}^2$ tangent to the identity. We will adapt the method of Hironaka for the resolution of singularities of algebraic varieties to our study. In particular, we introduce a new numerical invariant associated to such maps at the singularities, called “adapted order”, which behaves well under blow-ups. Though other methods for desingularizing such maps exist, our approach has the advantage that it has no essential obstruction for generalizing to higher dimensions.

1. Introduction

In [2], ABATE generalized the well-known Leau–Fatou Flower Theorem in one-dimensional holomorphic dynamics to dimension two. The proof of this generalization suggests an interesting connection between discrete and continuous local holomorphic dynamics. Following this connection, some new results in the study of holomorphic maps, which have similar counterparts in the study of vector fields, have been obtained (see e.g. [3], [8], [9]).

We studied holomorphic maps of $\mathbb{C}^n$ tangent to the identity with absolutely isolated singularities in [9]. However, the method used in [9] does not apply to the case when the singularities are not absolutely isolated. For the general case, we
propose to use the idea of Hironaka for the resolution of singularities of algebraic varieties ([6], [7]). In this paper, we deal with dimension two. A similar approach in the study of plane vector fields has been carried out by CANO ([5]).

Our main result is the following reduction theorem (see Section 2 for precise definitions).

**Theorem 1.** Let $f$ be a holomorphic map of $\mathbb{C}^2$, tangent to the identity at an isolated fixed point. Then there exists a finite sequence of blow-ups, which reduces $f$ to a map whose adapted order at each of its singularities is less than or equal to one.

## 2. Reduction of singularities

Let us first recall some basic definitions (cf. [3]). For simplicity, we give all definitions in dimension two.

Let $M$ be a two-dimensional complex manifold and $f$ a holomorphic self-map of $M$ with $p \in M$ as a fixed point. Assume that $f$ is tangent to the identity at $p$, that is $df_p = \text{id}$. Write $f = (f_1, f_2)$, with $f_j(z) = z_j + g_j(z)$. The order of $f$ at $p$ is $\nu(f, p) = \min\{\nu(g_1), \nu(g_2)\}$, where $\nu(g_j)$ is the least $i \geq 0$ such that $P_{j,i}$ is not identically zero in the homogeneous expansion of $g_j$, $g_j = P_{j,0} + P_{j,1} + \ldots$, with $\deg P_{j,i} = i$ or $P_{j,i} \equiv 0$. We always assume that $\nu(f, p) < \infty$. Set $l = \gcd(g_1, g_2)$ and $g_j = l g_j^o$, with both $l$ and $g_j^o$ defined up to units in $\mathcal{O}_{M,p}$. The pure order of $f$ at $p$ is $\nu_o(f, p) = \min\{\nu(g_1^o), \nu(g_2^o)\}$. We say that $p$ is a singular point or a singularity of $f$ if $\nu_o(f, p) \geq 1$.

Let $P = (P_1, P_2)$ be a 2-tuple of homogeneous polynomials of degree $\nu$ in $\mathbb{C}^2$. A characteristic direction for $P$ is a vector $v \in \mathbb{P}^1$ such that $P(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. It is a nondegenerate characteristic direction if $\lambda \neq 0$, and degenerate otherwise. A characteristic direction for $P$ at $p$ is a characteristic direction for $P_{\nu(f,p)} = (P_{1,\nu(f,p)}, P_{2,\nu(f,p)})$. A singular direction for $f$ at $p$ is a characteristic direction for $P_{\nu(f,p)} = (P_{1,\nu(f,p)}, P_{2,\nu(f,p)})$. Here $P_{j,\nu(f,p)} = P_{j,\nu_o(f,p)} R_\sigma$, with $R_\sigma$ being the first nonzero term in the homogeneous expansion of $l$ and $\sigma = \nu(f, p) - \nu_o(f, p)$. The set of singular directions is clearly either a discrete set of points of $\mathbb{P}^1$ or the whole of $\mathbb{P}^1$. If the set of singular directions is a discrete set, we say that $f$ is non-dicritical at $p$, otherwise we say that $f$ is dicritical at $p$.

Let $\pi : \tilde{M} \to M$ be the blow-up of $M$ at $p$. Then, there exists a unique map $\tilde{f}$, the blow-up of $f$ at $p$, such that $\pi \circ \tilde{f} = f \circ \pi$ (see [1]). By [3, Lemma 2.2], if $p$ is non-dicritical then a direction $v \in \mathbb{P}^1$ is singular for $f$ if and only if it is a singularity of $\tilde{f}$. And by [9, Lemma 2.5], if $p$ is dicritical then the pure order of $\tilde{f}$
Reduction of singularities of holomorphic maps of $\mathbb{C}^2$ tangent to the identity at any of its singularities is less than the pure order of $f$ at $p$. Thus the crucial part of the proof for Theorem 1 is to deal with non-dicritical points. Therefore, without loss of generality, we will assume that no dicritical points occur during the blow-ups.

We now start our investigation of holomorphic maps of $\mathbb{C}^2$ tangent to the identity at an isolated fixed point, which we assume to be the origin $O$.

First of all, note that the pure order is not a “good” numerical invariant, since it might increase after blow-ups at non-dicritical points. Therefore we are going to use the adapted order instead. Let $M$ be a two-dimensional manifold and $f$ a holomorphic self-map of $M$ pointwise fixing $S \subset M$, where $S$ is an analytic variety of $M$ with only normal crossings as singularities. (For our purpose, we have either $f$ being the map we start with and $S = O$ or $f$ being the blow-up map and $S$ being the exceptional divisor.) Let $p \in S$ be a singularity of $f$ and let $e_p$ be the number of irreducible components of $S$ through $p$. Then $e_p = 0$, 1 or 2.

If $e_p = 0$, choose local coordinates $(x, y)$ and write $f$ at $p$ as

$$\begin{align*}
x_1 &= x + a(x, y) \\
y_1 &= y + b(x, y).
\end{align*}$$

(1)

If $e_p = 1$, choose local coordinates $(x, y)$ such that the irreducible component is given by $\{x = 0\}$ (or $\{y = 0\}$). Write $f$ at $p$ as

$$\begin{align*}
x_1 &= x + x^\kappa a(x, y) \\
y_1 &= y + x^\kappa b(x, y),
\end{align*}$$

(2) or

$$\begin{align*}
x_1 &= x + y^\tau a(x, y) \\
y_1 &= y + y^\tau b(x, y).
\end{align*}$$

(3)

If $e_p = 2$, choose local coordinates $(x, y)$ such that the two irreducible components are given by $\{x = 0\}$ and $\{y = 0\}$ respectively. Write $f$ at $p$ as

$$\begin{align*}
x_1 &= x + x^\kappa y^\tau a(x, y) \\
y_1 &= y + x^\kappa y^\tau b(x, y).
\end{align*}$$

(4)

Here $\kappa, \tau \geq 1$ are the biggest possible powers and $a(x, y)$ and $b(x, y)$ are relatively prime in $\mathcal{O}_{M, p}$. The adapted order $\nu_1(f, p)$ of $f$ at $p$ is then equal to $\min\{\nu(a), \nu(b)\}$.

Remark 2. Under our assumptions, the blow-up map will always be “tangential” along a fixed component, and thus can be written in the form of either
In the “non-tangential” case, one can define “adapted order” to be just “pure order”. (For the definition of “tangential” and the dynamics in case the fixed point of the original map is not isolated, see e.g. [4]) One readily checks that the adapted order is well-defined (cf. [2, Lemma 2.2]). And it is easy to verify that if \( p \) is a singularity of \( f \) and \( q \) is a singularity of \( \tilde{f} \), the blow-up of \( f \) at \( p \), then \( \nu_1(\tilde{f}, q) \leq \nu_1(f, p) \). (To appreciate the difference between \( \nu_0 \) and \( \nu_1 \), one can look at the blow-up \( x = \tilde{x}\tilde{y}, y = \tilde{y} \) at \( O \) for \( x_1 = x + x^3, y_1 = y + x(x + y^2) \).)

If \( f \) is of the form (2) (resp. (3)) at \( p \) and \( \nu_1(f, p) = \nu(b) \) (resp. \( \nu_1(f, p) = \nu(a) \)), then \( p \) is said to be of type zero. Otherwise \( p \) is said to be of type one.

For an element \( g \in \mathcal{O}_{M,p} \) with order \( \nu(g) \geq r, r \in \mathbb{N} \), denote by \( \mathcal{T}(g) \) the affine plane \( \mathbb{C}^2 \) if \( r < \nu(g) \) and the strict tangent space (i.e. the maximum linear subvariety of the tangent cone of \( g = 0 \) leaving it invariant by translations) if \( r = \nu(g) \).

Set \( \mu = \nu_1(f, p) \). We define the directrix \( \mathcal{T}(f, p) \) of \( f \) at \( p \) as follows. If \( e_p = 0 \) or 2, then

\[
\mathcal{T}(f, p) = \mathcal{T}^\mu(a) \cap \mathcal{T}^\mu(b).
\]

If \( p \) is of type zero (resp. one) and \( f \) is of the form (3) (resp. (2)), then

\[
\mathcal{T}(f, p) = \mathcal{T}(a).
\]

If \( p \) is of type zero (resp. one) and \( f \) is of the form (2) (resp. (3)), then

\[
\mathcal{T}(f, p) = \mathcal{T}(b).
\]

By definition, we always have that \( \dim \mathcal{T}(f, p) \leq 1 \). If \( \dim \mathcal{T}(f, p) = 1 \), the directrix defines a closed point \( \mathbb{P}(\mathcal{T}(f, p)) \) in the exceptional divisor of the blow-up with center \( p \). We have the following

**Lemma 3.** Let \( \tilde{f} \) be the blow-up of \( f \) at \( p \). If \( q \) is a closed point of the exceptional divisor such that \( \nu_1(\tilde{f}, q) = \nu_1(f, p) \), then \( \dim \mathcal{T}(f, p) = 1 \) and \( q = \mathbb{P}(\mathcal{T}(f, p)) \).

**Proof.** Assume that \( f \) is of the form (2) and \( p \) is of type zero. Set \( \mu = \nu_1(f, p) \) and write \( b_\mu(x, y) \) for the leading homogeneous polynomial in the homogeneous expansion of \( b(x, y) \).

In the chart \( (x = \tilde{x}, y = \tilde{x}(\tilde{y} + \zeta)) \) centered in \( q \), \( \tilde{f} \) is of the form

\[
\begin{align*}
\tilde{x}_1 &= \tilde{x} + \tilde{x}^{\kappa+\mu-1} \cdot O(\tilde{x}), \\
\tilde{y}_1 &= \tilde{y} + \tilde{x}^{\kappa+\mu-1} \cdot (b_\mu(1, \tilde{y} + \zeta) + O(\tilde{x}))
\end{align*}
\]
Reduction of singularities of holomorphic maps of $\mathbb{C}^2$ tangent to the identity 541

Since $\nu_1(\tilde{f}, q) = \mu$, we have $\nu(b_\mu(x, 1, \tilde{y} + \zeta)) = \mu$. This implies that $b_\mu(x, y) = c(y - \zeta x)^\mu$, where $c$ is a constant. Therefore $\mathfrak{T}(f, p) = \{y - \zeta x = 0\}$ and $q = \mathbb{P}(\mathfrak{T}(f, p))$.

In the chart $(x = \tilde{x}\tilde{y}, y = \tilde{y})$ centered in $q$, $f$ is of the form
\[
\begin{align*}
\tilde{x}_1 &= \tilde{x} + \tilde{z}^{k+1} \cdot (\tilde{y} + O(\tilde{y}), \\
\tilde{y}_1 &= \tilde{y} + \tilde{z}^{k+1} \cdot O(\tilde{y})
\end{align*}
\]

Since $\nu_1(\tilde{f}, q) = \mu$, we have $\nu(b_\mu(x, 1)) = \mu$. This implies that $b_\mu(x, y) = cx^\mu$, where $c$ is a constant. Therefore $\mathfrak{T}(f, p) = \{x = 0\}$ and $q = \mathbb{P}(\mathfrak{T}(f, p))$.

The argument for other cases is similar and we leave it to the interested reader.

Due to the above lemma, we will then focus on points $p$ with dim $\mathfrak{T}(f, p) = 1$ and $e_p \geq 1$. In suitable coordinates, the map $f$ takes exactly one of the following forms:

I. (2) type zero $\quad \mathfrak{T}(f, p) = \mathbb{P}^1$.
II. (3) type zero $\quad \mathfrak{T}(f, p) = \mathbb{P}^2$.
III. (2) type one $\quad \mathfrak{T}(f, p) = \mathbb{P}^2$.
IV. (3) type one $\quad \mathfrak{T}(f, p) = \mathbb{P}^2$.
V. (4) $\zeta \in \mathbb{C} \quad \mathfrak{T}(f, p) = \{y - \zeta x = 0\}$.

As in [7], for a power series $g = \sum_{i,j} g_{i,j} x^i y^j \in \mathbb{C}[x, y]$ and $r \in \mathbb{N}$, set
\[
\gamma^r(g; x, y) = \min \left\{ \frac{i}{r-j} : j < r, \ g_{i,j} \neq 0 \right\}.
\]

The following facts are easily verified:

\[
\gamma^r(g; x, y) < 1 \iff \nu(g) < r, \quad (5)
\]

and if $\nu(g) = r$, then

\[
\gamma^r(g; x, y) > 1 \iff \mathfrak{T}(g) = \{y = 0\}
\]

and

\[
\gamma^r(\tilde{g}; \tilde{x}, \tilde{y}) = \gamma^r(g; x, y) = 1, \quad \tilde{g} = g \circ \pi \cdot \tilde{x}^{-r} (\pi : (\tilde{x}, \tilde{y}) \mapsto (x, y) = (\tilde{x}, \tilde{y})). \quad (6)
\]

We define $\gamma(f, p; x, y)$ to be equal to

\[
\begin{align*}
\min\{\gamma^\mu(ya; x, y), \gamma^\mu(b; x, y)\}, & \quad \text{if I}, \\
\min\{\gamma^\mu(a; x, y), \gamma^\mu(yb; x, y)\}, & \quad \text{if II}, \\
\min\{\gamma^{\mu+1}(ya; x, y), \gamma^{\mu+1}(b; x, y)\}, & \quad \text{if III}, \\
\min\{\gamma^{\mu+1}(a; x, y), \gamma^{\mu+1}(yb; x, y)\}, & \quad \text{if IV}, \\
\min\{\gamma^\mu(a; x, y), \gamma^\mu(b; x, y)\}, & \quad \text{if V}.
\end{align*}
\]
Since $a$ and $b$ are relatively prime, one has that $\gamma(f, p; x, y) < \infty$ if $\mu \geq 2$.

Set $\gamma = \gamma(f, p; x, y)$. If $\gamma \in \mathbb{N}$, a $\gamma$-preparation is a change of coordinates of the form $(x' = x, y' = y + \lambda x^\gamma)$, $\lambda \in \mathbb{C}$. One can make successive $\gamma$-preparations to increase $\gamma(f, p; x, y)$. If $\mu \geq 2$, this increase is finite since $a$ and $b$ are relatively prime in $O_{M,p}$, if and only if they are relatively prime in the completion $\hat{O}_{M,p}$. We say that $f$ is $\gamma$-prepared with respect to $(x, y)$ if one of the following holds:

(i) $\gamma \notin \mathbb{N}$;
(ii) $\gamma \in \mathbb{N}$ and $f$ takes the form II, IV or V;
(iii) $\gamma \in \mathbb{N}$ and $\gamma$ does not increase after any $\gamma$-preparation.

We then define $\gamma(f, p)$ to be the minimum $\gamma(f, p; x, y)$, where $(x, y)$ runs over all $\gamma$-prepared situations. (If $\mu = 1$, one may have that $\gamma(f, p) = \infty$.)

Consider a sequence of blow-ups

$$p = p_0 \xleftarrow{\pi_0} p_1 \xleftarrow{\pi_1} \cdots,$$

where $p_0$ is the center of $\pi_0$ and lying in the exceptional divisor of $\pi_{i-1}$. Let $f_i$ be the blow-up map at $p_i$. We say that the sequence (7) is stationary of order $\mu$ if $\mu = \nu_i(f_i, p_i)$ remains constant. By Lemma 3, there is at most one possible stationary sequence for each $p_0$.

One readily checks that, in a stationary sequence, the forms of blow-up maps change as follows:

\[
\begin{array}{c|c}
I & I \\
II & V \\
III & I, II, III \\
IV & V \\
V & I, II, III, V \\
\end{array}
\]

Lemma 4. Let $\{p_i\}$ be a stationary sequence of order $\mu$. If $f_i$ takes the form II at $p_i$ for some $i > 0$, then

(i) the stationary sequence terminates at $p_i$, if $\mu \geq 2$;
(ii) the stationary sequence terminates at $p_{i+1}$, if $\mu = 1$.

Proof. If $e_{p_{i-1}} \geq 1$ then, since $f_i$ takes the form II at $p_i$, we know that $f_{i-1}$ takes the form III or V at $p_{i-1}$.

First, suppose that $f_{i-1}$ takes the form III at $p_{i-1}$. We can then write $f_{i-1}$ as

\[
\begin{align*}
x_1 &= x + x^\alpha a(x, y) \\
y_1 &= y + x^\beta b(x, y),
\end{align*}
\]
Reduction of singularities of holomorphic maps of $\mathbb{C}^2$ tangent to the identity

where $a(x, y) = cy^\mu + O(\mu + 1)$ with $c \neq 0$ and $b(x, y) = O(\mu + 1)$. After the blow-up ($\tilde{x} = x, \tilde{y} = y/x$) and then interchanging $\tilde{x}$ and $\tilde{y}$, we see that $f_i$ is of the form

\[
\begin{cases}
\tilde{x}_1 = \tilde{x} + \tilde{y}^{\kappa+\mu} \tilde{a}(\tilde{x}, \tilde{y}) \\
\tilde{y}_1 = \tilde{y} + \tilde{y}^{\kappa+\mu} \tilde{b}(\tilde{x}, \tilde{y}),
\end{cases}
\]

where $\tilde{a}(\tilde{x}, \tilde{y}) = dy^\mu + c\tilde{x}\nu + O(\mu + 1)$ with $d \neq 0$ and $\tilde{b}(\tilde{x}, \tilde{y}) = c\tilde{x}^{\mu} + O(\tilde{y})$.

After the blow-up ($\tilde{x} = x, \tilde{y} = y/x$), we can then write $f_{i+1}$ as

\[
\begin{cases}
\tilde{x}_1 = \tilde{x} + \tilde{x}^{\kappa+2\mu-1} \tilde{y}^{\kappa+\mu} \tilde{a}(\tilde{x}, \tilde{y}) \\
\tilde{y}_1 = \tilde{y} + \tilde{x}^{\kappa+2\mu-1} \tilde{y}^{\kappa+\mu} \tilde{b}(\tilde{x}, \tilde{y}),
\end{cases}
\]

where $\tilde{a}(\tilde{x}, \tilde{y}) = dy^\mu + c\tilde{x} + O(x^2)$ and $\tilde{b}(\tilde{x}, \tilde{y}) = -dy^\mu + (c - e)\tilde{x} + O(x^2) + O(\tilde{y})$.

We see that $\nu_1(f_{i+1}, p_{i+1}) = 1$. Therefore, if $\mu \geq 2$ then the stationary sequence terminates at $p_i$. If $\mu = 1$, we have $\dim \Sigma(f_{i+1}, p_{i+1}) = 0$ and thus the stationary sequence terminates at $p_{i+1}$ by Lemma 3.

Second, suppose that $f_{i-1}$ takes the form $V$ at $p_{i-1}$. We can then write $f_{i-1}$ as

\[
\begin{cases}
x_1 = x + x^{\kappa} y^{\tau} a(x, y) \\
y_1 = y + x^{\kappa} y^{\tau} b(x, y),
\end{cases}
\]

where $a(x, y) = c(y - \zeta x)^\mu + O(\mu + 1)$ and $b(x, y) = d(y - \zeta x)^\mu + O(\mu + 1)$, with $|c| + |d| \neq 0$ and $\zeta \neq 0$. After the blow-up ($\tilde{x} = x, \tilde{y} = y/x - \zeta$) and then interchanging $\tilde{x}$ and $\tilde{y}$, we see that $f_i$ is of the form

\[
\begin{cases}
\tilde{x}_1 = \tilde{x} + \tilde{y}^{\kappa+\tau+\mu} (\tilde{x} + \zeta)^{\tau+1} \tilde{a}(\tilde{x}, \tilde{y}) \\
\tilde{y}_1 = \tilde{y} + \tilde{y}^{\kappa+\tau+\mu} (\tilde{x} + \zeta)^{\tau+1} \tilde{b}(\tilde{x}, \tilde{y}),
\end{cases}
\]

where $\tilde{b}(\tilde{x}, \tilde{y}) = e\tilde{y}^{\mu} + O(\mu + 1)$ with $e \neq 0$ and $\tilde{b}(\tilde{x}, \tilde{y}) = c\tilde{x}^{\mu} + O(\tilde{y})$ with $c \neq 0$ (since we necessarily have $c = d$). We then see that we can argue exactly as above.

Now suppose that $e_{p_{i-1}} = 0$. We then necessarily have $i - 1 = 0$. By Lemma 3, we can write $f_0$ in suitable local coordinates $(x, y)$ as

\[
\begin{cases}
x_1 = x + a(x, y) \\
y_1 = y + b(x, y),
\end{cases}
\]

where $a(x, y) = cy^\mu + O(\mu + 1)$ and $b(x, y) = dy^\mu + O(\mu + 1)$ with $|c| + |d| \neq 0$. Since $f_1$ takes the form $II$ at $p_1$, we necessarily have $d = 0$ (and thus $c \neq 0$) and
after the blow-up \((\tilde{x} = x, \tilde{y} = y/x)\) and then interchanging \(\tilde{x}\) and \(\tilde{y}\), we can write \(f_1\) as

\[
\begin{align*}
\tilde{x}_1 &= \tilde{x} + \tilde{y}^{\mu-1}\hat{a}(\tilde{x}, \tilde{y}) \\
\tilde{y}_1 &= \tilde{y} + \tilde{y}^{\mu-1}\tilde{b}(\tilde{x}, \tilde{y}),
\end{align*}
\]

where \(\hat{a}(\tilde{x}, \tilde{y}) = e\tilde{y}^\mu + O(\mu + 1)\) with \(e \neq 0\) and \(\tilde{b}(\tilde{x}, \tilde{y}) = c\tilde{x}^\mu + O(\tilde{y})\). We can then again argue as above.

This completes the proof. \(\square\)

**Lemma 5.** Let \(\{p_i\}\) be a stationary sequence of order \(\mu\). If \(f_i\) takes the form I (resp. III, resp. IV, resp. V) at \(p_i\) and \(f_{i+1}\) takes the form I (resp. III, resp. V, resp. V) at \(p_{i+1}\), then

\[
\gamma(f_{i+1}, p_{i+1}) \leq \gamma(f_i, p_i) - 1.
\]

**Proof.** Assume, without loss of generality, that \(\gamma(f_i, p_i) < \infty\).

Let \((x, y)\) be local coordinates around \(p_i\) such that \(f_i\) is \(\gamma\)-prepared with respect to \((x, y)\), where \(\gamma = \gamma(f_i, p_i; x, y)\). In each of the cases we are concerned with, the coordinates centered in \(p_{i+1}\) are \((\tilde{x} = x, \tilde{y} = y/x)\). By (6), we have that

\[
\gamma(f_{i+1}, p_{i+1}; \tilde{x}, \tilde{y}) = \gamma(f_i, p_i; x, y) - 1.
\]

It is not difficult to see that \(f_{i+1}\) is \((\gamma - 1)\)-prepared with respect to \((\tilde{x}, \tilde{y})\) if \(f_i\) is \(\gamma\)-prepared with respect to \((x, y)\). Therefore the lemma follows from the above equality and the definition of \(\gamma(f, p)\). \(\square\)

For stationary sequences of order greater or equal to two, we have the following

**Theorem 6.** All the possible stationary sequences of order greater or equal to two have finite length.

**Proof.** If \(f_i\) takes the form III or V at \(p_i\) and \(f_{i+1}\) takes the form II at \(p_{i+1}\), then, by Lemma 4 (i), the stationary sequence terminates at \(p_{i+1}\).

If \(f_i\) takes the form I (resp. III, resp. IV, resp. V) at \(p_i\) and \(f_{i+1}\) takes the form I (resp. III, resp. V, resp. V) at \(p_{i+1}\), then, by Lemma 5, we have

\[
\gamma(f_{i+1}, p_{i+1}) \leq \gamma(f_i, p_i) - 1.
\]

We can now conclude using the above inequality and (5). \(\square\)

It is obvious that Theorem 1 is a corollary of the above theorem. To finish the paper, we give a brief discussion on stationary sequences of order one.
Lemma 7. Assume that $f$ takes the form $I$ at $p$. Let $(x, y)$ be local coordinates around $p$ such that $f$ is $\gamma$-prepared with respect to $(x, y)$ for $\gamma = \gamma(f, p; x, y)$. Set $\gamma' = \gamma(f, p)$ and assume that $\gamma > \gamma'$. Then $\gamma' \in \mathbb{N}$ and there exists a change of coordinates $(x', y') = (x, y) + \lambda x^\gamma$ such that $\gamma(f, p; x', y') = \gamma'$ and that $f$ is $\gamma'$-prepared with respect to $(x', y')$.

Proof. By the assumption, we can write $f$ as

$$
\begin{aligned}
x_1 &= x + x^\kappa a(x, y) \\
y_1 &= y + x^\kappa b(x, y),
\end{aligned}
$$

where $a(x, y) = O(\mu)$ and $b(x, y) = cy^\mu + O(\mu + 1)$, with $\mu = \nu_1(f, p)$ and $c \neq 0$.

Choose local coordinates $(\bar{x}, \bar{y})$ such that $\gamma(f, p; \bar{x}, \bar{y}) = \gamma'$ and $f$ is $\gamma'$-prepared with respect to $(\bar{x}, \bar{y})$. Since $f$ takes the form $I$ at $p$, we have that $\bar{x} = u \cdot x$, $\bar{y} = v \cdot y + w \cdot x^n$, where $u, v, w \in \mathcal{O}_{M, p}$ with $u(0,0) \neq 0$ and $v(0,0) \neq 0$. Since $\gamma > \gamma'$, we also have that $w(0,0) \neq 0$.

Since $b(x, y) = cy^\mu + O(\mu + 1)$, the term $\bar{x}^n \bar{y}^m$ appears in the expression of $\bar{y}_1$. Therefore, we have $n \geq \gamma'$. Since terms $x^i y^j$ create terms $\bar{x}^{i+kn} \bar{y}^l$ with $k + l = j$ and $i > \gamma'(\mu - j)$, we have

$$
\frac{i + kn}{\mu - l} > \frac{\gamma'(\mu - k - l) + kn}{\mu - l} = \gamma' + \frac{k(n - \gamma')}{\mu - l} \geq \gamma'.
$$

Therefore, we actually have $\gamma' = n \in \mathbb{N}$.

Now consider the change of coordinates $(x' = x, y' = y + \lambda x^\gamma)$, with $\lambda \neq 0$. By the above discussion, we have that $\gamma(f, p; x', y') = \gamma'$. Due to the presence of the term $x^n y^\mu$, we see that $\gamma'$ does not increase after any $\gamma'$-preparation. Therefore $f$ is $\gamma'$-prepared with respect to $(x', y')$. \hfill \Box

For stationary sequences of order equal to one, we have the following

Theorem 8. The length of a stationary sequence of order one is infinite if and only if there exists a step $p_i$ such that $f_i$ takes the form $I$ at $p_i$ and $\gamma(f_i, p_i) = \infty$. Moreover, if $i \geq l$ then $f_i$ takes the form $I$ at $p_i$ and $\gamma(f_i, p_i) = \infty$.

Proof. Assume that $\{p_i\}$ is a stationary sequence of order one with infinite length. If $\gamma(f_i, p_i) < \infty$ for all $i$, then we can show as in Theorem 6 that the stationary sequence has finite length, a contradiction. Therefore, we have that $\gamma(f_i, p_i) = \infty$ for some $l$. Since $\nu_1(f_i, p_i) = 1$, by the definition of $\gamma(f, p)$, one readily checks that $f_i$ takes the form $I$ at $p_i$. If $\gamma(f_i, p_i) < \infty$ for some $i > l$, then by Lemma 5 and (5) we have that the stationary sequence has finite length, a contradiction. Therefore, the last statement holds.
546  F. Rong : Reduction of singularities of holomorphic maps of \( \mathbb{C}^2 \) tangent…

For the converse, assume that \( f_1 \) takes the form I at \( p_l \) and \( \gamma(f_1, p_l) = \infty \) for some \( l \). Suppose that \( \gamma(f_{l+1}, p_{l+1}) = \gamma < \infty \). We can choose local coordinates \((x, y)\) such that \( \gamma(f_l, x, y) > \gamma + 1 \). Then \( \gamma(f_{l+1}, x, y) > \gamma \) for \((\tilde{x} = x, \tilde{y} = y/x)\). By Lemma 7, there exists a change of coordinates \((\tilde{x}' = \tilde{x}, \tilde{y}' = \tilde{y} + \lambda \tilde{x}')\) such that \( \gamma(f_{l+1}, \tilde{x}', \tilde{y}') = \gamma \) and that \( f_{l+1} \) is \( \gamma \)-prepared with respect to \((\tilde{x}', \tilde{y}')\). Now consider the change of coordinates \((x' = x, y' = y + \lambda x')\). We have that \( \gamma(f_l, x', y') = \gamma \) and that \( f_l \) is \((\gamma+1)\)-prepared with respect to \((x', y')\), a contradiction. Therefore, we must have \( \gamma(f_{l+1}, p_{l+1}) = \infty \).

\( \square \)

Remark 9. In [9], we gave “final forms” of singularities which are persistent under blow-ups. In the two-dimensional case, it is easy to see that a simple point of type A) takes the form I with \( \gamma(f, p) = \infty \) and a simple point of type B) and a non-dicritical simple corner has adapted order 0 (see [9] for precise definitions).

References

[8] F. Rong, Robust parabolic curves in \( \mathbb{C}^m \) (\( m \geq 3 \)), Houston J. Math. 36 (2010), 147–155.

FENG RONG
DEPARTMENT OF MATHEMATICS
SHANGHAI JIAO TONG UNIVERSITY
800 DONG CHUAN ROAD
SHANGHAI, 200240
P.R. CHINA

E-mail: frong@sjtu.edu.cn
URL: wu.math.sjtu.edu.cn/faculty/frong

(Received November 22, 2011; revised December 9, 2012)