Graphs of Riemann Zeta Function

As a complex valued function of a complex variable, the graph of the Riemann zeta function $\zeta(s)$ lives in four dimensional real space. To get an idea of what the function looks like, we must do something clever.

Level Curves

The real and imaginary parts of $\zeta(s)$ are each real valued functions; we can think of the graphs of each one as a surface in three dimensional space. Rather than look at two surfaces simultaneously, we can view the level curves for the two surfaces. The level curves are curves in the $s$ plane showing points of constant height on the surface, as on a contour map.

Level curves for $\text{Re}(\zeta(s))$ are shown above with the solid lines; the red curve is $\text{Re}(\zeta(s))=0$, the black curves represent values other than zero. Level curves for $\text{Im}(\zeta(s))$ are shown above with dotted lines; the green curve is $\text{Im}(\zeta(s))=0$, the black curves represent values other than zero. The function $\zeta(s)$ is real on the real axis, thus $\text{Im}(\zeta(s))=0$ there. Zeros of $\zeta(s)$ are points in the plane where both $\text{Re}(\zeta(s))=0$ and $\text{Im}(\zeta(s))=0$; these are points where the red and green curves cross. You can see the
trivial zeros at the negative even integers, and the first nontrivial zero at \( s=1/2+i*14.135... \), this is the point in the plane \((1/2, 14.135...).\) You can also see the pole at \( s=1.\)

**Argument in Color**

Another possibility is to view the complex number \( w=\zeta(s) \), itself a point in the plane, as a vector in polar coordinates. The angle, or argument of \( \zeta(s) \) is a number between 0 and 2\( \pi \) for each complex \( s \). We can interpret this angle as a color on the color wheel, and plot a pixel with that color at the corresponding point in the domain, that is, in the \( s \) plane. In this representation, a zero of \( \zeta(s) \) is a point where all the colors come together; you can see both trivial zeros and the first three non-trivial zeros. Each of these is a simple zero (as the Riemann Hypothesis predicts); going around the point once we see each of the colors exactly once. Observe the colors also come together at the pole \( s=1; \) but with a difference. At the pole one circles the color wheel in the opposite direction.
The Mathematica code which produced this is

```mathematica
Show[
  Graphics[
    RasterArray[
      Table[
        Hue[Mod[3Pi/2 + Arg[Zeta[sigma + I t]], 2Pi]/(2Pi)],
        {t, -4.5, 30, .1}, {sigma, -11, 12, .1}]
      ],
    AspectRatio -> Automatic
  ]
]
```