Simple waves and characteristic decompositions of quasilinear hyperbolic systems in two independent variables

Wancheng Sheng

Department of Mathematics, Shanghai University
(Joint with Yanbo Hu)

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A simple wave is defined as a flow in a region whose image is a curve in the phase space.

It plays an important role in the theories of gas dynamics and fluid mechanics.

Simple waves play a fundamental role in describing and building up solutions of flow problems. (pp. 59-60)
Simple wave

Hyperbolic systems in two independent variables

\[ u_x + A(u)u_y = 0 \]

where \( A(u) = (a_{ij}(u))_{n \times n}, \ u = (u_1, \cdots, u_n)^\top. \) The real and distinct eigenvalues

\[ \lambda_1(u) < \cdots < \lambda_n(u). \]

Lax 1957 and Dafermos 2000

The states in a domain adjacent to a domain of constant state is always a simple wave by using the Riemann invariants.

Remark

The treatment is invalid when the matrix \( A \) depends on \( x \) and \( y \) as well as \( u \).
Characteristic decompositions—an example

A classical 1-D wave equation

\[ u_{tt} - c^2 u_{xx} = 0 \]

with constant speed \( c \), which has an interesting decomposition

\[ (\partial_t \pm c \partial_x)(\partial_t \mp c \partial_x)u = 0. \]

One can rewrite them as

\[ \partial_+ R = 0 \quad \text{and} \quad \partial_- S = 0, \]

where

\[ R = \partial_- u := \partial_t u - c \partial_x u, \quad S = \partial_+ u := \partial_t u + c \partial_x u. \]
### Characteristic decompositions—quasilinear equations

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The general $2 \times 2$ strictly hyperbolic system

\[
\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_y = 0, \tag{1}
\]

where the coefficients $a_{ij} = a_{ij}(x, y, u, v), i, j = 1, 2$, which satisfy

\[(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) > 0.\]

The coefficient matrix of the system has two eigenvalues

\[
\lambda_{\pm} = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2},
\]

which are solutions to the characteristic equation

\[
\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.
\]
The characteristic form of system (1) is

\[ \partial_{\pm} u + \frac{\lambda_{\pm} - a_{22}}{-a_{21}} \partial_{\pm} v = 0, \]  

(2)

where \( \partial_{\pm} := \partial_x + \lambda_{\pm} \partial_y \). Here we assume without loss of generality that \( a_{21} \neq 0 \). Denote

\[ A_1 := a_{11x} + a_{11} a_{11y} + a_{21} a_{12y}, \]
\[ A_2 := a_{12x} + a_{12} a_{11y} + a_{22} a_{12y}, \]
\[ A_3 := a_{21x} + a_{11} a_{21y} + a_{21} a_{22y}, \]
\[ A_4 := a_{22x} + a_{12} a_{21y} + a_{22} a_{22y}, \]

where \( u \) and \( v \) are parameters in \( a_{ij} = a_{ij}(x, y, u, v) \) \( (i, j = 1, 2) \).
If the coefficients $a_{ij}$ ($i, j = 1, 2$) satisfy

$$A_1 = A_2 = A_3 = A_4 = 0,$$

then there hold

$$\partial_\pm \lambda_\pm = \frac{(\lambda_+ - a_{22})\lambda_\pm u + a_{21}\lambda_\pm v}{\lambda_+ - a_{22}} \partial_\pm u,$$

and $\partial_\mp \partial_\pm u = h_\pm \partial_\pm u$ for some suitable factors $h_\pm$. Consequently, the role of above equations is to ensure that the simple wave region be covered by a family of straight characteristics.


Characteristic decomposition of the $2 \times 2$ quasilinear strictly hyperbolic systems
If the sufficient condition (4) does not satisfy, we will obtain more general results.


(commutator relation) For any quantity $l = l(x, y)$, there holds

$$
\partial_- \partial_+ l - \partial_+ \partial_- l = \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_- - \lambda_+} (\partial_- l - \partial_+ l).
$$

Taking $l = u$, we get

$$
\partial_\pm \partial_{\mp} u + \frac{\partial_+ \lambda_- - \partial_- \lambda_+}{\lambda_+ - \lambda_-} \partial_{\mp} u = \frac{\lambda_- - a_{22}}{\lambda_- - \lambda_+} \left( \partial_{\mp} \left( \frac{\lambda_- - a_{22}}{a_{21}} \right) \frac{a_{21}}{\lambda_- - a_{22}} \partial_{\pm} u + \frac{\lambda_- - a_{22}}{a_{21}} \partial_{\mp} \left( \frac{a_{21}}{\lambda_- - a_{22}} \right) \partial_{\mp} u \right).
$$
We compute
\[
\partial_{\pm} \left( \frac{\lambda_{\pm} - a_{22}}{a_{21}} \right) = \partial_{\pm} (\lambda_{\pm} - a_{22}) a_{21} - (\lambda_{\pm} - a_{22}) \partial_{\pm} a_{21} \\
= m_{\pm} \partial_{\pm} u - \frac{a_{21}(a_{22x} + \lambda_{\pm} a_{22y}) + (\lambda_{\pm} - a_{22})(a_{21x} + \lambda_{\pm} a_{21y}) - a_{21}(\lambda_{\pm} x + \lambda_{\pm} y)}{a_{21}^2}
\]

where
\[
m_{\pm} = a_{21} \left( \lambda_{\pm} u + \frac{a_{21}}{\lambda_{\pm} - a_{22}} \lambda_{\pm} v - a_{22} u - a_{22} v \frac{a_{21}}{\lambda_{\pm} - a_{22}} \right) - (\lambda_{\pm} - a_{22}) \left( a_{21} u + a_{21} v \frac{a_{21}}{\lambda_{\pm} - a_{22}} \right) \frac{a_{21}}{a_{21}^2}.
\]
Then there hold

$$\partial_\pm \left( \frac{\lambda_\pm - a_{22}}{a_{21}} \right) = m_\pm \partial_\pm u$$

if the coefficients $a_{ij}$ ($i, j = 1, 2$) satisfy the following equations:

$$a_{22x} + \lambda_\pm a_{22y} + \frac{\lambda_\pm - a_{22}}{a_{21}} (a_{21x} + \lambda_\pm a_{21y}) = \lambda_\pm x + \lambda_\pm \lambda_\pm y.$$
It follows that

\[ B_1 + \lambda \pm B_2 = 0, \]

where

\[ B_1 = a_{21}(a_{11}a_{22} - a_{12}a_{21})x + (a_{11} + a_{22})(a_{22}a_{21}x - a_{21}a_{22}x) \]
\[ - (a_{11}a_{22} - a_{12}a_{21})[(a_{11} - a_{22})a_{21}y - (a_{11} - a_{22})_y a_{21} + 2a_{21}x] \]

\[ B_2 = a_{21}(a_{11}a_{22} - a_{12}a_{21})y - a_{21}(a_{11}x + a_{22}x) + 2a_{21}a_{22}x - 2a_{22}a_{21}x \]
\[ - (a_{11} + a_{22})(a_{21}x + a_{21}a_{22}y - a_{22}a_{21}y) + (a_{11} + a_{22})[2a_{21}x + 2a_{21}a_{22}y - 2a_{22}a_{21}y - (a_{11} + a_{22})a_{21}y - a_{21}(a_{11}y + a_{22}y)] + 2[(a_{11} + a_{22})^2 \]
\[ - (a_{11}a_{22} - a_{12}a_{21})]a_{21}y. \]
Thus, from $\lambda_+ - \lambda_- \neq 0$, we have $B_1 = B_2 = 0$, which can be simplified as

\[
\begin{cases}
  a_{21}A_2 - a_{12}A_3 = 0, \\
  a_{21}(A_1 - A_4) + (a_{22} - a_{11})A_3 = 0.
\end{cases}
\]

Therefore there hold the identities

\[
\partial_+ \partial_\pm u = h_\pm \partial_\pm u
\]

for some suitable factors $h_\pm$ if system (5) is satisfied.
2 × 2 strictly hyperbolic system—more general one.

**Theorem 1.**

There hold

\[ \partial_{\pm} u = \frac{\lambda_{\mp} - a_{22}}{a_{21}} \partial_{\pm} v, \]

and

\[ \partial_{\mp} \partial_{\pm} u = h_{\pm} \partial_{\pm} u \]

for some factors \( h_{\pm} \) if the coefficients \( a_{ij} \ (i, j = 1, 2) \) with \( a_{21} \neq 0 \) satisfy system

\[
\begin{align*}
    a_{21} A_2 - a_{12} A_3 &= 0, \\
    a_{21} (A_1 - A_4) + (a_{22} - a_{11}) A_3 &= 0.
\end{align*}
\]
Simple wave adjacent to a constant state

Theorem 2.

Adjacent to a constant state of equations (1) is a simple wave in which the variables \((u,v)\) are constant along a family of characteristics if the coefficients \(a_{ij} \ (i,j = 1,2)\) with \(a_{21} \neq 0\) satisfy system

\[
\begin{align*}
    a_{21}A_2 - a_{12}A_3 &= 0, \\
    a_{21}(A_1 - A_4) + (a_{22} - a_{11})A_3 &= 0.
\end{align*}
\]

Remark

Similar arguments can be made for the case \(a_{21} = 0\).
Theorem 2.

A simple wave region to be covered by one family of straight characteristics if and only if the coefficients \(a_{ij}\) \((i, j = 1, 2)\) satisfy

\[ A_1 = A_2 = A_3 = A_4 = 0. \]

where

\[ A_1 := a_{11}x + a_{11}a_{11}y + a_{21}a_{12}y, \quad A_2 := a_{12}x + a_{12}a_{11}y + a_{22}a_{12}y, \]
\[ A_3 := a_{21}x + a_{11}a_{21}y + a_{21}a_{22}y, \quad A_4 := a_{22}x + a_{12}a_{21}y + a_{22}a_{22}y. \]
In fact, if
\[ \lambda_{\pm x} + \lambda_{\pm} \lambda_{\pm y} = 0, \quad (6) \]
then
\[ \partial_{\pm} \lambda_{\pm} = \frac{(\lambda_{\mp} - a_{22}) \lambda_{\pm u} + a_{21} \lambda_{\pm v}}{\lambda_{\mp} - a_{22}} \partial_{\pm} u. \]

Thus, the condition (6) can be rewritten as
\[ \left( (a_{11} + a_{22})(a_{11} + a_{22})y + (a_{11} + a_{22})x - (a_{11} a_{22} - a_{12} a_{21})y \right) \lambda_{\pm} \]
\[ - [(a_{11} a_{22} - a_{12} a_{21})x + (a_{11} a_{22} - a_{12} a_{21})(a_{11} + a_{22}) y] = 0, \]
from which we get after some simplifications
\[ \left\{ \begin{array}{l}
A_1 + A_4 = 0, \\
(a_{11} - a_{22})A_1 + a_{21} A_2 + a_{12} A_3 = 0.
\end{array} \right. \]
Simple wave with straight characteristics

Then, we have

\[
\begin{pmatrix}
0 & a_{21} & -a_{12} & 0 \\
a_{21} & 0 & a_{22} - a_{11} & -a_{21} \\
1 & 0 & 0 & 1 \\
a_{11} - a_{22} & a_{21} & a_{12} & 0
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{pmatrix} = 0.
\]

Therefore, we find

\[A_1 = A_2 = A_3 = A_4 = 0\]

by the fact that the determinant of coefficient matrix

\[a_{21}[(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})] \neq 0.\]
Theorem 3. Assume that the coefficients $a_{ij}$ ($i, j = 1, 2$) with $a_{21} \neq 0$ satisfy

$$\begin{align*}
    a_{21}A_2 - a_{12}A_3 &= 0, \\
    a_{21}(A_1 - A_4) + (a_{22} - a_{11})A_3 &= 0.
\end{align*}$$

Then $(u, v)(x, y)$ is a simple wave solution of (1).
Simple wave solutions

\[
\begin{align*}
    & y \\
    & 0 \\
    & x
\end{align*}
\]

\[c_{+}(s)\]
1. Pseudo-steady Euler equations

In 2-D isentropic irrotational ideal flow in the self-similar plane \((\xi, \eta) = (x/t, y/t)\), the equations of motion is

\[
\begin{bmatrix}
u \\
v
\end{bmatrix}_{\xi} + \begin{bmatrix}
-2UV & \frac{c^2-V^2}{c^2-U^2} \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}_{\eta} = 0,
\tag{7}
\]

where \(c\) is the speed of sound satisfying the pseudo-Bernoulli’s law

\[
\frac{c^2}{\gamma - 1} + \frac{U^2 + V^2}{2} = -\varphi,
\]

\((U, V) = (u - \xi, v - \eta)\) is the pseudo-velocity, and \(\varphi = \varphi(\xi, \eta)\) is the pseudo-potential such that \(\varphi_{\xi} = U, \varphi_{\eta} = V\). Taking \(u\) and \(v\) are parameters in \(c\), we can directly obtain \(c_{\xi} = 0\) and \(c_{\eta} = 0\).
Then we have

\[ a_{11}\xi + a_{11}a_{11}\eta + a_{21}a_{12}\eta = \left( \frac{-2(u - \xi)(v - \eta)}{c^2 - (u - \xi)^2} \right) \xi 
+ \frac{-2(u - \xi)(v - \eta)}{c^2 - (u - \xi)^2} \cdot \left[ \frac{-2(u - \xi)(v - \eta)}{c^2 - (u - \xi)^2} \right] \eta 
- \frac{c^2 - (v - \eta)^2}{c^2 - (u - \xi)^2} \eta 
= 2(v - \eta)\left[ c^2 - (u - \xi)^2 \right] + 4(u - \xi)^2(v - \eta) \]

\[ \frac{1}{[c^2 - (u - \xi)^2]^2} \]

\[ + \frac{-2(u - \xi)(v - \eta)}{c^2 - (u - \xi)^2} \cdot \frac{2(u - \xi)[c^2 - (u - \xi)^2]}{[c^2 - (u - \xi)^2]^2} 
- \frac{2(v - \eta)}{c^2 - (u - \xi)^2} = 0, \]
1. Pseudo-steady Euler equations

and

\[ a_{12} \xi + a_{12} a_{11} \eta \]

\[ = \left[ \frac{c^2 - (v - \eta)^2}{c^2 - (u - \xi)^2} \right] \xi + \frac{c^2 - (v - \eta)^2}{c^2 - (u - \xi)^2} \cdot \left[ \frac{-2(u - \xi)(v - \eta)}{c^2 - (u - \xi)^2} \right] \eta \]

\[ = -2(u - \xi) \left[ \frac{c^2 - (v - \eta)^2}{[c^2 - (u - \xi)^2]^2} \right] + \frac{c^2 - (v - \eta)^2}{c^2 - (u - \xi)^2} \cdot \frac{2(u - \xi)}{c^2 - (u - \xi)^2} \]

\[ = 0, \]

which imply the coefficients of equations (7) satisfy system (3).
2. The generalized UTSD system

The generalized UTSD system

\[(A(U) - xJ(U))U_x + (B(U) - yJ(U))U_y = 0,\] (8)

where \(U = [u, v]^\top\), \(A(U) = (a_{ij}(U))\), \(B(U) = (b_{ij}(U))\), \(J(U) = (j_{ij}(U))\), \(i, j = 1, 2\).

This kind of system was used in dealing with the unsteady transonic small disturbance (UTSD) equations by Čanić and Keyfitz.


*Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problems*
Assume without loss of generality that $|A(U) - xJ(U)| \neq 0$, then we can rewrite (8) in a new form

\[
\begin{bmatrix}
u \\
v
\end{bmatrix}_x + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} u \\
v
\end{bmatrix}_y = 0, \quad (9)
\]
2. The generalized UTSD system

where

\[
\begin{align*}
c_{11} &= \frac{(a_{22} - x_{j22})(b_{11} - y_{j11}) - (a_{12} - x_{j12})(b_{21} - y_{j21})}{(a_{11} - x_{j11})(a_{22} - y_{j22}) - (a_{12} - x_{j12})(a_{21} - y_{j21})}, \\
c_{12} &= \frac{(a_{22} - x_{j22})(b_{12} - y_{j12}) - (a_{12} - x_{j12})(b_{22} - y_{j22})}{(a_{11} - x_{j11})(a_{22} - y_{j22}) - (a_{12} - x_{j12})(a_{21} - y_{j21})}, \\
c_{21} &= \frac{(a_{11} - x_{j11})(b_{21} - y_{j21}) - (a_{21} - x_{j21})(b_{11} - y_{j11})}{(a_{11} - x_{j11})(a_{22} - y_{j22}) - (a_{12} - x_{j12})(a_{21} - y_{j21})}, \\
c_{22} &= \frac{(a_{11} - x_{j11})(b_{22} - y_{j22}) - (a_{21} - x_{j21})(b_{12} - y_{j12})}{(a_{11} - x_{j11})(a_{22} - y_{j22}) - (a_{12} - x_{j12})(a_{21} - y_{j21})}.
\end{align*}
\]

We find that the coefficients $c_{ij}, i, j = 1, 2$ of equations (9) satisfy system (5) by direct computation.
Thank you!