Exponential extinction time of the contact process on finite graphs

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Introduction
History of contact processes

- First introduced by T. E. Harris (1974).
- A model to describe the spread of diseases.
- Two classical books:
Basic definitions of contact process

- $G = (V, E)$: a connected undirected graph.
- The process $(\xi_t : t \geq 0)$: a continuous-time Markov process.
- State space: $\{A : A \subseteq V\}$.
- At each $t$, each vertex is either healthy or infected. $\xi_t$ is the collection of infected vertices at time $t$.
- Transition rates:
  \[
  \begin{align*}
  \xi_t &\to \xi_t \setminus \{x\} \text{ for } x \in \xi_t \text{ at rate } 1, \\
  \xi_t &\to \xi_t \cup \{x\} \text{ for } x \notin \xi_t \text{ at rate } \lambda \cdot |\{y \in \xi_t : x \sim y\}|.
  \end{align*}
  \]
- $(\xi_t^A : t \geq 0)$: the process with initial state $A$.
- Absorbing state: $\emptyset$. 

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Contact process on infinite graphs

- We say that the process \textit{survives} if the infection will persist with positive probability; otherwise we say that it \textit{dies out}.

- \textbf{survival} \begin{cases} \text{strong survival} \\ \text{weak survival} \end{cases}

- Strong survival: every site will be infected infinitely many times with positive probability.

- Weak survival: the infection will persist with positive probability, but every site will be infected only \textit{finite} times with probability 1. (Infection moves away to \textit{infinity}.)
Two critical values

\[
\begin{align*}
\lambda_1 &:= \inf\{\lambda : \text{the process survives}\}, \\
\lambda_2 &:= \inf\{\lambda : \text{the process survives strongly}\}.
\end{align*}
\]

Three phases:

\[
\begin{align*}
\xi_t &\text{ dies out if } \lambda < \lambda_1, \\
\xi_t &\text{ survives weakly if } \lambda_1 < \lambda < \lambda_2, \\
\xi_t &\text{ survives strongly if } \lambda > \lambda_2.
\end{align*}
\]
Two critical values

- On integer lattices $\mathbb{Z}^d$ ($d \geq 1$), $\lambda_1 = \lambda_2 \triangleq \lambda_c$.
- On homogeneous trees $T_d$ ($d \geq 3$), $\lambda_1 < \lambda_2$.

References:


If $G$ is a finite graph, then the contact process on $G$ must die out.

We are interested in the extinction time from full occupancy. Denote

$$\tau = \tau(G) = \inf\{t \geq 0 : \xi_t^V = \emptyset\}.$$
Contact process on finite subgraphs of $\mathbb{Z}^d$ (subcritical case)

- Consider the graph $\{0, \ldots, n\}^d$ (viewed as a subgraph of $\mathbb{Z}^d$) and the distribution of $\tau$ for this graph, as $n$ goes to infinity.
- If $\lambda < \lambda_c$, then $\tau / \log n$ converges in probability to a constant.

References:


Contact process on finite subgraphs of $\mathbb{Z}^d$ (supercritical case)

- If $\lambda > \lambda_c$, then $\log E[\tau]/n^d$ converges to a positive constant, and $\tau/E[\tau]$ converges in distribution to the unit exponential distribution.

References:


In the supercritical case, the order of magnitude of the extinction time is exponential in the number of vertices of the graph; the process is said to exhibit **metastability**, meaning that it persists for a long time in a state that resembles an equilibrium and then quickly moves to its true equilibrium (\(\emptyset\) in this case).
Contact process on finite subgraphs of $\mathbb{Z}^d$ (critical case)

- If \( d = 1 \) and \( \lambda = \lambda_c \), then \( \tau/n \to \infty \) and \( \tau/n^4 \to 0 \) in probability.

Reference:

Fix $d \geq 2$, let $\mathbb{T}_d^h$ be the finite subgraph of $\mathbb{T}_d$ defined by considering up to $h$ generations from the root and again take the contact process started from full occupancy on this graph, with associated extinction time $\tau$.

If $\lambda < \lambda_2$, then there exist constants $c, C > 0$ such that $P(ch \leq \tau \leq Ch) \to 1$ as $h \to \infty$. 
If $\lambda > \lambda_2$, then for any $\sigma < 1$ there exist $c_1, c_2 > 0$ such that $P \left[ \tau > c_1 e^{c_2 (\sigma d)^h} \right] \to 1$ as $h \to \infty$.

This implies that $\tau$ is at least as large as a stretched exponential function of the number of vertices, $(d + 1)^h$.

Reference:

As far as we know, no rigorous results are available concerning finite graphs which are not regular.
Main results
Main results

For $n \in \mathbb{N}$ and $d > 0$, let $\Lambda(n, d)$ be the set of all trees with $n$ vertices and degree bounded by $d$, and let $\mathcal{G}(n, d)$ be the set of graphs having a spanning tree in $\Lambda(n, d)$.

**Theorem 1.** For any $d \geq 2$ and $\lambda > \lambda_c(\mathbb{Z})$, there exists $c > 0$ such that

$$
\lim_{n \to \infty} \inf_{T \in \Lambda(n,d)} P [\tau_T \geq e^{cn}] = 1.
$$

In particular,

$$
\lim \inf_{n \to \infty} \inf_{T \in \Lambda(n,d)} \frac{\log E[\tau_T]}{n} \geq c.
$$
Main results

**Theorem 2.** Let $d \geq 2$, $\lambda > \lambda_c(\mathbb{Z})$, and $(G_n)_{n \in \mathbb{N}}$ be a sequence of graphs with $G_n \in \mathcal{G}(n, d)$. Then the distribution of $\tau_{G_n}/E[\tau_{G_n}]$ converges to the unitary exponential distribution as $n$ tends to infinity.

**Reference:**

Idea of proof
Lemma. For a tree $T \in \Lambda(n, d)$, there exists an edge whose removal separates $T$ into two subtrees $T_1$ and $T_2$ both of size at least $\lfloor n/d \rfloor$.

Proposition. For $n$ large enough, let $T \in \Lambda(n, d)$ be split into two subtrees $T_1, T_2$ as described by the above lemma. Then we have

$$E[\tau_T] \geq n^{-9} E[\tau_{T_1}] E[\tau_{T_2}]$$
Application
Application–Contact process on NSW random graphs

- The graph: $G^n := (V^n, E^n)$.  
- Vertex set: $V^n := \{1, 2, \cdots, n\}$.  
- Degree of vertex $i$ is denoted by $d_i$ ($i = 1, 2, \cdots, n$), which follows  
  (1) $d_1, d_2, \cdots, d_n$ i.i.d.  
  (2) $p_k (:= P(d_1 = k)) \sim Ck^{-\alpha}$ ($\alpha > 2, C > 0$) if $k$ is large.
Application–Contact process on NSW random graphs

How can we construct the graph once given a **suitable** realization of the degree sequence $(d_1(\omega), d_2(\omega), \cdots, d_n(\omega))$?
Application–Contact process on NSW random graphs

  each vertex $i$ was issued with $d_i$ half edges and these half edges were matched up in a uniformly chosen manner.

- If $\alpha > 3$, then $P(\text{no loops or multiple edges}) \rightarrow 1$ as $n \rightarrow \infty$. If $2 < \alpha \leq 3$, not so.

- We treat both cases in our work.
Assumptions and notation

• Assumptions:
  (1) $p_0 = p_1 = p_2 = 0$;
  (2) Conditioned on $E_n := \{d_1 + \cdots + d_n \text{ is even}\}$.

\[
\lim_{n \to \infty} P(E_n) = \frac{1}{2}
\]
Theorem 3. For any $\lambda > 0$, there exists $c > 0$ such that

$$P [ \tau_{G^n} \geq e^{cn}] \to 1 \text{ as } n \to \infty.$$ 

As a result, the critical infection parameter for these graphs is 0.
Application–Contact process on NSW random graphs

- **Remark.** Previous result: For any $\lambda > 0$ and any $\delta > 0$,

  \[ P \left[ \tau_{G^n} \geq e^{n^{1-\delta}} \right] \to 1 \text{ as } n \to \infty. \]

Reference:

Application–Contact process on NSW random graphs

- **Idea of proof.** Treat the “stars” as a single point and use the results of Theorems 1 and 2.
Further work
The study of metastable densities for the contact processes on NSW random graphs.

Reference:
Thank you!

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