Comparison morphisms and Hochschild cohomology

Shanghai Conference on Representation Theory of Algebras

Guodong Zhou

Ecole Polytechnique Fédérale de Lausanne, Switzerland

October 5, 2011
joint work with Jue Le

- Introduction
- Comparison morphisms
- Hochschild cohomology
- Monomial algebras
- Prospective
Given projective resolutions

\[ \cdots \xrightarrow{d_{n+1}^P} P_n \xrightarrow{d_n^P} \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} M \twoheadrightarrow 0 \]

\[ \cdots \xrightarrow{d_{n+1}^Q} Q_n \xrightarrow{d_n^Q} \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} N \twoheadrightarrow 0 \]

then one can lift \( f \) to a chain map.
Given projective resolutions

\[ \cdots \xrightarrow{d_{n+1}^P} P_n \xrightarrow{d_n^P} \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} M \xrightarrow{} 0 \]

\[ \cdots \xrightarrow{d_{n+1}^Q} Q_n \xrightarrow{d_n^Q} \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} N \xrightarrow{} 0 \]

then one can lift \( f \) to a chain map.
Given projective resolutions

\[
\cdots \xrightarrow{d_{n+1}^P} P_n \xrightarrow{d_n^P} \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} \mathcal{M} \xrightarrow{\theta} 0
\]

\[
\cdots \xrightarrow{d_{n+1}^Q} Q_n \xrightarrow{d_n^Q} \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} \mathcal{N} \xrightarrow{\theta} 0
\]

then one can lift \( f \) to a chain map.

Goal: Find an effective method for the construction of the lift and apply it to computations of Hochschild cohomology.
Let $A$ be a ring and $p : M \rightarrow N$ a surjection of $A$-modules.
Suppose that we have a set-theoretic section $t$ of $p$. 

\[
\begin{array}{c}
\xymatrix{A \ar[d]^f & \ar[l]_p M \ar[d] \ar[r] & N \ar[r] & 0 \ar[l]_t }
\end{array}
\]
We can define $\bar{f}(1) = t \circ f(1)$ and extend it $A$-linearly.
Comparison morphisms: Setwise homotopy

Let $k$ be a field and $A$ a $k$-algebra. Given a resolution

$$\cdots \xrightarrow{d_{n+1}^Q} Q_n \xrightarrow{d_n^Q} \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} N \rightarrow 0.$$  

**Definition (Ning Bian, Guanglian Zhang, Pu Zhang 2010)**

A setwise homotopy over this resolution is linear maps (even set-theoretic maps) $\{t_i : Q_i \rightarrow Q_{i+1}, i \geq -1\}$ (where $Q_{-1} := N$) such that

$$d_0^Q t_{-1} = Id_N \quad \text{and} \quad t_{n-1} d_n^Q + d_{n+1}^Q t_n = Id_{Q_n} \quad \text{for} \quad n \geq 0.$$
Given projective resolutions

\[
\cdots \xrightarrow{d_{n+1}^P} P_n \xrightarrow{d_n^P} \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} M \xrightarrow{f} 0
\]

\[
\cdots \xrightarrow{d_{n+1}^Q} Q_n \xrightarrow{d_n^Q} \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} N \xrightarrow{0}
\]

Given a setwise homotopy over the second resolution, then we can construct recursively a chain map \( f_* : P_* \rightarrow Q_* \) which lifts \( f \).
Supposons que pour chaque $n \geq 0$, $P_n = A^{(X_n)}$.

\[ \cdots \xrightarrow{d_{n+1}^P} P_n \xrightarrow{d_n^P} \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} M \xrightarrow{f} 0 \]

\[ \cdots \xrightarrow{d_{n+1}^Q} Q_n \xrightarrow{d_n^Q} \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} N \xrightarrow{0} \]

Pour $x \in X_0$, définir $f_0(x) = t_{-1} \circ f \circ d_0^P(x)$ et l'extender $A$-linéairement.

Pour $n \geq 1$, définir récursivement $f_n(x)$ pour $x \in X_n$:
Supposons that for each $n \geq 0$, $P_n = A^{(X_n)}$.

For $x \in X_0$, define $f_0(x) = t_{-1}f_{d_0^P}(x)$ and extend it $A$-linearly.
Comparison morphisms: Free resolution version

Supposons that for each $n \geq 0$, $P_n = A^{(X_n)}$.

\[
\begin{array}{cccccccc}
\cdots & d_{n+1}^P & \rightarrow & P_n & \rightarrow & \cdots & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & M & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & d_{n+1}^Q & \rightarrow & Q_n & \rightarrow & \cdots & \rightarrow & Q_1 & \rightarrow & Q_0 & \rightarrow & N & \rightarrow & 0 \\
\end{array}
\]

For $x \in X_0$, define $f_0(x) = t_{-1}fd_0^P(x)$ and extend it $A$-linearly. For $n \geq 1$, define recursively $f_n(x)$ for $x \in X_n$:
Supposons que pour chaque $n \geq 0$, $P_n = A(\mathcal{X}_n)$.

$\cdots \xrightarrow{d_{n+1}^P} P_n \xrightarrow{d_n^P} \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} M \xrightarrow{f} 0$

$\cdots \xrightarrow{d_{n+1}^Q} Q_n \xrightarrow{d_n^Q} \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} N \xrightarrow{f} 0$

Pour $x \in \mathcal{X}_0$, définissez $f_0(x) = t_{-1} f d_0^P(x)$ et étendez-le $A$-linéairement. Pour $n \geq 1$, définissez récursivement $f_n(x)$ pour $x \in \mathcal{X}_n$:
Suppose that $A$ is a semiperfect ring. The method also works when $P_\ast$ is a projective resolution and in this case we need to modify the setwise homotopy $t_\ast$. 
Hochschild cohomology: The bar resolution

Let $A$ be a $k$-algebra.

The bar resolution $(\text{Bar}_*(A), d_*)$ is a projective resolution of $A$ as bimodules.
Hochschild cohomology: The bar resolution

Let $A$ be a $k$-algebra.

The bar resolution $(\text{Bar}_*(A), d_*)$ is a projective resolution of $A$ as bimodules.

$\text{Bar}_n(A) = A \otimes A^\otimes n \otimes A$, $n \geq 0$, and

the differential $d_n: \text{Bar}_n(A) \rightarrow \text{Bar}_{n-1}(A)$

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for $n \geq 1$ and $d_0: \text{Bar}_0(A) = A \otimes A \rightarrow A$ is the multiplication of $A$. 
Hochschild cohomology: The bar resolution

Let $A$ be a $k$-algebra.

The bar resolution $(\text{Bar}_*(A), d_*)$ is a projective resolution of $A$ as bimodules.

$$\text{Bar}_n(A) = A \otimes A^\otimes n \otimes A, \quad n \geq 0,$$

and the differential $d_n : \text{Bar}_n(A) \to \text{Bar}_{n-1}(A)$

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for $n \geq 1$ and $d_0 : \text{Bar}_0(A) = A \otimes A \to A$ is the multiplication of $A$.

This is a complex, and it is exact, as there is a self-homotopy

$$s_n : \text{Bar}_n(A) \to \text{Bar}_{n+1}(A), \quad a_0 \otimes \cdots \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n+1},$$

which are homomorphisms of right $A$-modules.
Let $A$ be a $k$-algebra.
Hochschild cohomology: Basic definitions

Let $A$ be a $k$-algebra.

\[ HH^*(A) = HH^*(A, A) = \text{Ext}^*_{Ae}(A, A) \]
Hochschild cohomology: Basic definitions

Let $A$ be a $k$-algebra.

$$HH^*(A) = HH^*(A, A) = Ext^*_A(A, A)$$

$$C^*(A, A) = Hom_A^e(Bar_*(A), A)$$
Hochschild cohomology: Basic definitions

Let $A$ be a $k$-algebra.

$$HH^*(A) = HH^*(A, A) = \text{Ext}^*_A(A, A)$$

$$C^*(A, A) = \text{Hom}_A(\text{Bar}_*(A), A)$$

$$C^n(A, A) = \text{Hom}_A(\text{Bar}_n(A), A) \simeq \text{Hom}_k(A^{\otimes n}, A)$$
Let $A$ be a $k$-algebra.

$HH^*(A) = HH^*(A, A) = \text{Ext}^*_A(A, A)$

$C^*(A, A) = \text{Hom}_A(\text{Bar}_*(A), A)$

$C^n(A, A) = \text{Hom}_A(\text{Bar}_n(A), A) \cong \text{Hom}_k(A^\otimes n, A)$

The differential $\delta : C^n(A, A) \to C^{n+1}(A, A)$ is given by

$$\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes a_{n+1})$$

$$+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}$$
Let $f \in C^n(A, A)$ and $g \in C^m(A, A)$. Then $f \smile g \in C^{n+m}(A, A)$ defined as follows:

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{n+m}) := f(a_1 \otimes \cdots \otimes a_n)g(a_{n+1} \otimes \cdots \otimes a_{n+m})$$
Hochschild cohomology: Lie bracket

Let $f \in C^n(A, A)$ and $g \in C^m(A, A)$. Then $[f, g] \in C^{n+m-1}(A)$ is defined as follows:

If $n \geq 1$, for $1 \leq i \leq n,$

$$f \circ_i g(a_1 \otimes \cdots \otimes a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes \cdots \otimes a_{n+m-1});$$

if $n = 0$, $f \circ_i g = 0.$
Hochschild cohomology: Lie bracket

Let \( f \in C^n(A, A) \) and \( g \in C^m(A, A) \). Then \([f, g] \in C^{n+m-1}(A)\) is defined as follows:

If \( n \geq 1 \), for \( 1 \leq i \leq n \),

\[
f \circ_i g(a_1 \otimes \cdots \otimes a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes \cdots \otimes a_{n+m-1});
\]

if \( n = 0 \), \( f \circ_i g = 0 \).

Define

\[
f \circ g = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g
\]

\[
[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f
\]
Hochschild cohomology: Gerstenhaber’s work 1963

Hochschild cohomology

$$HH^*(A) = \bigoplus_{i=0}^{\infty} HH^i(A) = \bigoplus_{i=0}^{\infty} \text{Ext}^i_{A^e}(A, A)$$

has
Hochschild cohomology

\[ HH^\ast(A) = \bigoplus_{i=0}^{\infty} HH^i(A) = \bigoplus_{i=0}^{\infty} \text{Ext}_{A^e}^i(A, A) \]

has

\[ \langle, \rangle : HH^n(A) \times HH^m(A) \to HH^{n+m}(A) \]
Hochschild cohomology: Gerstenhaber’s work 1963

Hochschild cohomology

\[ HH^*(A) = \bigoplus_{i=0}^{\infty} HH^i(A) = \bigoplus_{i=0}^{\infty} \text{Ext}^i_{Ae}(A, A) \]

has

- \( \lor : HH^n(A) \times HH^m(A) \to HH^{n+m}(A) \)
- \( [ \, , \, ] : HH^n(A) \times HH^m(A) \to HH^{n+m-1}(A) \)
Hochschild cohomology: Gerstenhaber’s work 1963

Hochschild cohomology

\[ HH^*(A) = \bigoplus_{i=0}^{\infty} HH^i(A) = \bigoplus_{i=0}^{\infty} \text{Ext}_A^i(A, A) \]

has

\[ \circlearrowright: HH^n(A) \times HH^m(A) \rightarrow HH^{n+m}(A) \]

\[ [\ , \ ]: HH^n(A) \times HH^m(A) \rightarrow HH^{n+m-1}(A) \]

such that

\[ (HH^*(A), \circlearrowright) \text{ is graded commutative} \]
Hochschild cohomology

\[ HH^*(A) = \bigoplus_{i=0}^{\infty} HH^i(A) = \bigoplus_{i=0}^{\infty} \text{Ext}^i_{A^e}(A, A) \]

has

- \( \cup : HH^n(A) \times HH^m(A) \to HH^{n+m}(A) \)
- \( [ , ] : HH^n(A) \times HH^m(A) \to HH^{n+m-1}(A) \)

such that

- \( (HH^*(A), \cup) \) is graded commutative
- \( (HH^*(A), [ , ]) \) is a graded Lie algebra
Hochschild cohomology: Problems and solutions

Problems:

- The bar resolution is large
Problems:

- The bar resolution is large
- The above structures are defined by using the (reduced) bar resolution

Solutions:

- Construct a smaller resolution.
- Construct comparison morphisms between the bar resolution and the smaller resolution.

Remark

Let $P^* \rightarrow A$ be a projective bimodule resolution. As resolution of one-sided modules, it splits, so there is always a setwise homotopy over it which consists of right (or left) $A$-module homomorphisms.
Hochschild cohomology: Problems and solutions

Problems:

- The bar resolution is large
- The above structures are defined by using the (reduced) bar resolution

Solutions:
Hochschild cohomology: Problems and solutions

Problems:

- The bar resolution is large
- The above structures are defined by using the (reduced) bar resolution

Solutions:

- Construct a smaller resolution.
Problems:

- The bar resolution is large
- The above structures are defined by using the (reduced) bar resolution

Solutions:

- Construct a smaller resolution.
- Construct comparison morphisms between the bar resolution and the smaller resolution

Remark:

Let $P^* \to A$ be a projective bimodule resolution. As resolution of one-sided modules, it splits, so there is always a setwise homotopy over it which consists of right (or left) $A$-module homomorphisms.
Hochschild cohomology: Problems and solutions

Problems:
- The bar resolution is large
- The above structures are defined by using the (reduced) bar resolution

Solutions:
- Construct a smaller resolution.
- Construct comparison morphisms between the bar resolution and the smaller resolution

Remark
Let $P_* \rightarrow A$ be a projective bimodule resolution. As resolution of one-sided modules, it splits, so there is always a setwise homotopy over it which consists of right (or left) $A$-module homomorphisms.
Let $A = kQ/I$ be a monomial algebra, i.e. $I$ is generated by paths of length $\geq 2$. 
Let $A = kQ/I$ be a monomial algebra, i.e. $I$ is generated by paths of length $\geq 2$.

Bardzell 1997 constructed a minimal projective bimodule resolution of a monomial algebra.
Let $A = kQ/I$ be a monomial algebra, i.e. $I$ is generated by paths of length $\geq 2$.

Bardzell 1997 constructed a minimal projective bimodule resolution of a monomial algebra.

Sköldberg 2008 constructed a setwise homotopy over Bardzell minimal resolution.
Let $A = kQ/I$ be a monomial algebra, i.e. $I$ is generated by paths of length $\geq 2$.

Bardzell 1997 constructed a minimal projective bimodule resolution of a monomial algebra.

Sköldberg 2008 constructed a setwise homotopy over Bardzell minimal resolution.

We can construct comparison morphisms between Bardzell’s resolution and the reduced bar resolution.
Monomial algebras: Bardzell vers bar

Let $A = kQ/I$ be a monomial algebra, i.e. $I$ is generated by paths of length $\geq 2$.

Bardzell 1997 constructed a minimal projective bimodule resolution of a monomial algebra.

Sköldberg 2008 constructed a setwise homotopy over Bardzell minimal resolution.

We can construct comparison morphisms between Bardzell’s resolution and the reduced bar resolution.

We can give explicit formulae for the cup product and the Lie bracket over the Hochschild cohomology of a monomial algebra.
Monomial algebras: Special cases known

- Truncated quiver algebras
  - comparison morphisms and cup product (Ames, Cagliero, Tirao 2009)
  - Lie bracket (Yunge Xu, Chao Zhang 2010)

- Quadratic monomial algebras
  - comparison morphisms, cup product and Lie bracket (Bustamente 2006)
Conjecture (Bustamente 2006)

Let $A = kQ/I$ be a monomial triangular algebra. Then the cup product over $HH^{\geq 1}(A)$ is trivial.
Prospective: Bustamente’s conjecture

Conjecture (Bustamente 2006)

Let $A = kQ/I$ be a monomial triangular algebra. Then the cup product over $HH_{\geq 1}(A)$ is trivial.

Special cases known:

- Truncated quiver algebras (Ames, Cagliero, Tirao 2009)
- Triangular string algebras (Bustamente 2006)
(D-)Koszul algebras

▶ Koszul algebras (Priddy 1970)
Prospective: (D-)Koszul algebras

(D-)Koszul algebras

- Koszul algebras (Priddy 1970)
- (D-)Koszul algebras (Berger 2001, Green, Marcos, Martinez-Villa, Pu Zhang 2004)
Prospective: (D-)Koszul algebras

(D-)Koszul algebras
  ▶ Koszul algebras (Priddy 1970)
  ▶ (D-)Koszul algebras (Berger 2001, Green, Marcos, Martinez-Villa, Pu Zhang 2004)

We should expect to recover
  ▶ Cup product for Koszul algebras (Burweitz, Green, Snashall, Solberg 2008)
Prospective: (D-)Koszul algebras

(D-)Koszul algebras

- Koszul algebras (Priddy 1970)
- (D-)Koszul algebras (Berger 2001, Green, Marcos, Martinez-Villa, Pu Zhang 2004)

We should expect to recover

- Cup product for Koszul algebras (Burweitz, Green, Snashall, Solberg 2008)
- Cup product for (d-)Koszul algebras (Yunge Xu, Huali Xiang 2011)
Thank you!